

# COMBINATORIAL MATHEMATICS

©Douglas B. West

## Preface

viii

## Chapter 0 – Introduction

2

Combinatorial Problems: 2. Sets and Relations: 3. Graphs: 5. Paths, Trees, Connection, and Cycles: 6. Structures and Methods: 9. Complexity: 11.

## Part I — Enumeration

15

### Chapter 1 – Combinatorial Arguments.

16

- 1.1. Classical Models. 17  
 Elementary Principles: 17. Words, Sets, and Multisets: 20. Exercises: 25.
- 1.2. Identities. 30  
 Lattice Paths: 30. Delannoy Numbers: 34. Exercises: 38.
- 1.3. Applications. 44  
 Graphs and Trees: 44. Multinomial Coefficients: 46. The Ballot Problem: 48. Catalan Numbers: 51. Exercises: 56.

### Chapter 2 – Recurrence Relations.

62

- 2.1. Obtaining Recurrences. 64  
 One Parameter, One Sequence: 64. Variations: 69. Exercises: 72.
- 2.2. Elementary Solution Methods. 81  
 The Characteristic Equation Method: 81. The Generating Function Method: 87. Exercises: 94.
- 2.3. Further Topics. 98  
 The Substitution Method: 98. Asymptotic Analysis: 101. The WZ Method (optional): 104. Exercises: 110.

### Chapter 3 – Generating Functions.

114

- 2.1. Ordinary Generating Functions. 114  
 Modeling Counting Problems: 115. Permutation Statistics: 120. Exercises: 126.
- 3.2. Coefficients and Applications. 132

Operations and Summations: 133. Snake Oil: 138. Further Applications (optional): 140. Exercises: 142.

### 3.3. Exponential Generating Functions. 148

Modeling Labeled Products: 148. EGF Analogues of OGF Applications: 154. The Exponential Formula: 157. The Lagrange Inversion Formula (optional): 160. Exercises: 164.

### 3.4. Partitions of Integers. 170

Generating Function Methods: 170. Ferrers Diagrams: 174. Bulgarian Solitaire (optional): 177. Distribution Models (summary): 180. Exercises: 183.

## Chapter 4 – Further Topics.

188

- 4.1. The Inclusion-Exclusion Principle. 188  
 The Basic Principle: 188. Restricted Permutations: 196. Signed Involutions: 200. Determinants and Path Systems: 203. Exercises: 210.
- 4.2. Pólya–Redfield Counting. 218  
 Burnside’s Lemma: 219. The Pattern Inventory: 222. Classical Cycle Indices: 227. Exercises: 229.
- 4.3. Permutations and Tableaux. 233  
 The Hook-Length Formula: 233. The RSK Correspondence: 238. Switching  $P$ -Symbol and  $Q$ -Symbol: 244. Jeu de Taquin: 248. Exercises: 254.

## Part II — Graphs

257

### Chapter 5 – First Concepts for Graphs.

258

- 5.1. Definitions and Examples. 258  
 Graphs and Subgraphs: 258. Isomorphism: 262. The Petersen Graph and Hypercubes: 263. Exercises: 267.
- 5.2. Vertex Degrees. 273  
 The Degree-Sum Formula: 273. Degree Lists: 275. Extremal Problems: 277. Directed Graphs: 278. Exercises: 280.
- 5.3. Connection and Decomposition. 284  
 Components and Walks: 284. Cycles and Cut-Edges: 287. Eulerian Circuits: 290. Exercises: 292.
- 5.4. Trees and Distance. 297  
 Properties of Trees: 298. Distance and Diameter: 300. Optimization on Weighted Graphs: 302. Exercises: 306.

