Mathematical Thinking

Problem-Solving and Proofs

Second Edition

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PRENTICE HALL
Upper Saddle River, NJ 07458

Library of Congress Cataloging-in-Publication Data

D'Angelo, John P.
Mathematical thinking: problem-solving and proofs / John P. D'Angelo, Douglas B. West.–2nd ed.
p. cm.
Includes bibliographical references and index.
ISBN 0-13-014412-6
1. Mathematics. 2. Problem solving. I. West, Douglas Brent. II. Title.

QA39.2 .D25 2000
510–dc21
99-050074

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Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-014412-6

Prentice-Hall International, (UK) Limited, *London*
Prentice-Hall of Australia Pty Limited, *Sydney*
Prentice-Hall Canada Inc., *Toronto*
Prentice-Hall Hispanoamericana, (S.A.) *Mexico*
Prentice-Hall of India Provate Limited, *New Delhi*
Pearson Education Asia Pte. Ltd.
Editora Prentice-Hall do Vrasil, Ltda, *Rio d Janeiro*

To all who enjoy mathematical puzzles,
and to our loved ones,
who tolerate our enjoyment of them

Contents

Preface for the Instructor

This book arose from discussions about the undergraduate mathematics curriculum. We asked several questions. Why do students find it difficult to write proofs? What is the role of discrete mathematics? How can the curriculum better integrate diverse topics? Perhaps most important, why don't students enjoy and appreciate mathematics as much as we might hope?

Upperclass courses in mathematics expose serious gaps in the preparation of students; the difficulties are particularly evident in elementary real analysis courses. Such courses present two obstacles to students. First, the concepts of analysis are subtle; it took mathematicians centuries to understand limits. Second, proofs require both attention to exposition and a different intellectual attitude from computation. The combination of these difficulties defeats many students. Basic courses in linear or abstract algebra pose similar difficulties and can be overly formal.

Due to their specialized focus, upperclass courses cannot adequately address the issue of careful exposition. If students first learn techniques of proof and habits of careful exposition, then they will better appreciate more advanced mathematics when they encounter it.

The excitement of mathematics springs from engaging problems. Students have natural mathematical curiosity about problems such as those listed in the Preface for the Student. They then care about the techniques used to solve them; hence we use these problems as a focus of development. We hope that students and instructors will enjoy this approach as much as we have.

A course introducing techniques of proof should not specialize in one area of mathematics; later courses offer ample opportunities for specialization. This book considers diverse problems and demonstrates relationships among several areas of mathematics. One of the authors studies complex analysis in several variables, the other studies discrete mathematics. We explored the interactions between discrete and continuous mathematics to create a course on problem-solving and proofs.

When we began the revisions for the second edition, neither of us had any idea how substantial they would become. We are excited about the improvements. Our primary aim has been to make the book easier to use by making the treatment more accessible to students, more mathematically coherent, and better arranged for the design of courses. In the remainder of this preface we discuss the changes in more detail; here we provide a brief summary.

- We added almost 300 exercises; many are easy and/or check basic understanding of concepts in the text.
- We added sections called "How to Approach Problems" in Chapters 1–5 and 13–14 to help students get started on the exercises.
- We greatly expanded Appendix B: "Hints for Selected Exercises".
- Chapters 1–4 form the core of a coherent "Transition" course that can be completed in various ways using initial sections of other chapters.
- The real number system is the starting point. All discussion of the construction of $\mathbb{R}$ from $\mathbb{N}$ is in Appendix A.
- Induction comes earlier, immediately following the background material discussed in Chapters 1 and 2.
- Individual chapters have a sharper focus, and the development flows more smoothly from topic to topic.
- Terms being defined are in bold type, mostly in Definition items.
- The language is friendlier, the typography better, and the proofs a bit more patient, with more details.

Content and Organization

Our text presents elementary aspects of algebra, number theory, combinatorics, and analysis. We cover a broad spectrum of material that illustrates techniques of proof and emphasizes interactions among the topics.

