MATH 012 Test 3 Key

No Graphing Calculator Portion

Note that I cannot graph with this typesetter, so I will try to describe the graphs as best as I can for problems that require graphing.

1. Graph the function $y = -\log_3(x+1) - 2$ using transformations, starting with the appropriate basic graph. Label each graph with the appropriate equation.

Solution: The basic graph, $y = \log_3 x$, has a vertical asymptote at $x = 0$ and passes through the points $(1, 0)$ and $(3, 1)$. The next graph should be $y = \log_3(x+1)$, which is the previous graph shifted 1 to the left. The next graph should be $y = -\log_3(x+1)$, which is the previous graph reflected across the x axis. The final graph, that of $y = -\log_3(x+1) - 2$, is the previous graph shifted 2 down.

2. Determine all intercepts and asymptotes of $y = \dfrac{x^2 - 10x + 9}{x + 1}$.

Solution: Note that $x^2 - 10x + 9$ factors as $(x-1)(x-9)$. Thus, the x intercepts are at $(1, 0)$ and at $(9, 0)$. The y intercept is at $(0, 9)$. Since $x + 1$ is in the denominator, the graph has a vertical asymptote $x = -1$. Since the degree of the numerator is exactly one more than the degree of the denominator, the graph has a slant asymptote and thus does not have a horizontal asymptote. The slant asymptote can be obtained by using synthetic division:

-1	1	-10	9
		-1	11
	1	-11	20

Thus, the graph has a slant asymptote at $y = x - 11$.

b. Graph the function $y = \dfrac{x^2 - 10x + 9}{x + 1}$.

Solution: By splitting up $\dfrac{(x-1)(x-9)}{x+1}$, it can be determined that the expression is negative when $x < -1$, positive when $-1 < x < 1$, negative when $1 < x < 9$, and positive when $x > 9$. Thus, as x approaches -1 from the left, y approaches $-\infty$, and as x approaches -1 from the right, y approaches ∞. Therefore, the graph is below the slant asymptote when $x < -1$ and is above the slant asymptote when $x > -1$.

3. a. Factor $2x^3 - 3x^2 - 8x + 12$ by grouping.

Solution:

$$\begin{aligned} 2x^3 - 3x^2 - 8x + 12 &= x^2(2x-3) - 4(2x-3) \\ &= (x^2-4)(2x-3) \\ &= (x-2)(x+2)(2x-3) \end{aligned}$$

b. Graph the function $y = 2x^3 - 3x^2 - 8x + 12$. Determine all intercepts.

Solution: Because the degree is 3 and the leading coefficient is positive, the graph is below the x axis to the far left and is above the x axis to the far right.

By part a., the x intercepts are at $(-2, 0)$, $\left(\frac{3}{2}, 0\right)$, and $(2, 0)$. The y intercept is at $(0, 12)$. The graph starts from below the x axis, crosses the x axis at $(-2, 0)$, crosses the y axis at $(0, 12)$, crosses below the x axis at $\left(\frac{3}{2}, 0\right)$, and crosses the x axis at $(2, 0)$.

4. Molly is trying to obtain information about the graph of the polynomial function $f(x) = 2x^5 - x^4 - 2x^3 + x^2 - 4x + 2$. She performs the following calculations.

1|	2	-1	-2	1	-4	2
		2	1	-1	0	-4
	2	1	-1	0	-4	-2

2|	2	-1	-2	1	-4	2
		4	6	8	18	28
	2	3	4	9	14	30

a. Use the remainder theorem to calculate $f(1)$ and $f(2)$.

Solution: By the remainder theorem, $f(1) = -2$ and $f(2) = 30$.

b. Using the intermediate value theorem, what information about the real roots of f can Molly deduce?

Solution: By the intermediate value theorem, f has a real root between 1 and 2.

c. Using the upper bound theorem, what information about the real roots of f can Molly deduce?

Solution: By the upper bound theorem, f has no real root larger than 2.

Graphing Calculator Portion

1. Determine all real zeroes of the function $f(x) = 2x^4 - x^3 - 7x^2 + 3x + 3$. Do not round.

Solution:

1|	2	-1	-7	3	3
		2	1	-6	-3
	2	1	-6	-3	0

Thus, 1 is a real zero of f. Synthetically dividing $2x^3 + x^2 - 6x - 3$ by $x + \frac{1}{2}$ would be successful, but I prefer to factor.

$$\begin{aligned} 2x^3 + x^2 - 6x - 3 &= x^2(2x+1) - 3(2x+1) \\ &= (x^2 - 3)(2x+1) \end{aligned}$$

Since $2x + 1$ is a factor of f, then $\frac{-1}{2}$ is a zero of f.

The factor $x^2 - 3$ has zeroes $\sqrt{3}$ and $-\sqrt{3}$.

Thus, the real zeroes of f are 1, $\frac{-1}{2}$, $\sqrt{3}$, and $-\sqrt{3}$.

2. Harry wants start a savings account. Bank A offers an interest rate of 8.55% compounded quarterly, and Bank B offers an interest rate of 8.5% compounded continuously. He has \$1500 that he is willing to invest, and he plans on making no withdrawals from this account for ten years. At which bank should Harry start a savings account? Justify your answer.

