Math 347 C1

HOUR EXAM I

27 June 2006

SOLUTIONS

1. Let $\langle a \rangle$ be a sequence satisfying $a_1 = 1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for $n \in N$.

   SOLUTION: We use strong induction on $P(n) : a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.
   We will show that $P(k+1) : a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$ follows from $P(k) : a_k = 3 \cdot 2^{k-1} + 2(-1)^k$ and $P(k-1) : a_{k-1} = 3 \cdot 2^{k-2} + 2(-1)^{k-1}$, using the given relation $a_n = a_{n-1} + 2a_{n-2}$.
   Using the formula directly above and the induction hypothesis we have
   $a_{k+1} = 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-1}) = 3 \cdot 2^k + 2(-1)^k + 4(-1)^{k-1} = 3 \cdot 2^k + 2(-1)^{k+1}$, since $2(-1)^k + 4(-1)^{k-1} = -2(-1)^{k+1} + 4(-1)^{k+1} = 2(-1)^{k+1}$.
   Thus $a_{k+1} = 3 \cdot 2^k + 2(-1)^{k+1}$, which is precisely the statement $P(k+1)$. It follows from the Principle of Mathematical Induction that $P(n)$ is true for all $n \in N$.

2. Find all integer solutions to the inequality: $a^2 b > 2a$.

   SOLUTION: i) $a \neq 0$, since otherwise we get $0 > 0$, which is false.
   We must examine 2 cases.

   ii) Suppose $a > 0$. Then $ab > 2$ and therefore $b > 0$. Thus the inequality is true for all positive values of a, b except $(a = 1, b = 1), (a = 1, b = 2), (a = 2, b = 1)$. The inequality is therefore true for all of the integral points in first quadrant (not including the axes) except for the 3 points $(1, 1), (1, 2), (2, 1)$.

   iii) Suppose $a < 0$. Then $ab < 2$. This is true for all $a < 0$, and all $b \geq 0$. Also for one point in the third quadrant: $(-1, -1)$.

3 . Let $S = \{x \in R | x^2 > x + 6\}$ and let $T = \{x \in R | x > 3\}$. Determine whether the following statements are true.

   a) $T \subseteq S$.

   b) $S \subseteq T$.

   SOLUTION: Let $u \in T$. Then $u > 3$. In order for u to be an element of S, we need to show that $u^2 > u + 6$. But $x^2 > x + 6 \iff x^2 - x - 6 > 0 \iff (x - 3)(x + 2) > 0$. Now since $u > 3$, it follows that $u - 3 > 0$ and $u + 2 > 5 > 0$. Therefore $(u - 3)(u + 2) > 0$. Hence $u \in S$ and so $T \subseteq S$.

   Now let $v \in S$. Then $(v - 3)(v + 2) > 0$ and so either both of these factors are positive or both are negative. In particular, if $v = -3$, then $(v-3)(v+2) > 0$ and so $v \in S$, but $v \notin T$. Hence $S \subseteq T$ is false.

4. Let $f : A \to B$ and $g : B \to A$, with $f(g(y)) = y$, for all $y \in B$.

   Answer the following question either by providing a proof or by giving a counterexample.

   i) Is f is surjective?

   ii) Is f injective?

   SOLUTION: i) Since $f(g(y)) = y$ for all $y \in B$, f is surjective.
   ii) f is NOT injective. Example: Let $A = \{u, v\}$ and $B = \{w\}$, with $f(u) = w, f(v) = w$ and $g(w) = u$. Then clearly f is not injective and f satisfies the condition $f(g(y)) = y$ since the only element in B is w and $f(g(w)) = f(u) = w$.

5. For $k \in N$ determine all ordered pairs (x, y) of real numbers which satisfy

$$\sum_{j=0}^{2k} x^{2k-j} y^j = 0.$$

   SOLUTION: By Lemma 3.13,

$$(x-y) \sum_{j=0}^{2k} x^{2k-j} y^j = x^{2k+1} - y^{2k+1}.$$

   Since we are given that

$$\sum_{j=0}^{2k} x^{2k-j} y^j = 0,$$

   it follows that $x^{2k+1} = y^{2k+1}$ and thus $x = y$. In that case

$$\sum_{j=0}^{2k} x^{2k-j} y^j = (2k+1)x^{2k} = 0,$$

   whence $x = y = 0$, the only solution to the given equation.