Part I (Elementary Concepts) begins by deriving the quadratic formula and using it to motivate the axioms for the real numbers, which we agree to assume. We discuss inequalities, sets, logical statements, and functions, with careful attention to the use of language. Chapter 1 establishes the themes of mathematical discussion: numbers, sets, and functions. We added lively material on inequalities and level sets. The background terminology about functions moved to Chapter 1. The more abstract discussion of injections and surjections appears in Chapter 4, introduced by the base q representation of natural numbers. This allows induction to come early; the highlight of Part I is the use of induction to solve engaging problems. Part I ends with an optional treatment of the Schroeder-Bernstein Theorem.

Part II (Properties of Numbers) studies $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$. We explore elementary counting problems, binomial coefficients, permutations (as functions), prime factorization, and the Euclidean algorithm. Equivalence relations lead to the discussion of modular arithmetic. We emphasize geometric aspects of the rational numbers. Features include Fermat's Little Theorem (with several proofs), the Chinese Remainder Theorem, criteria for irrationality, and the description of Pythagorean triples.

Part III (Discrete Mathematics) explores more subtle combinatorial arguments. We consider conditional probability and discrete random variables, the pigeonhole principle, the inclusion-exclusion principle, graph theory, and recurrence relations. Highlights include Bertrand's Ballot Problem (Catalan numbers), Bayes' Theorem, Simpson's Paradox, Euler's totient function, Hall's Theorem on systems of distinct representatives, Platonic solids, and the Fibonacci numbers. With the focus on probability in Chapter 9, the optional discussion of generating functions has moved to the end of Chapter 12, where it is used to solve recurrences.

Part IV (Continuous Mathematics) begins with the Least Upper Bound Property for $\mathbb{R}$ and its relation to decimal expansions and uncountability of $\mathbb{R}$. We prove the Bolzano-Weierstrass Theorem and use it to prove that Cauchy sequences converge. We develop the theory of calculus: sequences, series, continuity, differentiation, uniform convergence, and the Riemann integral. We define the natural logarithm via integration and the exponential function via infinite series, and we prove their inverse relationship. Defining sine and cosine via infinite series, we use results on interchange of limiting operations to verify their properties (we do not rely on geometric intuition for technical statements). We include convex functions and van der Waarden's example of a continuous and nowhere differentiable function, but we omit many applications covered adequately in calculus courses, such as Taylor polynomials, analytic geometry, Kepler's laws, polar coordinates, and physical interpretations of derivatives and integrals. Finally, we develop the properties of complex numbers and prove the Fundamental Theorem of Algebra.

In Appendix A we develop the properties of arithmetic and construct the real number system using Cauchy sequences. There we begin with $\mathbb{N}$ and subsequently construct $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. This foundational material establishes the properties of the real number system that we assume in the text. We leave this material to Appendix A because most students do not appreciate it until after they become familiar with writing proofs. Beginning instead by assuming the real numbers makes the theoretical development flow smoothly and keeps the interest of the students.

Chapters 1 and 2 provide the language for subsequent mathematical work. Formal discussion of mathematical language is problematic; students master techniques of proofs through examples of usage, not via memorization of terminology and symbolism of formal logic. Instead of formal manipulation of logical symbols, we emphasize the understanding of words. After the discussion in Chapter 2 that emphasizes the *use* of logic, familiarity with logical concepts is conveyed by repeated use throughout the book. Chapter 2 can be treated lightly in class; students can refer to it when they need help manipulating logical statements.

The rearrangement of material in Part I makes it more accessible to students and avoids using results before they can be proved. Students find induction easier and less abstract than bijections, and now it comes first. Placing the basic language about functions in Chapter 1 allows them to be used as a precise concept in Chapter 2, allows us to prove needed statements about them by induction in Chapter 3, and permits a sharper focus on the properties of injections and surjections in Chapter 4.

The material in Part II has been reorganized to give the chapters a clearer focus and to place the more fundamental material early in each chapter. Instead of combining cardinality and counting in Chapter 5, the material on cardinality has moved to Chapter 4 to better illuminate the properties of bijections. The discussion of binomial coefficients is in Chapter 5; in the first edition some of this was in Chapter 9. Chapter 5 also has new material on permutations that further explores aspects of functions. Because students have trouble producing combinatorial proofs, we provide additional examples here in the Approaches section.