<b>Chapter 6 – Matchings.</b>	<b>312</b>
6.1. Matching in Bipartite Graphs.	312
Hall's Theorem: 313. Min-Max Relations: 317. Exercises: 320.	
6.2. Matching in General Graphs.	324
Tutte's 1-Factor Theorem: 325. General Factors of Graphs: 328. Exercises: 330.	
6.3. Algorithmic Aspects.	336
Augmenting Paths: 336. Fast Bipartite Matching (optional): 338. Weighted Bipartite Matching (optional): 340. Stable Matchings (optional): 345. Exercises: 347.	
<b>Chapter 7 – Connectivity and Cycles.</b>	<b>352</b>
7.1. Connectivity Parameters.	352
Separating Sets: 352. Edge Cuts: 356. Blocks: 359. Exercises: 360.	
7.2. Properties of $k$ -Connected Graphs.	364
Menger's Theorem: 365. Applications of Menger's Theorem: 368. 3-Connected and 2-Connected Graphs: 372. $k$ -Connected Orientations (optional): 375. Exercises: 381.	
7.3. Spanning Cycles.	385
Necessary Conditions: 386. Sufficient Conditions: 389. Long Cycles (optional): 394. Exercises: 399.	
<b>Chapter 8 – Coloring.</b>	<b>406</b>
8.1. Vertex Coloring.	406
Upper Bounds: 408. Triangle-Free Graphs: 411. Exercises: 414.	
8.2. Structural Aspects.	418
Color-Critical Graphs: 418. List Coloring: 420. Forced Subgraphs (optional): 424. Exercises: 429.	
8.3. Edge-Coloring and Perfection.	435
Examples: 436. Vizing's Theorem: 438. Perfect Graphs: 441. Exercises: 448.	
<b>Chapter 9 – Planar Graphs.</b>	<b>456</b>
9.1. Embeddings and Euler's Formula.	456
Drawings and Duals: 456. Euler's Formula: 463. Exercises: 466.	
9.2. Structure of Planar Graphs.	470
Kuratowski's Theorem: 471. The Separator Theorem (optional): 476. Exercises: 482.	

9.3. Coloring of Planar Graphs.	484
5-Colorable and 5-Choosable: 485. The Four Color Problem: 487. Discharging (optional): 490. Edge-Colorings and Spanning Cycles: 501. Exercises: 504.	

## **Part III — Sets** **511**

### **Chapter 10 – Ramsey Theory.** **512**

10.1. The Pigeonhole Principle.	512
Classical Applications: 513. Monotone Sublists: 518. Pattern-Avoiding Permutations (optional): 520. Large Girth and Chromatic Number: 523. Edge-Coloring of Hypergraphs (optional): 528. Exercises: 531.	
10.2. Ramsey's Theorem.	535
The Main Theorem: 535. Applications: 538. Ramsey Numbers: 541. Graph Ramsey Theory: 544. Exercises: 547.	
10.3. Further Topics (optional).	550
Infinite Sets: 550. The Canonical Ramsey Theorem: 553. Van der Waerden's Theorem: 557. Exercises: 565.	

### **Chapter 11 – Extremal Problems.** **568**

11.1. Forced Subgraphs.	568
Turán's Theorem: 568. Erdős–Stone Theorem: 573. Linear Ramsey for Bounded Degree: 578. Proof of the Regularity Lemma: 580. Exercises: 584.	
11.2. Extremal Set Theory.	588
The Kruskal–Katona Theorem: 588. Antichains and Intersecting Families: 592. Chvátal's Conjecture: 596. Exercises: 599.	
11.3. Matroids.	603
Hereditary Systems and Examples: 604. Axiomatics of Matroids: 610. Duality and Minors: 614. The Span Function: 620. Matroid Intersection: 622. Matroid Union: 627. Exercises: 630.	

