

Math 247 - Instructions to Instructors

Courses in mathematics at the 300-level require acquaintance with basic language and techniques of mathematics. The suggested course in which to obtain this is Math 247. For this reason, consistency in the training provided by Math 247 is needed; it is important that instructors honor the course description. In addition, high standards must be maintained so that students obtain a realistic view of the level of care needed to be successful in mathematics and so that students who don't have the ability or determination to be successful will get that message at an early stage.

The official course description of Math 247:

Fundamental ideas used in many areas of mathematics. Topics will include: techniques of proof, mathematical induction, binomial coefficients, rational and irrational numbers, the least upper bound axiom for real numbers, and a rigorous treatment of convergence of sequences and series. This will be supplemented by the instructor from topics available in the various texts. Students will regularly write proofs emphasizing precise reasoning and clear exposition.

Math 247 should prepare students to read and write mathematics at the advanced undergraduate level. “Techniques of proof” refers to logical statements, quantifiers, direct and indirect proof, etc. Mathematical induction should not need to be taught in later courses; it is imperative that it be understood clearly in Math 247. The material on binomial coefficients involves the idea of bijection. Divisibility and equivalence relations are valuable additional topics for giving students sufficient mathematical literacy. “Rational and irrational numbers” includes the existence of irrational numbers. The least upper bound axiom is essential to understanding the properties of real numbers and convergence. The material earlier in the course helps prepare students to understand the meaning of convergence and related proof techniques.

Students cannot develop the skills to write proofs correctly without substantial written homework and careful feedback. Instructors and graders in this course should provide written feedback to help students improve their presentation of mathematical arguments. Both the mathematical ideas and the quality of the presentation should be evaluated. Adequate practice and feedback requires substantial written homework, perhaps five written arguments per week. The unusual declaration of proof-writing activities in the course description emphasizes that this is an important and necessary part of the course.

Rigorous standards of evaluation should be maintained. For most students, this is the first course where they are asked to express mathematical ideas clearly. Honest grading should make it clear that this is important and should convey to students who cannot do it that they should consider another major before it is too late. The grade of *A* should be given only to students who perform extremely well. Grades of *A* in calculus courses do not guarantee that a student will excel in 247 or in later mathematics.

Math 247 - Philosophy (personal opinions of D.B. West)

Goals and Syllabus

The most important function of Math 247 is to develop students' ability to think logically; in particular, to read, understand, and write mathematics. To this end, basic tools of mathematical reasoning are emphasized. Manipulation of logical statements, elementary proof techniques and set operations, functions, and induction fall into this category.

With this focus, it has been said that the course has no content, or that its content is unimportant. This is nonsense. Discussing mathematical reasoning in the abstract, with no motivating context, is a horrendous bore. Focusing on the manipulation of truth tables will drive students away. Students will not expend effort to understand mathematical reasoning and why statements need to be proved unless they see that these techniques are of some use or can enable them to do things they find interesting.

Some faculty believe that the most elegant context for conveying mathematical techniques is a foundations course that constructs the number systems. We thought so, too, until we tried it. Such a course is extremely elegant — for a mathematician. Unfortunately, it ignores the audience. Only someone who is already a mathematician appreciates and sees value in such constructions. The students in Math 247 are not yet mathematicians. They become frustrated when told they cannot divide or take square roots because the course has not yet justified them. This extinguishes natural curiosity and enthusiasm. For this reason, the course must start with the real numbers available.

Math 247 should give students the tools of mathematical reasoning while preserving and enhancing their excitement about mathematics so that later they can appreciate abstract mathematics. Students get excited about mathematical reasoning when they learn how to use it to solve problems that interest them. This gives them the motivation to put forth the effort needed to learn the language and techniques of elementary mathematics. “Proof” is then viewed as a rigorous explanation of why an interesting fact is true.

Given that students are spending a semester developing their skills in elementary reasoning (and careful writing), it would be silly to pass the time on random facts with no use in the curriculum. Thus the motivating problems should be carefully selected to develop useful mathematics in their solutions.

The most obvious application of the techniques developed is in the 300-level math courses. We developed our version of Math 247 from the idea that students encounter trouble in the transition from calculus to 300-level courses if they are simultaneously asked for the first time both to deal with careful logical arguments and to absorb abstract mathematical concepts. (This is another reason why a course constructing the real numbers cannot succeed as a transition course.) Our aim was to develop familiarity with mathematical language and reasoning first, thereby giving students the tools to understand abstract concepts later.

Preparation for 300-level courses suggests applying the elementary concepts to topics like binomial coefficients, divisibility and congruence, countability and rational numbers, uncountability, and properties of the real numbers involving limits and convergence. There are many interesting motivating problems that facilitate development of these topics.

Some faculty argue that 247 should drill students on a very small set of topics and until they can do something well. I disagree, for two reasons. First, one can convey a progression of topics culminating in convergence and other aspects of analysis. Counting arguments increase familiarity with bijections, which is needed to understand countability. Counting and induction apply to divisibility, which leads naturally to modular arithmetic and rational numbers, both of which convey understanding of equivalence relations, which provide understanding of real numbers via Cauchy sequences. q -ary expansions and geometric sums (developed using induction) help to explain decimal expansions, and finite sums in general lead to series. The use of quantified statements throughout the first two-thirds of the semester prepares students to understand the epsilon-delta definition of limits and the related “epsilon/2 argument” (bounding an error by the sum of two small errors).