We reorganized Chapter 6 to start with divisibility and factorization, allowing the Euclidean algorithm and diophantine equations to be skipped. We also added an optional section on algebraic properties of (the ring of) polynomials in one variable. In Chapter 7 we separated the discussion of general equivalence relations from the discussion of congruence. We reorganized Chapter 8 to remove the construction of $\mathbb{Q}$, beginning instead with geometric aspects of rational numbers. We moved the material on probability to Chapter 9, which now focuses completely on this topic. This clarifies the treatment of conditional probability and random variables. We moved the optional section on generating functions from Chapter 9 to Chapter 12, where it is applied.

In Part IV, we provided more details in proofs, plus friendlier language and typesetting. The treatment of decimal expansions in 13 is more natural and more precise. In Chapter 14, the material on Cauchy Sequences now appears after the material on limits of sequences.

Pedagogy and Special Features

Certain pedagogical issues require careful attention. In order to benefit from this course, students need a sense of intellectual progress. An axiomatic development of the real numbers is painfully slow and frustrates students. They have learned algebraic computational techniques throughout their schooling, and it is important to build on this foundation. This dictates our starting point.

In Chapter 1 we list the axioms for the real numbers and their elementary algebraic consequences, and we accept them for computation and reasoning. We defer the construction of the real numbers and verification of the field axioms to Appendix A, for later appreciation. In the second edition, we have made this pedagogically valuable approach more firmly consistent, obtaining $\mathbb{N}$ within $\mathbb{R}$ in Chapter 3 and moving the details of the rational number system from Chapter 8 to Appendix A. This simplifies the treatment of induction and eliminates most comments (and student uncertainty) about what we do and do not know at a given time. We exclude the use of calculus until it is developed in Part IV.

The exercises are among the strongest features of this book. Many are fun, some are routine, and some are difficult. Exercises designated by "$(-)$" are intended to check understanding of basic concepts; they require neither deep insight nor long solutions. The "$(+)$" problems are more difficult. Those designated by "(!)" are especially interesting or instructive. Most exercises emphasize thinking and writing rather than computation. The understanding and communication of mathematics through problem-solving should be the driving force of the course.

We have reorganized the exercises and added many, especially of the "$(-)$" type. We increased the number of exercises by 60% in Parts I–II and 40% overall; there are now well over 900 exercises. We have gathered the routine exercises at the beginning of the exercise sections. Usually a line of dots separates these from the other exercises to assist the instructor in selecting problems; after the dots the exercises are ordered roughly in parallel to the presentation of material in the text. Many of the exercise sets also have true/false questions, where students are asked to decide whether an assertion is true or false and then to provide a proof or a counterexample.

The purpose of the exercises is to encourage learning, not to frustrate students. Many of the exercises in the text carry hints; these represent what we feel will be helpful to most students. Appendix B contains more elementary hints for many problems; these are intended to give students a starting point for clearer thinking if they are completely stumped by a problem. We have expanded Appendix B so that now we give hints for more than half of the problems in the book.

We have also added sections called "How to Approach Problems" in Chapters 1–5 and 13–14. These are the chapters emphasized in courses with beginning students. In these sections, we summarize some thoughts from the chapters and provide advice to help students avoid typical pitfalls when starting to solve problems. The discussion here is informal.

The Preface to the Student lists many engaging problems. Some of these begin chapters as motivating "Problems"; others are left to the exercises. Solutions of such problems in the text are designated as "Solutions". Items designated as "Examples" are generally easier than those designated as “Solutions” or “Applications”. “Examples” serve primarily to illustrate concepts, whereas “Solutions” or “Applications” employ the concepts being developed and involve additional reasoning.

Students have some difficulty recognizing what material is important. The book has two streams of material: the theoretical mathematical development and its illustrations or applications. “Definitions”, “Propositions”, “Lemmas”, “Theorems”, and “Corollaries” are set in an indented style. Students may use these results to solve problems and may want to learn them. Other items generally provide examples or commentary.

This book does not assume calculus and hence in principle can be used in a course taught to freshmen or to high school students. It does require motivation and commitment from the students, since problems can no longer be solved by mimicking memorized computations. The book is appropriate for students who have studied standard calculus and wonder why the computations work. It is ideal for beginning majors in mathematics and computer science. Readers outside mathematics who enjoy careful thinking and are curious about mathematics will also profit by it. High school teachers of mathematics may appreciate the interaction between problem-solving and theory.