### **Chapter 12 – Partially Ordered Sets.** **636**

12.1. Structure of Posets.	636
Definitions and Examples: 636. Dilworth's Theorem: 642. Beyond Dilworth (optional): 644. Exercises: 647.	
12.2. Symmetric Chains and LYM Orders.	651

Ranked and Graded Posets: 651. Symmetric Chain Decompositions: 654. LYM and Sperner Properties: 660. Products of LYM Orders (optional): 665. Exercises: 667.	
12.3. Linear Extensions & Dimension.	673
Order Dimension. 673. Computation and Bounds: 677. Exercises: 683.	
12.4. Special Families of Posets.	686
Semiororders and Interval Orders: 687. Bipartite Posets: 692. Exercises: 700.	

## **Chapter 13 – Combinatorial Designs. 704**

13.1. Arrangements.	704
Latin Squares: 704. Block Designs: 708. Symmetric Designs: 711. Hadamard Matrices: 715. Exercises: 720.	
13.2. Projective Planes.	722
Relation to Designs: 722. Applications to Extremal Problems: 727. Difference Sets: 734. Exercises: 738.	
13.3. Further Constructions.	741
Steiner Triple Systems: 742. Constructions of Other Designs: 746. Resolvable Designs and Other Tools: 748. The Euler Conjecture (optional): 751. Exercises: 755.	

## **Part IV — Methods 759**

### **Chapter 14 – The Probabilistic Method. 760**

14.1. Existence and Expectation.	760
Probability Spaces and Inequalities: 761. Existence Arguments: 763. Random Variables: 767. Exercises: 772.	
14.2. Refinements of Basic Methods.	778
Deletions and Alterations: 778. The Symmetric Local Lemma: 782. The General Local Lemma: 786. Derandomization: 789. Exercises: 791.	
14.3. Moments and Thresholds.	794
“Almost Always”: 795. Threshold Functions: 799. Sharp Thresholds (optional): 803. Evolution and Graph Parameters: 806. Exercises: 812.	
14.4. Concentration Inequalities.	816
Chebyshev and Chernoff Bounds: 817. Martingales: 822. Exercises: 829.	

### **Chapter 15 – Linear Algebra. 832**

15.1. Dimension and Polynomials.	832
The Polynomial Method: 835. Families with Restricted Intersections: 837. Combinatorial Nullstellensatz: 844. The Alon–Tarsi Theorem: 849. Other Applications to Graphs: 853. Exercises: 857.	
15.2. Matrices.	861
Determinants: 861. Permanents and Planar Graphs: 868. Van der Waerden’s Conjecture: 872. Exercises: 879.	
15.3. Eigenvalues.	882
Spectra of Graphs: 883. Eigenvalues and Graph Parameters: 885. Regular and Strongly Regular Graphs: 888. Laplacian Eigenvalues and Expanders: 894. Exercises: 898.	
15.4. Other Topics (optional).	902
Lattices: 902. Distributive Lattices: 906. Correlational Inequalities: 911. The XYZ Inequality: 916. Möbius Inversion: 920. Exercises: 928.	

### **Chapter 16 – Geometry and Topology. 934**

16.1. Graph Drawings.	934
Embeddings on Grids: 934. Crossing Number: 942. Exercises: 950.	
16.2. Combinatorial Topology.	952
Sperner’s Lemma and Bandwidth: 952. Equivalent Topological Lemmas: 957. The Borsuk–Ulam Theorem: 961. Kneser Conjecture & Gale’s Lemma: 967. Ham Sandwiches & Bisections: 972. Borsuk’s Conjecture: 975. Exercises: 978.	
16.3. Volumes and Containment.	980
Monotone Subsequences: 980. Balanced Comparisons: 982. Containment Orders: 989. Exercises: 994.	

### **Appendix 1 - Glossary of Terms 996**

### **Appendix 2 - Summary of Notation 1024**

### **Appendix 3 - Hints to Exercises (later)**

### **Appendix 4 - References (later)**

### **Appendix 5 - Author Index (later)**

### **Index (presently just points to Definitions) 1030**

## PREFACE

Combinatorics is now a mature discipline. Although some see it as a maelstrom of isolated problems, it has central themes, techniques, and results that make it a surprisingly coherent subject. Meanwhile, it still rewards its students with endless discovery and delight.

