Key
MATH 012 Final
Graphing Calculator Portion

1. Convert 24° to radians. Do not round.

Solution:

$$24^\circ = \frac{24^\circ \cdot \pi}{180^\circ} = \frac{2\pi}{15}$$

2. Determine the value of $\cos 1$. Round to the nearest hundredth.

Solution: Since there is nothing after the 1, the angle is in radians. Thus, $\cos 1 \approx 0.54$.

3. Determine the **exact** value of the following expression:

$$\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$$

Solution:

$$\begin{aligned}\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

4. Determine the real and imaginary part of $(2-3i)(1+4i)$.

Solution:

$$\begin{aligned}(2-3i)(1+4i) &= 2+8i-3i-12i^2 \\ &= 2+5i-12(-1) \\ &= 14+5i\end{aligned}$$

Thus, the real part is 14 and the imaginary part is 5.

5. Convert the decimal $0.\overline{79}$ to a fraction. Show all work.

Solution:

$$\begin{aligned}x &= 0.\overline{79} \\ 100x &= 79.\overline{79} \\ 99x &= 79 \\ x &= \frac{79}{99}\end{aligned}$$

6. Solve the following system of equations for x and y:

$$\begin{cases} 2x + 3y = 1 \\ 3x - y = -4 \end{cases}$$

Solution: Multiplying the third equation by 3 yields the following system:

$$\begin{cases} 2x + 3y = 1 \\ 9x - -3y = -12 \end{cases}$$

Adding these two equations yields:

$$11x = -11$$

Thus, $x = -1$. Plugging this back into $3x - y = -4$ yields:

$$\begin{aligned} 3(-1) - y &= -4 \\ -3 - y &= -4 \\ -y &= -1 \\ y &= 1 \end{aligned}$$

The answer as an ordered pair is $(-1, 1)$.

7. Solve the inequality $|3t-2|+6 > 7$ for t. Put the answer in interval notation.

Solution:

$$\begin{aligned} |3t-2|+6 &> 7 \\ |3t-2| &> 1 \end{aligned}$$

$$\begin{array}{rlcrl} 3t-2 & > 1 & \text{or} & 3t-2 & < -1 \\ 3t & > 3 & \text{or} & 3t & < 1 \\ t & > 1 & \text{or} & t & < \frac{1}{3} \end{array}$$

The answer in interval notation is $\left(-\infty, \frac{1}{3}\right) \bigcup \left(1, \infty\right)$.

8. If $f(x) = \lfloor x \rfloor$ and $g(x) = 2\sin x + 3$, determine $(f \circ g)\left(\frac{\pi}{4}\right)$.

Solution:

$$\begin{aligned} (f \circ g)\left(\frac{\pi}{4}\right) &= f\left(g\left(\frac{\pi}{4}\right)\right) \\ &= f\left(2\sin\left(\frac{\pi}{4}\right) + 3\right) \\ &= f\left(2\left(\frac{\sqrt{2}}{2}\right) + 3\right) \\ &= f(\sqrt{2} + 3) \\ &= \lfloor \sqrt{2} + 3 \rfloor \\ &= 4 \end{aligned}$$

9. Is the function $f(x) = \dfrac{\sin x}{x^2 + |x| + 1}$ even, odd, or neither? Explain.

Solution: Since $f(-x) = \dfrac{\sin(-x)}{(-x)^2 + |-x| + 1} = \dfrac{-\sin x}{x^2 + |x| + 1} = -f(x)$, $f(x)$ is odd.

10. Let θ be an angle in the third quadrant. Write $\tan\theta$ as an expression in terms of $\cos\theta$.

Solution: Recall the equations $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$.

By the second equation $\sin^2\theta = 1 - \cos^2\theta$. Since θ is in the third quadrant, $\sin\theta$ is negative. Therefore, $\sin\theta = -\sqrt{1 - \cos^2\theta}$. Plugging this into the first equation yields $\tan\theta = \dfrac{-\sqrt{1 - \cos^2\theta}}{\cos\theta}$.

