Math 347 C1

HOUR EXAM I

30 June 2009

SOLUTIONS

1. Let a be a non-zero real number. Show that

$$a^2 + \frac{1}{a^2} \geq 2.$$

SOLUTION Assume the opposite:

$$a^2 + \frac{1}{a^2} < 2.$$

Then, since $a^2 > 0$ we can multiply by a^2 without changing the direction of the inequality. Thus

$$a^4 + 1 < 2a^2,$$

whence

$$a^4 - 2a^2 + 1 < 0.$$

But $a^4 - 2a^2 + 1 = (a^2 - 1)^2 \geq 0$ for all values of a, a contradiction. Hence the opposite of our assumption is true, that is,

$$a^2 + \frac{1}{a^2} \geq 2.$$

2. Let $p(x)$ be a polynomial of degree $r > 0$ with real coefficients,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r$$

with $a_i \in R$.
Let A be the sum of the coefficients of $p(x)$ with even subscripts and let B be the sum of the coefficients of $p(x)$ with odd subscripts. Show that

$$A = \frac{1}{2}(p(1) + p(-1))$$

and

$$B = \frac{1}{2}(p(1) - p(-1)).$$

SOLUTION

$$p(1) = a_0 + a_1 + a_2 + \ldots + a_r$$

and

$$p(-1) = a_0 - a_1 + a_2 - \ldots \pm a_r.$$

Adding these two equations we get

$$p(1) + p(-1) = 2a_0 + 2a_2 + \ldots + 2a_r$$

when r is even and

$$p(1) + p(-1) = 2a_0 + 2a_2 + \ldots + 2a_{r-1}$$

when r is odd.
In each case, the right hand side is just $2A$ and so

$$A = \frac{1}{2}(p(1) + p(-1))$$

.
Similarly,

$$B = \frac{1}{2}(p(1) - p(-1)).$$

3. Show by induction that

$$\prod_{i=2}^{n}(1 - \frac{1}{i}) = \frac{1}{n}$$

for all natural numbers $n \geq 2$.
SOLUTION If $n = 2$, then $1 - \frac{1}{2} = \frac{1}{2}$ which settles the case for $n = 2$.
Suppose

$$\prod_{i=2}^{k}(1 - \frac{1}{i}) = \frac{1}{k}$$

and consider

$$\prod_{i=2}^{k+1}(1 - \frac{1}{i}) = (1 - \frac{1}{k+1})(\prod_{i=2}^{k}(1 - \frac{1}{i})) = (1 - \frac{1}{k+1})(\frac{1}{k})$$

by the induction assumption. But $(1 - \frac{1}{k+1})(\frac{1}{k}) = (\frac{k+1-1}{k+1})(\frac{1}{k}) = (\frac{k}{k+1})(\frac{1}{k}) = \frac{1}{k+1}$.
Thus

$$\prod_{i=2}^{k+1}(1 - \frac{1}{i}) = \frac{1}{k+1}$$

and the result holds by the Theorem of Mathematical Induction.

4. Let S and R be mathematical statements.
Show that the truth value of S is the same as the truth value of $(\neg S \Rightarrow (R \wedge \neg R))$, regardless of the truth values of S and R.
Explain your answer.
SOLUTION In order for $(R \wedge \neg R)$ to be true, both R and $\neg R$ would have to be true. This is impossible and hence $(R \wedge \neg R)$ is false regardless of the truth value of R.
If S is true, then the implication has a false hypothesis and a false conclusion and hence is true.
If S is false, then the implication has a true hypothesis and a false conclusion and hence is false.

5. Let R be the set of real numbers and let R^+ be the set of non-negative real numbers.
a) Let $f : R \rightarrow R$ be the function defined by $f(x) = x^2$. Show that f is not a bijection.
b) Let $g : R^+ \rightarrow R^+$ be the function defined by $g(x) = x^2$. Show that g is a bijection.
SOLUTION a) Since $f(1) = f(-1) = 1$, it follows that f is not a bijection.
b) To show that g is a bijection we have to show that for each element, say b, in the target of g, there is exactly one element in the domain which is mapped onto b. Thus, let $b \in R^+$. Since b is a non-negative real number, there is exactly one positive square root of b, which is in the domain of g. Thus $g(\sqrt{b}) = (\sqrt{b})^2 = b$ and so g is a bijection.