This book introduces the reader to a substantial portion of combinatorics. Being a general textbook, it is not exhaustive in its topics, results, or bibliography. On the other hand, it is thorough enough to equip the reader with the tools needed to read or do research in combinatorics or to apply combinatorics in other areas of mathematics and computer science. It assumes the maturity and sophistication of graduate students without assuming prior exposure to combinatorics. It assumes basic undergraduate mathematics, such as elementary set theory, induction, equivalence relations, limits, elementary calculus, and some linear algebra.

One book cannot include everything. More advanced material will appear in *The Art of Combinatorics*, a four-volume series of texts intended for researchers and for advanced courses in combinatorics.

### History and Rationale

Despite its fundamental nature and its explosive growth in the past fifty years, combinatorics still is not a standard part of mathematics instruction. Curricula (and mathematicians) are slow to change.

Many undergraduates encounter combinatorial ideas in courses on discrete mathematics, but such courses can be insubstantial. Serious undergraduate courses in combinatorics are rarely required for math majors. At the graduate level, requirements in combinatorics are essentially nonexistent. Nevertheless, it is an elegant and valuable subject.

In the early 1980's, I began to teach graduate courses in combinatorics at the University of Illinois. Although excellent books existed about many parts of combinatorics, every graduate-level textbook I could find omitted some substantial area I wanted to teach. Gathering material for a general textbook, I succumbed to the overabundance of riches before me. With so much beautiful material in combinatorics, the project grew and grew, and the courses grew along with it. The result was four rotating courses taught from four books, now called *The Art of Combinatorics*, each exploring part of combinatorics at an advanced level. That project still continues.

In 1996, I realized that this course structure served only students who were already committed to studying combinatorics in detail. For other students, an overview of combinatorics could have great value. An educated mathematician or theoretical computer scientist should know some algebra and analysis, and also such a person should be acquainted with fundamental combinatorics and its relationships to other areas of mathematics. Also, disparities in preparation of entering graduate students make a core course worthwhile to establish a common background before studying advanced material in combinatorics.

In 1997, I started a one-semester overview course to serve these goals. I extracted the fundamental material from *The Art of Combinatorics* to provide text material, and I reorganized it to emphasize the connections among the topics. The present book is the result of this process, with enough material to support a two-semester sequence. I confess that with so much beautiful combinatorics to choose from, I could not cut the book down to one semester. Nevertheless, one can craft a coherent one-semester course from this book in many ways.

### Usage

Most schools do not have many regular graduate courses in combinatorics. At such schools, this book is appropriate for a two-semester sequence. Instead of separating graph theory from other topics to make two courses, this text integrates the topics into a coherent whole.

This approach enables students whose main interest is outside combinatorics to acquire the fundamental material about enumeration, graphs, and sets in the first semester without continuing to the second. Also, some topics are best appreciated after seeing the fundamentals in several parts

of combinatorics. These must be left out of a beginning graduate course that studies only one part of combinatorics, unable to assume acquaintance with another part. Examples include the existence (Chapter 14) or construction (Chapter 10) of graphs with large chromatic number and girth, the application of projective planes to extremal graph problems (Chapter 13), and applications of matroids (Chapter 11). With the approach here, such applications enliven the second semester.

Nevertheless, the text can also be used separately for courses in graph theory and in “other” combinatorics. Below I will describe the extractions needed to separate the topics.

At schools with advanced courses in combinatorics, a one-semester core course is appropriate. To facilitate use in such courses, I have designated some sections and subsections as “optional”, and within others the items marked “\*” are optional. This is material that is more technical or advanced and can be skipped at first reading without loss of continuity. This also holds for “Remark” items that describe additional results.