11. If $g(x) = x^2 - 2x + 8$, determine and simplify the following expression:

$$\frac{g(x+h) - g(x)}{h}$$

Solution:
$$\begin{aligned}
\frac{g(x+h) - g(x)}{h} &= \frac{(x+h)^2 - 2(x+h) + 8 - (x^2 - 2x + 8)}{h} \\
&= \frac{x^2 + 2hx + h^2 - 2x - 2h + 8 - x^2 + 2x - 8}{h} \\
&= \frac{2hx + h^2 - 2h}{h} \\
&= 2x + h - 2
\end{aligned}$$

12. Solve the equation $\sqrt{3x+4} - \sqrt{x+1} = 3$ for x.

Solution:
$$\begin{aligned}
\sqrt{3x+4} - \sqrt{x+1} &= 3 \\
\sqrt{3x+4} &= 3 + \sqrt{x+1} \\
3x + 4 &= (3 + \sqrt{x+1})^2 \\
3x + 4 &= 9 + 6\sqrt{x+1} + x + 1 \\
3x + 4 &= x + 10 + 6\sqrt{x+1} \\
2x - 6 &= 6\sqrt{x+1} \\
x - 3 &= 3\sqrt{x+1} \\
(x-3)^2 &= 9(x+1) \\
x^2 - 6x + 9 &= 9x + 9 \\
x^2 - 15x &= 0 \\
x(x - 15) &= 0 \\
x = 0 \quad &\text{or} \quad x = 15
\end{aligned}$$

Checking $x = 0$: $\sqrt{3 \cdot 0 + 4} - \sqrt{0 + 1} = \sqrt{4} - \sqrt{1} = 2 - 1 = 1 \neq 3$
Checking $x = 15$: $\sqrt{3 \cdot 15 + 4} - \sqrt{15 + 1} = \sqrt{49} - \sqrt{16} = 7 - 4 = 3$.

Thus, the only solution is $x = 15$.

13. a. Determine the domain of $y = \log_3 x - \log_3(1-x)$.

Solution: Because of the logarithms, it must be the case that $x > 0$ and $1-x > 0$. The second inequality yields $x < 1$. Combining this with the first inequality yields $0 < x < 1$.

b. Determine the inverse of $y = \log_3 x - \log_3(1-x)$.

Solution: This is accomplished by switching x and y, then solving for y.

$$\begin{aligned}
x &= \log_3 y - \log_3(1-y) \\
x &= \log_3\left(\frac{y}{1-y}\right) \\
3^x &= \left(\frac{y}{1-y}\right) \\
3^x(1-y) &= y \\
3^x - 3^x y &= y \\
3^x &= 3^x y + y \\
3^x y + y &= 3^x \\
(3^x+1)y &= 3^x \\
y &= \frac{3^x}{3^x+1}
\end{aligned}$$

c. What is the range of $y = \log_3 x - \log_3(1-x)$? Explain.

Solution: Since the domain of the inverse function is $\mathbb{R}$, the range of the original function is $\mathbb{R}$.

14. Determine all real zeroes of the polynomial function $p(x) = 2x^6 - 3x^5 - 2x^4 - 8x^2 + 12x + 8$. Show all work. Do not round.

Solution: There are many ways to work this problem. Factoring by grouping right at the beginning is quite effective:

$$\begin{aligned}
2x^6 - 3x^5 - 2x^4 - 8x^2 + 12x + 8 &= x^4(2x^2 - 3x - 2) - 4(2x^2 - 3x - 2) \\
&= (x^4 - 4)(2x^2 - 3x - 2) \\
&= (x^2+2)(x^2-2)(2x^2 - 3x - 2) \\
&= (x^2)(x+\sqrt{2})(x-\sqrt{2})(x-2)(2x+1)
\end{aligned}$$

Note that $x^2 + 2$ has no real roots. Thus, the real roots of $p(x)$ are:

$$-\sqrt{2},\ \sqrt{2},\ 2,\ \text{and}\ \frac{-1}{2}$$

15. Samuel and Samantha can mow the lawn in 30 minutes if they work together. Samuel can mow the lawn by himself in half the time that it takes Samantha to mow the lawn by herself. How long does it take Samuel to mow the lawn by himself?

Solution: Let t be the time that it takes Samuel to mow the lawn by himself. Since Samantha takes twice as long, the time that it takes her to mow the lawn by herself is $2t$. Thus, Samuel's and Samantha's rates are $\frac{1}{t}$ and $\frac{1}{2t}$, respectively.

	rate	time	work
Samuel	$\frac{1}{t}$	30	$\frac{30}{t}$
Samantha	$\frac{1}{2t}$	30	$\frac{15}{t}$

$$\begin{aligned} \frac{30}{t} + \frac{15}{t} &= 1 \\ \frac{45}{t} &= 1 \\ t &= 45 \text{ minutes} \end{aligned}$$

16. Determine two positive real numbers whose sum is 100 such that their product is maximal.

Solution: Let x be the first number. Since the sum of the two numbers is 100, the second number must be $100 - x$. Their product is $x(100 - x) = 100x - x^2$, which is a quadratic function of x. Since $a = -1 < 0$, the function has a maximum. It occurs at $x = \frac{-100}{2(-1)} = 50$. When $x = 50$, $100 - x = 100 - 50 = 50$. Thus, the two numbers whose sum is 100 with maximal product are 50 and 50.