Math 347 C1

HOUR EXAM II

18 July 2006

SOLUTIONS

1. a) Find the *gcd* $(8332, 181)$ and express it as a linear combination of 8332 and 181.
   b) Solve: $181x \equiv 1 \; mod \; 8332$.

   SOLUTION: a) Using the Euclidean Algorithm, we get:
   $8332 = 181 \cdot 46 + 6$
   $181 = 6 \cdot 30 + 1$
   Therefore $gcd\,(8332, 181) = 1$.
   Now $1 = 181 + 6(-30)$
   $1 = 181 + (8332 - 181 \cdot 46)(-30)$
   $1 = 181(1381) + 8332(-30)$
   b) Using the result above,
   $181(1381) = 1 + 8332(30)$
   $181(1381) \equiv 1 \; mod \; 8332$.
   Therefore $x = 1381$ is a solution to the congruence $181x \equiv 1 \; mod \; 8332$.

2. How many $n-$digit sequences, using the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, are there in which the digits 1, 2, and 3 all appear.

   SOLUTION: Let A_1 be the set of such sequences that don't contain 1. Let A_2 be the set of such sequences that don't contain 2. Let A_3 be the set of such sequences that don't contain 3. Let S be the set of all such sequences that contain 1, 2 and 3. Since there are 10^n $n-$digit sequences with no restrictions, it follows from the Inclusion-Exclusion Principle that

   $$|S| = 10^n - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|.$$

   Since $|A_1| = |A_2| = |A_3| = 9^n$ and $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 8^n$ and $|A_1 \cap A_2 \cap A_3| = 7^n$, then

   $$|S| = 10^n - 3 \cdot 9^n + 3 \cdot 8^n - 7^n.$$

3 . Let c be an integer and let $f(x) = x^6 + cx^5 + 1$. Prove that $f(x)$ has rational roots if and only if $c = \pm 2$.

   SOLUTION: By the Rational Zeros Theorem (8.16), if $x = \frac{u}{v}$ is a rational zero of f, with $u, v \in Z$, then $u|1$ and $v|1$. Therefore $x = \frac{u}{v} = \pm 1$.

If $x = 1$, then $f(1) = 1 + c + 1 = c + 2 = 0$ and therefore $c = -2$.
If $x = -1$, then $f(-1) = 1 - c + 1 = 2 - c = 0$ and therefore $c = 2$.
Conversely if $c = -2$, then $f(1) = 1 - 2 + 1 = 0$ and so 1 is a rational zero.
If $c = 2$, then $f(-1) = 1 - 2 + 1 = 0$ and so -1 is a rational zero.

4. Prove

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$$

by interpreting each binomial coefficient as the number of ways to choose a subset of a certain size from a set of a given size.

SOLUTION: $\binom{m+n}{k}$ is the number of ways to choose a subset of size k from a set of size $m+n$. Any set R of size $m+n$ can be written as the disjoint union of two sets, S and T, of size m and n respectively. Each subset of k elements chosen from R can be divided into i from S and $k-i$ from T, for all possible choices of $i = 0, 1, \ldots, k$. Each term on the right hand side of the equation counts how many ways there are to choose those k elements for all possible values of i.