Parts I and II contain a concise treatment of the basic material taught in most undergraduate courses on combinatorics and graph theory, presented with a deeper point of view for graduate students. Students with prior exposure to the subject also benefit from this discussion. Classical topics in graph theory reside in Part II, but interactions between graphs and other topics and techniques appear at various places in other chapters.

Later topics are more independent, but the order of presentation here works well. Part III can be viewed as a third introductory area; it considers basic questions about sets and order relations. The methods of Part IV are then applied to questions that arise in the combinatorial contexts introduced in Parts I–III.

In a two-semester sequence that assumes no prior combinatorics and treats the second semester as optional, one might in the first semester discuss most of the material not designated optional in Chapters 1–10, plus early portions of Chapters 12, 13, and 14. This is about half of the text. I take this approach in a one-semester introduction that prepares our students for four independent advanced courses.

The many exercises are applications of material in the text and/or exploration of further concepts. I have tried to indicate difficulty by marking easier problems with (–) and harder problems with (+). Problems of intermediate difficulty that seem particularly interesting or instructive are marked with (◊). There is much ambiguity in these designations, partly because the difficulty of finding a solution is not proportional to its length or its complexity. Thus these designations should be taken lightly.

## Organization

One can organize combinatorial material in many ways: by objects, by types of questions asked, by methods used, etc. In a broad overview, the connections among topics are as important as the groupings within topics.

Some presentations of elementary combinatorics begin with graph theory; the classical approach begins with enumeration. Natural enumerative questions arise in elementary graph theory, and some graph-theoretic arguments use basic techniques of enumeration. Nevertheless, in Part I there is little need for graph-theoretic material beyond the basic concepts of trees and isomorphism, which appear in Chapter 0.

Part I concisely presents the basic techniques involving bijective arguments, generating functions, recurrence relations, and the inclusion-exclusion principle. Inclusion/exclusion is expanded via the notion of signed involutions, leading to more recent material. Young tableaux and the elementary aspects of Pólya-Redfield counting appear here from a combinatorial point of view. Deeper algebraic aspects of enumeration are omitted.

Graph theory has become a huge subject; selecting fundamental core material is difficult. Part II pursues central themes of elementary graph theory while reaching important and classical results. Many large topics are mentioned at most in passing or in exercises; these include diameter, encodings, domination, decomposition, packings, genus, minors, nowhere-zero flows, generalized colorings, and structure of special families. Some aspects of graph theory that are closely related to other topics appear as applications in later chapters.

Part III explores perhaps our most general structural object: families of sets. Viewed as hypergraphs, this includes graphs and most of the other structures we study. In Part III, three loosely related aspects of set systems are considered: Ramsey theory, extremal set theory, and structural aspects of partially ordered sets. Omitted are many aspects of algebraic combinatorics that grow out of partially ordered sets and enumerative problems.

Part IV develops methods from probability, algebra, and geometry and applies them to questions concerning graphs and sets. Also there are some combinatorial applications to geometric questions. With the focus on methods and interaction, it is helpful to have available the terminology and basic results of graph theory and enumeration.

Many topics omitted here are explored at length in *The Art of Combinatorics*. Some of these also appear in my earlier *Introduction to Graph Theory*, which is a more detailed introduction to elementary graph theory than is presented here.

## Graph Theory vs. Combinatorics

As mentioned earlier, the text can be used for separate courses in graph theory and “other” combinatorics. Yet a third course emphasizing techniques is reasonable for students who have some acquaintance with graphs and sets. Although the text often mentions connections to other chapters, the chapters after Part II are relatively independent except for the background of language from the early parts.

A course in enumerative and set-theoretic combinatorics would emphasize Part I and Part III. In Part I, one needs only the definitions of graphs and trees for Cayley’s Formula in Section 1.3 and the counting of isomorphism classes in Section 4.2. The discussion of chromatic polynomials in Section 4.1 is an application of inclusion-exclusion that can be skipped. Part II can be skipped completely; the language of matching and coloring can be introduced as needed. In Part III, one can skip the graph-theoretic applications of the Pigeonhole principle in Section 10.1, graph Ramsey Theory in 10.2, and Turán’s Theorem and edge-coloring in 11.1. If time remains, one can include techniques from Chapter 14 or 15 or material on designs from Chapter 13. The material on linear extensions and containment orders in Chapter 16 is appealing but advanced.