The second author maintains a web site for this book with course materials, listing of errors or updates, etc. Please visit

http://www.math.uiuc.edu/~west/mt

Comments and corrections are welcome at west@math.uiuc.edu.

Design of Courses

We developed this book through numerous courses, beginning with a version we team-taught in 1991 at the University of Illinois. Various one-semester courses can be constructed from this material. The changes made for the second edition facilitate the design of courses.

Many schools have a one-semester “transition” course that introduces students to the notions of proof. Such a course should begin with Chapters 1–4 (omitting the Schroeder-Bernstein Theorem). Depending on the local curriculum and the students, good ways to complete such a course are with Chapters 5–8 or Chapters 13–14 (or both). The second edition makes these chapters more independent and places the more elementary material in each chapter near the beginning. This makes it easy to present just the fundamental material in each chapter. With good students, it is possible to present Chapters 1–10 and 13–15 in one semester, omitting the optional material.

A one-semester course on discrete mathematics that emphasizes proofs can cover Parts I–III, omitting most of Chapter 8 (rational numbers) and the more algebraic material from Chapters 6 and 7. Depending on the preparation of the students, Chapters 1–2 can be treated as background reading for a faster start. It should be noted that Part II maintains a more elementary atmosphere than Part III, and that the topics in Part III are more specialized.

A one-semester course in elementary analysis covers Chapters 3 and 4, perhaps some of Chapter 8 (many such courses discuss the rational numbers), and Chapters 13–17. Students should read Chapters 1 and 2 for background. This yields a thorough course in introductory analysis. The first author has twice taught successful elementary real analysis courses along these lines, covering chapters 13–17 completely after spending a few weeks on these earlier chapters.

The full text is suitable for a patient and thorough one-year course culminating in the Fundamental Theorem of Algebra.

Acknowledgments

Our preparation of the first edition was helped by comments from Art Benjamin, Dick Bishop, Kaddour Boukaabar, Peter Braunfeld, Tom Brown, Steve Chiappari, Everett Dade, Harold Diamond, Paul Drelles, Sue Goodman, Dan Grayson, Harvey Greenwald, Deanna Haunsperger, Felix Lazebnik, N. Tenney Peck, Steve Post, Sara Robinson, Craig Tovey, Steve Ullom, Josh Yulish, and other readers. The Mathematics Department of the University of Illinois gave us the opportunity to develop the course that inspired this book; we thank our students for struggling with preliminary versions of it. Our editor George Lobell provided the guidance and prodding needed to bring the book to its final form.

Additional comments for the preparation of the second edition were contributed by Charles Epstein, Dan Grayson, Corlis Johnson, Ward Henson, Ranjani Krishnan, Maria Muyot, Jeff Rabin, Mike Saks, Hector Sussmann, Steve Ullom, C.Q. Zhang. Many students who used the book spotted typographical errors or opaque passages and suggested additional improvements. The careful eye of our production editor Betsy Williams corrected many glitches and design problems.

The second edition was typeset using TEX, with illustrations created using the **gpic** program, a product of the Free Software Foundation. We thank Maria Muyot for assistance in the preparation of the index.

The authors thank their wives Annette and Ching, respectively, for their love, encouragement, and patience. The first author also thanks his children John, Lucie, and Paul for inspiration.

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Preface for the Student

This book demands careful thinking; we hope that it also is enjoyable. We present interesting problems and develop the basic undergraduate mathematics needed to solve them. Below we list 37 such problems. We solve most of these in this book, while at the same time developing enough theory to prepare for upperclass math courses.

In Chapters 1–5 and 13–14 we have included sections called "How to Approach Problems". These provide advice on what to do in solving problems and warnings on what not to do. The "Approaches" evolved from using the book in the classroom; we have learned what difficulties students encountered and what errors occurred repeatedly. We have also provided, in Appendix B, hints to many exercises. These hints are intended to get students started in the right direction when they don't know how to approach a problem.

