Math 315 D1

HOUR EXAM III

6 August 2003

SOLUTIONS

1. Identify the conic sections represented below by diagonalizing the defining matrices.
   a. $3x^2 + 8xy + 3y^2 - 28 = 0$
   b. $3x^2 + 4xy + 3y^2 - 28 = 0$

   SOLUTION: a) $3x^2 + 8xy + 3y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 28$. To find the eigenvalues of $\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$, consider $\begin{vmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 16 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$. Therefore the eigenvalues are 7 and -1 and this clearly defines a hyperbola.
   b) In this case the eigenvalues are 1 and 5 and so the curve is an ellipse.

2. Let A be any $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that if A is diagonalizable then $A = \lambda I$.

   SOLUTION: Since A is diagonalizable, there is a nonsingular matrix X such that $X^{-1}AX = D$, where D is a diagonal matrix whose diagonal elements are the eigenvalues of A. Since λ is the only eigenvalue, $D = \lambda I$. Hence

   $$X^{-1}AX = \lambda I.$$

   Now multiply this equation on the right by X^{-1} and on the left by X and we get

   $$A = X(\lambda I)X^{-1} = \lambda I$$

   since I commutes with every matrix.

3. Suppose the Gram-Schmidt process is applied to a set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, where $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent and $\mathbf{v_3} \in Span(\mathbf{v_1}, \mathbf{v_2})$. Describe the resulting vectors: $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. Explain.

   SOLUTION: Since $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent, the Gram-Schmidt process will produce the vectors $\{\mathbf{u_1}, \mathbf{u_2}\}$ which are orthonormal and satisfy $Span(\mathbf{u_1}, \mathbf{u_2}) = Span(\mathbf{v_1}, \mathbf{v_2})$. In order to compute $\mathbf{u_3}$ we need to find the projection, $\mathbf{p_2}$, of $\mathbf{v_3}$ onto $Span(\mathbf{u_1}, \mathbf{u_2})$. But $\mathbf{v_3} \in Span(\mathbf{v_1}, \mathbf{v_2}) = Span(\mathbf{u_1}, \mathbf{u_2})$ and so $\mathbf{v_3} = \mathbf{p_2}$. Hence $\mathbf{u_3}$ which is equal to $\mathbf{v_3} - \mathbf{p_2}$ divided by its length will be zero and so the final result of the Gram-Schmidt is just $\{\mathbf{u_1}, \mathbf{u_2}\}$.

4. Let $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Find a basis for each of the eigenspaces of A.
Is A diagonalizable? Explain.

<u>SOLUTION</u>: $\begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)(2-\lambda)(2-\lambda).$

Thus the eigenvalues are 1, 2, 3 and we need to describe the eigenvectors associated with each of these eigenvalues.

$\lambda = 1$

$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The solution vector is $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \\ 0 \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Therefore the eigenspace of $\lambda = 1$ is the one-dimensional space $Span\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$.

$\lambda = 2$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The solution vector is $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \end{pmatrix} = c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Therefore the eigenspace of $\lambda = 2$ is the one-dimensional space $Span\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right)$.

$\lambda = 3$

$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. The solution vector is $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{b}{2} \\ b \\ 0 \\ 0 \end{pmatrix} =$
$\frac{b}{2} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

Therefore the eigenspace of $\lambda = 3$ is the one-dimensional space $Span\left(\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}\right)$.

Since all eigenspaces yield only 3 linearly independent eigenvectors, the 4×4 matrix A is not diagonalizable.

5. Let $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ be an orthonormal basis of C^3. Let $\mathbf{y} = i\mathbf{x_1} + (2+i)\mathbf{x_2} + 2\mathbf{x_3}$ and $\mathbf{z} = \mathbf{x_1} + (3+i)\mathbf{x_3}$. Find
a. $\langle \mathbf{x_3}, \mathbf{y} \rangle, \langle \mathbf{y}, \mathbf{z} \rangle$.
b. $\|\mathbf{y}\|$ and $\|\mathbf{z}\|$.

<u>SOLUTION</u>: Since $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ is an orthonormal set, we use the fact that $\mathbf{x_j}^H \mathbf{x_k} = 0$ when $j \neq k$ and $\mathbf{x_j}^H \mathbf{x_j} = 1$.

a. $\langle \mathbf{x_3}, \mathbf{y} \rangle = \mathbf{y}^H \mathbf{x_3} = (-i\mathbf{x_1}^H + (2-i)\mathbf{x_2}^H + 2\mathbf{x_3}^H)\mathbf{x_3} = 2$.
$\langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{z}^H \mathbf{y} = (\mathbf{x_1}^H + (3-i)\mathbf{x_3}^H)(i\mathbf{x_1} + (2+i)\mathbf{x_2} + 2\mathbf{x_3}) = i + (3-i)2 = 6 - i$.
b. $\|\mathbf{y}\| = \sqrt{(i)(-i) + (2+i)(2-i) + 2^2} = \sqrt{10}$, $\|\mathbf{z}\| = \sqrt{1 + (3+i)(3-i)} = \sqrt{11}$.