Math 317 C1

HOUR EXAM I

24 June 2002

SOLUTIONS

1. Show that every common divisor of the positive integers a and b is a divisor of the greatest common divisor of a and b.

   SOLUTION: We know that there exist integers x and y such that

   $$(a, b) = ax + by.$$

   Hence if c is a common divisor of a and b, then c divides the right hand side above and therefore also the left hand side.

2. Prove that $(k + 1)^n \equiv kn + 1$ (modulo k^2) for all integers $n, k \geq 1$. Do NOT use induction.

   SOLUTION: The Binomial Theorem states

   $$(k+1)^n = k^n + \binom{n}{1}k^{n-1} + \binom{n}{2}k^{n-2} + \ldots + \binom{n}{n-2}k^2 + \binom{n}{n-1}k + 1.$$

   If $n = 1$, then there is nothing to prove. If $n \geq 2$, then every term in the expansion above except the last 2 terms is divisible by k^2. Hence

   $$(k+1)^n \equiv \binom{n}{n-1}k + 1 = nk + 1 (modulo\ k^2).$$

3. Find the (275, 47) and express it as a linear combination of 275 and 47.

   SOLUTION:

   $$275 = 47 \cdot 5 + 40$$

   $$47 = 40 \cdot 1 + 7$$

   $$40 = 7 \cdot 5 + 5$$

   $$7 = 5 \cdot 1 + 2$$

   $$5 = 2 \cdot 2 + 1.$$

   Hence (275, 47) = 1.
   Reversing the steps above we get

   $$1 = 20 \cdot 275 + (-117) \cdot 47.$$

4. Let $f : R \rightarrow R$ be the function satisfying $f(x) = x^2$, where R is the set of all real numbers.
a) Show that f is neither surjective nor injective.
b) Change only the codomain of f, giving the function f^*, so that now f^* is surjective.
c) Now change only the domain of f^*, giving the function f^{**}, so that now f^{**} is both injective and surjective.

<u>SOLUTION</u>: a) Since $f(-1) = f(1) = 1$, f is not injective. Since -1 is in the codomain of f, but there is no real number a satisfying $f(a) = a^2 = -1$, f is not surjective.

b) Replace R by $R^+ = \{r \in R \,|\, r \geq 0\}$ as the codomain, giving the function f^*. Then for each $a \in R^+$, $\sqrt{a} \in R$ and $f(\sqrt{a}) = a$. Thus f^* is surjective.

c) We need to change the domain of f^* so that not both a and $-a$ are in the domain. The simplest way to do this is to change the domain R to R^+, giving f^{**}. Then if $f^{**}(a) = f^{**}(b), a^2 = b^2$ and since $a, b \in R^+$, it follows that $a = b$. Thus f^{**} is injective. If $c \in R^+$, then $\sqrt{c} \in R^+$ and $f^{**}(\sqrt{c}) = c$. Thus f^{**} is surjective.

5. Show by Mathematical Induction, that for each integer $n \geq 1$, $n < 3^n$.

<u>SOLUTION</u>: Choosing $n = 1$, we get $1 < 3$, which is clearly true.
Now assume the result is true for $k \geq 1$. Then since $k < 3^k$, we can multiply by 3 and get

$$3k < 3^{k+1}.$$

Since $k \geq 1$, $2k > 1$ and so $3k > k + 1$. Thus $k + 1 < 3k < 3^{k+1}$, whence,

$$k + 1 < 3^{k+1}.$$

Hence $n < 3^n$ for all $n \geq 1$ by the Theorem of Mathematical Induction.