In a graph theory course, one would cover all of Part II, which does not require Part I except for occasional binomial coefficients and simple bijective arguments (counting two ways). From Part III one could use the material on extremal graph theory (Ramsey, Turán). Chapter 14 should be included, since probabilistic techniques are so effective with graphs. The material on matroids in Chapter 11 also would be appropriate, since only basic terminology associated with the containment order on subsets of a set is needed from Part III.

A course on methods of combinatorics would be appropriate for students who enter having some familiarity with basic enumeration and the terminology of graphs. Such a course might start with Chapter 10. These chapters on the average are longer than those in Parts I and II. Other material can be added as time and taste permit.

## Acknowledgments

When C.L. Liu heard in the mid-1980’s that I was accumulating text material on combinatorics, he showed me the lecture notes he had published as *Topics in Combinatorial Mathematics* (Math. Assoc. of America, 1972). These came from a summer seminar at Williams College in 1972 and were used in the combinatorics graduate course at the University of Illinois

that I inherited from him. He proposed that we work them into a polished textbook; thus began *The Art of Combinatorics*. As described earlier, that project grew beyond the confines of a single volume, and the present core text is closer to what Liu had in mind (probably still twice as big). I thank him for the suggestion that started the process.

Also worthy of mention is Liu’s earlier book *Introduction to Combinatorial Mathematics* (McGraw-Hill, 1968), which in 1972 introduced me to combinatorics. This book established the overall shape and subject matter for modern courses in combinatorics. Before it (at least in the U.S.) there was not much more than a compilation of chapters from eminent researchers who delivered a short course for engineers at UCLA (*Applied Combinatorial Mathematics*, 1964).

The presentation of this material has benefitted from comments contributed by many users and reviewers, including numerous students. Special thanks go to Garth Isaak, who used the book several times in pre-publication versions. Ira Gessel and Jay Goldman contributed many corrections and insightful comments concerning Part 1. John Ganci gave the book an extremely thorough reading, catching many glitches. Other reviewers contributing insightful comments have included Martin Aigner, Mike Albertson, Lowell Beineke, Miklós Bóna, Graham Brightwell, Lynne Butler, Jerry Griggs, Mike Jacobson, Jenő Lehel, Herbert Maier, Michael Molloy, Chris Rodger, Bruce Rothschild, László Szekely, and Wal Wallis.

I also thank generations of students who have labored with slowly evolving iterations of this material. Students who pointed out numerous typographical errors include Shivi Bansal, Alfio Giarlotta, Farzad Hassanzadeh, Darren Narayan, Radhika Ramamurthi, Prasun Sinha, and Reza Zamani.

This book has been typeset using  $\text{\TeX}$ . The scientific community owes a vast debt to Donald E. Knuth, the creator of  $\text{\TeX}$ . With brilliance, foresight, and generosity, he has provided a common language for the publication and communication of technical material that is now used all over the world. Besides its versatility and free availability, the incredible genius of it is that it runs amazingly fast.

Chris Hartman taught me `perl`, which I used to convert earlier `groff` files to  $\text{\TeX}$ .

## Feedback

I eagerly welcome comments on all aspects of this book. This includes selection and presentation of topics, errors in mathematics, attribution, or typography, missing items from the glossary or index (which presently are only in rough form), etc. I maintain a web page at

<http://www.math.uiuc.edu/~west/580/coerr.html>

listing typographical errors found in the current edition. Please send comments to

[west@math.uiuc.edu](mailto:west@math.uiuc.edu)

Douglas B. West  
*west@math.uiuc.edu*  
Urbana, IL