Math 347 C1

HOUR EXAM II

21 July 2009

SOLUTIONS

1. Consider the equation: $5x + 2y = 37$.
a) Find integers x, y, which satisfy the equation.
b) Find integers x, y, with y positive, which satisfy the equation. (If your answer to part a) has $y > 0$, then you are done.)

SOLUTION a) Since 5 and 2 are relatively prime, there are integers m and n such that $5m + 2n = 1$. By inspection, $m = 1$ and $n = -2$ work, i. e.

$$(5 \times 1) + (2 \times (-2)) = 1.$$

Then we can multliply this equation by 37 to get $(5 \times 37) + (2 \times (-2)37) = 37$, that is

$$(5 \times 37) + (2 \times (-74)) = 37.$$

Thus, $x = 37$ and $y = -74$ is a solution.
b) If we replace $y = -74$ by $y = (-74) + (5 \times 37)$ and $x = 37$ by $x = 37 - (2 \times 37)$, then we have $(5(37 - 2(37)) + (2(-74 + (5(37)) = 37$ (We have added and subtracted 10×37.) or

$$5(-37) + 2(111) = 37.$$

Thus $x = -37$ and $y = 111$.

2. a) Find a polynomial $p(x)$ of degree 3 with integral coefficients, such that $p(x) \equiv 0 \pmod 3$, for all $x \in Z$.
b) Is there a polynomial $q(x)$ of degree = 1 or 2 with integral coefficients, such that $q(x) \equiv 0 \pmod 3$, for all $x \in Z$? Explain.

SOLUTION a) Since 3 is a prime, we can invoke Fermat's Little Theorem and we get $x^3 \equiv x (\text{mod } 3)$ for all integers x. Thus the polynomial $p(x) = x^3 - x$ satisfies: $p(x) \equiv 0 \pmod 3$, for all $x \in Z$.
b) Let $q(x) = ax^2 + bx + c$ and suppose $q(0) \equiv q(1) \equiv q(2) \equiv 0$ (mod3). Then we have the equations

$$\begin{aligned} q(0) = c &\equiv 0 \\ q(1) = a + b + c &\equiv 0\,. \\ q(2) = 4a + 2b + c \equiv a + 2b + c &\equiv 0 \end{aligned}$$

This system of equations gives $c \equiv 0$, $a + b \equiv 0$ and $a + 2b \equiv 0$. These last two equations give $b \equiv 0$ and hence $a \equiv 0$, all (mod 3). Thus the polynomial $q(x)$

satisfying $q(0) \equiv q(1) \equiv q(2) \equiv 0 \pmod 3$ is the zero polynomial and hence no polynomial of degree 1 or 2 has that property.

3. Let A be a set of 5 elements and B a set of 3 elements, with $A \cap B = \emptyset$.

   Let $f : A \to [5]$ and $g : B \to [3]$ be injective functions.

   Let $h : A \cup B \to [8]$ be defined as follows.
   If $x \in A$, then $h(x) = f(x)$.
   If $x \in B$, then $h(x) = g(x) + 5$.

   Prove that h is injective.

   SOLUTION We need to show that if $h(x_1) = h(x_2)$, then $x_1 = x_2$.
   We must consider 3 cases.
   1. $x_1, x_2 \in A$. Then $h(x) = f(x)$ and since $f(x)$ is injective, $x_1 = x_2$.

   2. $x_1, x_2 \in B$. Then $h(x) = g(x)+5$ and so $h(x_1) = h(x_2) \Rightarrow g(x_1)+5 = g(x_2)+5 \Rightarrow g(x_1) = g(x_2)$ and since $g(x)$ is injective, $x_1 = x_2$.

   3. $x_1 \in A, x_2 \in B$. We show that, in this case, $h(x_1)$ cannot be equal to $h(x_2)$. For when $x_1 \in A$, then $h(x_1) = f(x_1) \leq 5$. While, if $x_2 \in B$, then $h(x_2) = g(x_2)+5 \geq 6$. Similarly for $x_1 \in B, x_2 \in A$.
   Thus $h(x)$ is injective.

4. Here is a way to count the number of subsets of size 2 in a set A of size n.

   Choose $a \in A$. Form all subsets of size 2 which contain a. How many such subsets are there?
   Now discard the element a from the set A, leaving the set A'. Choose some $a' \in A'$ and count how many subsets of size 2 there are in A' which contain a'?
   Continue this process as long as possible.
   Write an equation of the form: $\sum_{i=1}^{n} f(i) = \binom{n}{2}$ giving $f(i)$ explicitly. Explain why it is true.

   SOLUTION The first step in this process yields exactly $n-1$ subsets since we choose a single element a in A and then form subsets of size two by adding to this element every possible element from the remaining $n-1$ elements in A. At the next stage, we choose a different element from the remaining n - 1 elements in A, a' and again form all possible 2 elements subsets from A' that contain a'. There are exactly $n-2$ such subsets and all are different from the 2 element subsets chosen at stage 1. Thus the number of such subsets is given by $(n-1)+(n-2)+\ldots+1$. If we let $f(i) = n-i$, then the sum on the left is equal to $\binom{n}{2}$ on the right.

5. Consider the polynomial $f(x) = x^4 + 2x^3 + 5x^2 + 7x - 6$.

a) Find all rational numbers which COULD be zeros of $f(x)$, according to the Rational Zeros Theorem.

b) Find all rational zeros of $f(x)$. (You may use the fact that $f(-6) \neq 0$.)

c) (Extra Credit) Does $f(x)$ have any irrational zeros? Explain.

<u>SOLUTION</u> a) Since $c_0 = -6$ and $c_4 = 1$, it follows that a rational zero of $f(x)$ must be of the form $\frac{a}{b}$, in lowest terms, with $a|c_0$ and $b|c_4$. Hence the only possible solutions are $\pm 1, \pm 2, \pm 3, \pm 6$. Clearly, no positive value of $\frac{a}{b}$ gives $f(\frac{a}{b}) = 0$. Thus the only possible rational zeros are -1, -2, -3, -6.

b) $f(-1) = 1 - 2 + 5 - 7 - 6 = -9 \neq 0$.
$f(-2) = 16 - 16 + 20 - 14 - 6 = 0$.
$f(-3) = 81 - 54 + 45 - 21 - 6 = 45 \neq 0$.
$f(-6) \neq 0$.
Thus the only rational zero of $f(x)$ is $x = -2$.

c) Since -2 is a zero, it follows that $x+2$ is a factor. Divide $f(x)$ by $x+2$ and we get the polynomial $x^3 + 5x - 3$. We now note that $f(0) = -3$ and $f(1) = 3$. Therefore, since $f(x)$ is a continuous function on R, the graph of $f(x)$ crosses the x-axis for some value of x between $x = 0$ and $x = 1$. That value of x is a zero of $f(x)$ and it must be irrational by the Rational Zeros Theorem.