Math 347 C1

HOUR EXAM III

5 August 2009

SOLUTIONS

1. Let a, b, c be integers, relatively prime in pairs, that is, $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$. Suppose that $a^2 + b^2 = c^2$.
Show that a and b must have different parities.

   SOLUTION Assume that they have the same parity. Since $\gcd(a,b) = 1$, it follows that both must be odd. Thus let $a = 2r+1$ and $b = 2s+1$, for some r, s. Then

$$\begin{aligned} (2r+1)^2 + (2s+1)^2 &= c^2 \\ (4r^2+4r+1) + (4s^2+4s+1) &= c^2\,. \\ 4(r^2+s^2+r+s)+2 &= c^2 \end{aligned}$$

   Thus c^2 must be even and so c is even. Let $c = 2u$. Then

$$4(r^2+s^2+r+s)+2 = c^2 = (2u)^2 = 4u^2.$$

   Therefore, 4 must divide the left-hand side. But then $4|2$, a contradiction. Thus a and b have different parities.
ANOTHER SOLUTION: The Theorem on Pythagorean Triples states that either a or b must be of the form: $2rs$. Thus either a or b must be even. Since $\gcd(a,b)$ = 1, it follows that they have different parity.

2. Let n be a natural number and let A, B, C, D be finite sets satisfying the following properties.
i. Each set has exactly n elements.
ii. The intersection of each pair of these sets has exactly $n-1$ elements.
iii. The intersection of each triple of these sets has exactly $n-2$ elements.
iv. The set $A \cap B \cap C \cap D$ has exactly $n-3$ elements.

   Find the size of $A \cup B \cup C \cup D$ as a function of n.

   SOLUTION We need to apply the Inclusion-Exclusion Theorem.
Let $\mathbf{W} = |A| + |B| + |C| + |D|$.
Let $\mathbf{X} = |A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|$.
Let $\mathbf{Y} = |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$.
Let $\mathbf{Z} = |A \cap B \cap C \cap D|$.
Then the Inclusion-Exclusion Theorem states that
$|A \cup B \cup C \cup D| = \mathbf{W} - \mathbf{X} + \mathbf{Y} - \mathbf{Z}$.

It follows from i, ii, iii, and iv above that $\mathbf{W} = 4n$, that $\mathbf{X} = 6(n-1)$, that $\mathbf{Y} = 4(n-2)$ and that $\mathbf{Z} = n-3$.
Hence

$$A \cup B \cup C \cup D = 4n - 6(n-1) + 4(n-2) - (n-3) = n+1.$$

3. Solve the congruence $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 2, a_1 = 3$.

   <u>SOLUTION</u> Using the result of Theorem 12.22, consider the polynomial $x^2 - 6x + 9$. Clearly $x^2 - 6x + 9 = (x-3)^2$ and so $a_n = A3^n + Bn3^n$, and we have to determine the constants A and B.
   Since $a_0 = 2$ and $a_1 = 3$, we have $a_0 = 2 = A$ and $a_1 = 3 = A3 + B3$, whence $B = -1$. Therefore

   $$a_n = (2 \times 3^n) - (n \times 3^n) = 3^n(2-n).$$

4. Let $S = \{x \in R \mid 4x^2 > x^3 + 4x\}$.
   Determine whether S is bounded and find $sup(S)$ and $inf(S)$ when they exist.

   <u>SOLUTION</u>

   $$\begin{aligned} 4x^2 &> x^3 + 4x \\ 0 &> x^3 - 4x^2 + 4x \\ 0 &> x(x^2 - 4x + 4) \\ 0 &> x(x-2)^2 \end{aligned}$$

   Thus S consists of those real numbers x such that $x(x-2)^2 < 0$. Since $(x-2)^2 \geq 0$ for all x, it follows that S consists of all negative reals. That set is not bounded from below and its least upper bound is 0. Thus $inf(S)$ doesn't exist and $sup(S) = 0$.

5. If $a_1 = 1$ and $a_n = \sqrt{3a_{n-1} + 4}$ for $n > 1$, prove that $a_n < 4$ for every natural number n. (Hint: Use induction.)

   <u>SOLUTION</u> Use induction on n. Since $a_1 = 1, a_2 = \sqrt{3a_1 + 4} = \sqrt{3+4} = \sqrt{7} < 4$.
   Now assume $a_k = \sqrt{3a_{k-1} + 4} < 4$.
   Then $a_{k+1} = \sqrt{3a_k + 4} < \sqrt{(3 \times 4) + 4} = \sqrt{16} = 4$.
   By Mathematical induction, the result follows for all $n \geq 2$. But we are given that $a_1 = 1$, which completes the proof.