Many exercises are designated by "$(-)$", "(!)", or "$(+)$". The "$(-)$" exercises are intended to check understanding; a student who cannot do these is missing the basics. A student who can do an occasional "$(+)$" problem is showing some ability. The "(!)" problems are particularly instructive, important, or interesting; their difficulty varies. Many chapters contain true/false questions; here the student is asked to decide whether something is true and provide a proof or a counterexample.

This is a mathematics book that emphasizes writing and language skills. We do not ask that you memorize formulas, but rather that you learn to express yourself clearly and accurately. You will learn to solve mathematical puzzles as well as to write proofs of theorems from elementary algebra, discrete mathematics, and calculus. This will broaden your knowledge and improve the clarity of your thinking.

A proof is nothing but a complete explanation of why something is true. We will develop many techniques of proof. It may not be obvious what technique works in a given problem; we will sometimes give different proofs for a single result. Most students have difficulty when first asked to write proofs; they are unaccustomed to using language carefully and logically. Do not be discouraged; experience increases understanding and makes it easier to find proofs.

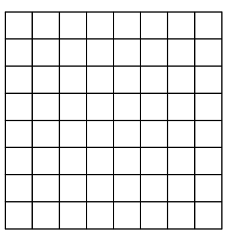
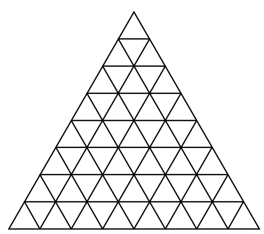
How can you improve your writing? Good writing requires practice. Writing out a proof can reveal hidden subtleties or cases that have been overlooked. It can also expose irrelevant thoughts. Producing a well-written solution often involves repeated revision. You must say what you mean and mean what you say. Mathematics encourages habits of writing precisely, because clear decisions can be made about whether sentences contain faulty reasoning. You will learn how to combine well chosen notation with clear explanation in sentences. This will enable you to communicate ideas concisely and accurately.

We invite you to consider some intriguing problems. We solve most of these in the text, and others appear as exercises.

1. Given several piles of pennies, we create a new collection by removing one coin from each old pile to make one new pile. Each original pile shrinks by one; 1, 1, 2, 5 becomes 1, 4, 4, for example. Which lists of sizes (order is unimportant) are unchanged under this operation?

2. Which natural numbers are sums of consecutive smaller natural numbers? For example, $30 = 9 + 10 + 11$ and $31 = 15 + 16$, but 32 has no such representation.

3. Including squares of sizes one-by-one through eight-by-eight, an ordinary eight-by-eight checkerboard has 204 squares. How many squares of all sizes arise using an n-by-n checkerboard? How many triangles of all sizes arise using a triangular grid with sides of length n?

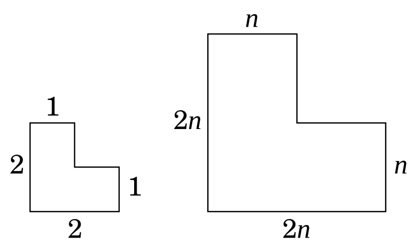



4. At a party with five married couples, no person shakes hands with his or her spouse. Of the nine people other than the host, no two shake hands with the same number of people. With how many people does the hostess shake hands?

5. We can tell whether two groups of weights have the same total weight by placing them on a balance scale. How many known weights are needed to balance each integer weight from 1 to 121? How should these weights be chosen? (Known weights can be placed on either side or omitted.)

6. Given a positive integer k, how can we obtain a formula for the sum $1^k + 2^k + \cdots + n^k$?

7. Is it possible to fill the large region below with non-overlapping copies of the small L-shape? Rotations and translations are allowed.



8. If each resident of New York City has 100 coins in a jar, is it possible that no two residents have the same number of coins of each type (pennies, nickels, dimes, quarters, half-dollars)?

9. How can we find the greatest common divisor of two large numbers without factoring them?

10. Why are there infinitely many prime numbers? Why are there arbitrarily long stretches of consecutive non-prime positive integers?

11. Consider a dart board having two regions, one worth a points and the other worth b points, where a and b are positive integers having no common factors greater than 1. What is the largest point total that cannot be obtained by throwing darts at the board?

12. A math professor cashes a check for x dollars and y cents, but the teller inadvertently pays y dollars and x cents. After the professor buys a newspaper for k cents, the remaining money is twice as much as the original value of the check. If $k = 50$, what was the value of the check? If $k = 75$, why is this situation impossible?