Solution: Depositing the \$1500 in Bank A would yield

$$\begin{aligned} A &= 1500\left(1 + \frac{0.0855}{4}\right)^{4\cdot 10} \\ &= \$3495.43 \end{aligned}$$

Depositing the \$1500 in Bank B would yield

$$\begin{aligned} A &= 1500e^{0.085\cdot 10} \\ &= \$3509.47 \end{aligned}$$

Thus, Bank B is the better choice.

3. Expand $\ln\left(\frac{(x^2+1)^3}{29z^5}\right)$ completely.

Solution: $3\ln(x^2+1) - \ln 29 - 5\ln z$

4. Solve $4^x - 3 \cdot 2^x = -2$ for x.

Solution:

$$
\begin{aligned}
4^x - 3 \cdot 2^x &= -2 \\
(2^2)^x - 3 \cdot 2^x + 2 &= 0 \\
(2^x)^2 - 3 \cdot 2^x + 2 &= 0 \\
\text{Let } u = 2^x \qquad u^2 - 3u + 2 &= 0 \\
(u-1)(u-2) &= 0
\end{aligned}
$$

$$
\begin{aligned}
u &= 1 \quad \text{or} \quad u = 2 \\
2^x &= 1 \quad \text{or} \quad 2^x = 2 \\
x &= 0 \quad \text{or} \quad x = 1
\end{aligned}
$$

It can be verified that $x = 0$ and $x = 1$ are indeed solutions to the original equation.

5. Solve $\log_9(x-5) + \log_9(x+3) = 1$ for x.

Solution:

$$
\begin{aligned}
\log_9(x-5) + \log_9(x+3) &= 1 \\
\log_9((x-5)(x+3)) &= 1 \\
\log_9(x^2 - 2x - 15) &= 1 \\
x^2 - 2x - 15 &= 9 \\
x^2 - 2x - 24 &= 0 \\
(x-6)(x+4) &= 0
\end{aligned}
$$

$$
x = 6 \quad \text{or} \quad x = -4
$$

Note that $x = -4$ is not in the domain of either $\log_9(x-5)$ or $\log_9(x+3)$ and thus is an extraneous solution.

It can be verified that $x = 6$ is indeed a solution to the original equation.

6. The half-life of strontium-90 is 28 years. How long will it take a 50 mg sample to decay to a mass of 32 mg? Round to the nearest hundredth of a year.

Solution:

$$
\begin{aligned}
32 &= 50\left(\frac{1}{2}\right)^{\frac{t}{28}} \\
\frac{32}{50} &= \left(\frac{1}{2}\right)^{\frac{t}{28}} \\
\ln\left(\frac{16}{25}\right) &= \frac{t}{28}\ln\left(\frac{1}{2}\right) \\
\ln 16 - \ln 25 &= \frac{-t}{28}\ln 2 \\
\ln 25 - \ln 16 &= \frac{t \ln 2}{28} \\
\frac{28(\ln 25 - \ln 16)}{\ln 2} &= t \\
t &\approx 18.03 \text{ years}
\end{aligned}
$$

This time, I will do the extra credit, as I think that these problems are extremely interesting and helpful.

1. Prove that $\log_b m + \log_b n = \log_b(mn)$. (*Suggestion:* Let $x = \log_b m$ and $y = \log_b n$, then use the definition of logarithm a few times.)

Solution: If $x = \log_b m$, then $b^x = m$. If $y = \log_b n$, then $b^y = n$. Thus, $b^x b^y = mn$. This simplifies to $b^{x+y} = mn$. Using the definition of logarithm yields $x + y = \log_b(mn)$. It follows that $\log_b m + \log_b n = \log_b(mn)$.

2. Construct a polynomial with integer coefficients that has both $\sqrt{2}$ and $\dfrac{1+\sqrt{5}}{2}$ as zeroes. Determine all the zeroes of the constructed polynomial.

Solution: First, a polynomial with integer coefficients that has $\sqrt{2}$ as a zero will be constructed.

$$\begin{aligned} x &= \sqrt{2} \\ x^2 &= 2 \\ x^2 - 2 &= 0 \end{aligned}$$

Next, a polynomial with integer coefficients that has $\dfrac{1+\sqrt{5}}{2}$ as a zero will be constructed.

$$\begin{aligned} x &= \frac{1+\sqrt{5}}{2} \\ 2x &= 1+\sqrt{5} \\ 2x-1 &= \sqrt{5} \\ (2x-1)^2 &= 5 \\ (2x-1)(2x-1) &= 5 \\ 4x^2-4x+1 &= 5 \\ 4x^2-4x-4 &= 0 \\ x^2-x-1 &= 0 \end{aligned}$$

(The polynomial in the second to last equation is acceptable, but dividing by four makes the polynomial simpler.)

Thus, polynomial with integer coefficients that has both $\sqrt{2}$ and $\dfrac{1+\sqrt{5}}{2}$ as zeroes is

$$\begin{aligned} (x^2-2)(x^2-x-1) &= x^4 - x^3 - x^2 - 2x^2 + 2x + 2 \\ &= x^4 - x^3 - 3x^2 + 2x + 2 \end{aligned}$$

The zeroes of this polynomial are $\sqrt{2}$, $-\sqrt{2}$, $\dfrac{1+\sqrt{5}}{2}$, and $\dfrac{1-\sqrt{5}}{2}$. This can be deduced by determining the roots of the polynomials x^2-2 and x^2-x-1.