13. Must there be a Friday the 13th in every year?

14. When two digits in the base 10 representation of an integer are interchanged, the difference between the old number and the new number is divisible by nine. Why?

15. A positive integer is **palindromic** if reversing the digits of its base 10 representation does not change the number. Why is every palindromic integer with an even number of digits divisible by 11?

16. What are all the integer solutions to $42x + 63y = z$? To $x^2 + y^2 = z^2$?

17. Given a prime number L, for which positive integers K can we express the rational number K/L as the sum of the reciprocals of two positive integers?

18. Are there more rational numbers than integers? Are there more real numbers than rational numbers? What does "more" mean for these sets?

19. Can player A have a higher batting average than B in day games and in night games but a lower batting average than B over all games?

Player	Day	Night	Overall
A	.333	.250	.286
B	.300	.200	.290

20. Suppose A and B gamble as follows: On each play, each player shows 1 or 2 fingers, and one pays the other x dollars, where x is the total number of fingers showing. If x is odd, then A pays B; if x is even, then B pays A. Who has the advantage?

21. Suppose candidates A and B in an election receive a and b votes, respectively. If the votes are counted in a random order, what is the probability that candidate A never trails?

22. Can the numbers $0, \ldots, 100$ be written in some order so that no 11 positions contain numbers that successively increase or successively decrease? (An increasing or decreasing set need not occupy consecutive positions or use consecutive numbers.)

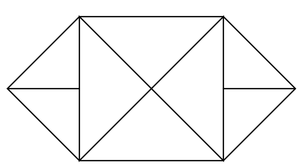
23. Suppose each dot in an n by n grid of dots is colored black or white. How large must n be to guarantee the existence of a rectangle whose corners have the same color?

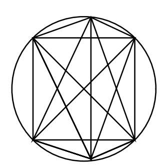
24. How many positive integers less than 1,000,000 have no common factors (greater than 1) with 1,000,000?

25. Suppose n students take an exam, and the exam papers are handed back at random for peer grading. What is the probability that no student gets his or her own paper back? What happens to this probability as n goes to infinity?

26. There are n girls and n boys at a party, and each girl likes some of the boys. Under what conditions is it possible to pair the girls with boys so that each girl is paired with a boy that she likes?

27. A computer plotter must draw a figure on a page. What is the minimum number of times the pen must be lifted while drawing the figure?





28. Consider n points on a circle. How many regions are created by drawing all chords joining these points, assuming that no three chords have a common intersection?

29. A Platonic solid has congruent regular polygons as faces and has the same number of faces meeting at each vertex. Why are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron the only ones?

30. Suppose n spaces are available for parking along the side of a street. We can fill the spaces using Rabbits, which take one space, and/or Cadillacs, which take two spaces. In how many ways can we fill the spaces? In other words, how many lists of 1's and 2's sum to n?

31. Repeatedly pushing the "x^2" button on a calculator generates a sequence tending to 0 if the initial positive value is less than 1 and tending to ∞ if it is greater than 1. What happens with other quadratic functions?

32. What numbers have more than one decimal representation?

33. Suppose that the points in a tennis game are independent and that the server wins each point with probability p. What is the probability that the server wins the game?

34. How is $\lim_{n\to\infty}(1 + x/n)^n$ relevant to compound interest?

35. One baseball player hits singles with probability p and otherwise strikes out. Another hits home runs with probability $p/4$ and otherwise strikes out. Assume that a single advances each runner by two bases. Compare a team composed of such home-run hitters with a team composed of such singles hitters. Which generates more runs per inning?

36. Let $T_1, T_2, \ldots$ be a sequence of triangles in the plane. If the sequence of triangles converges to a region T, can we then conclude that $\text{Area}(T) = \lim_{n\to\infty} \text{Area}(T_n)$?

37. Two jewel thieves steal a circular necklace with $2m$ gold beads and $2n$ silver beads arranged in some unknown order. Is it always true that there is a way to cut the necklace along some diameter so that each thief gets half the beads of each color? Does a heated circular wire always contains two diametrically opposite points where the temperature is the same? How are these questions related?