Second, Math 247 is the gateway to mathematics. While we hope that maintaining high standards in this course will scare away unqualified students (while still giving them some skills and good habits they can use), we also hope that it will invite good students to explore mathematics more deeply. The course enticed some of our best students of recent years to become math majors. It is the last stop before specialization, the last chance for substantial illustration of how different parts of mathematics fit together. It helps a student discover which parts of mathematics he or she likes best and wants to explore more deeply. Making the course focus on analysis or on algebra or on number theory or on formal logic misses a golden opportunity.

So, what should Math 247 accomplish? The students should understand mathematical statements (alternating quantifiers) and their negation, conditional statements and their converse and contrapositive, direct and indirect proof and proof by case analysis, induction and strong induction, functions (especially bijections), the meaning of binomial coefficients, divisibility, modular arithmetic and equivalence relations, rational and irrational numbers, the notion of a real number as a series, uncountability of the real numbers, the least upper bound axiom, the epsilon/delta definition of limits and the idea of the epsilon/2 argument, convergence of sequences and series, and the neighborhood and sequential versions of continuity up to the Intermediate Value Theorem.

Except for a few key results in the analysis area that are tools, such as the Monotone Convergence Theorem, most of the course should aim to develop tools of reasoning for the purpose of explaining phenomena that are already familiar at some level to many students. Material on divisibility and rational numbers is of this sort. Similarly, amassing a list of convergence tests for series is not so important, but understanding the meaning of convergence and the techniques for proving convergence tests is.

If I had to list key theorems that the students should be taught, I would include the fact that polynomials of degree d have at most d real zeros, the uniqueness of base q expansion of natural numbers, the countability of $\mathbb{N} \times \mathbb{N}$ and \mathbb{Q} , the Binomial Theorem, the prime factorization of integers, the operations of modular arithmetic, Fermat’s Little Theorem, the Rational Zeros Theorem (rational solutions of polynomials), the uncountability of the real numbers, the Monotone Convergence Theorem, the Bolzano-Weierstrass Theorem, the Cauchy Convergence Criterion (and its application to convergence of series), and the Intermediate Value Theorem. This does not fill the semester, and I cover more when I teach the course.

Pedagogy

Many faculty have pointed out the importance of maintaining high standards in this course. I agree that it is imperative to give students realistic expectations about their success in mathematics and to encourage those who do very poorly not to continue to higher courses. A faculty member who routinely gives all *As* is abdicating his or her educational responsibility. The potential negative repercussions of giving honest grades can be overcome by the positive responses that students have to effective teaching. We need to provide instructors with techniques they can use to help their students learn better and enjoy the process. Many of these take extra time, but perhaps the enterprise is worth it.

Students cannot write a mathematical statement properly and understand the effects of word order if they cannot write an English sentence properly. It is imperative to require substantial written work in this course and to maintain high standards for the quality of presentation. Careful feedback is needed to show students how to improve their presentation. Students would learn a lot if they had to revise a written solution until their presentation said what they intended. Unfortunately, repeated submission increases the time required of instructors or graders.

The process of revision and exercising care about writing would be greatly facilitated if students wrote homework solutions using computers. All students have computer accounts now, and the sooner they learn good habits of editing, the better. Having the words on the computer greatly facilitates the process of revision, and it also increases the desire to improve the presentation to match the improved appearance.

Producing solutions to mathematical problems is a two-phase process. The second phase, discussed above, is producing a clear and complete written presentation. The first phase is understanding the mathematical ideas and finding a proof. This relates to the issue of content of the course. I believe that students will not appreciate the techniques of mathematical reasoning solely by being shown and regurgitating examples of them. They must also experience the process of using those techniques to discover proofs.

Learning to discover proofs is a difficult process requiring considerable experience. Students need help to develop this experience. In undergraduate proof-based courses, I offer optional collaborative study sessions in which students work together in small groups to understand the ideas in the homework problems and develop solutions (the homework problems apply ideas from class but involve multiple steps and are challenging). I circulate and listen, answering questions and providing guidance when students get stuck. Students work out ideas and proofs together but write up the proofs on their own.

Speaking mathematics helps one see it in different ways; explaining a proof to someone else tests one's understanding and may suggest a better way to write it. This enables stronger students to benefit from participating. Weaker students benefit by seeing the thinking process of others, and gradually they begin to generate ideas of their own. Hearing the discussion keeps them engaged and encourages them to ask about ideas that they don't understand.

These study sessions enable each student to obtain the level of assistance that he or she needs to understand and learn. The best students don't need it and usually prefer to work on their own. Many students have said that such sessions were essential to their success in the course. They also help the instructor learn which problems are hard, where

students get stuck, and how to help students learn to develop approaches to problems.

The study sessions can be viewed as highly efficient office hours, since discussion occurs in different groups simultaneously, and the atmosphere encourages students to answer each other's questions. Study sessions handle most questions that would be asked in office hours (or not asked at all), but some students still use office hours. Instructors should also encourage questions by email; many who would not come to office hours use this medium.

When students ask questions, especially about homework, instructors should respond by correcting misconceptions and then ask questions that students should ask themselves to approach the problem. The aim is to break it down, as they should, into steps they can see how to do. Some students find this frustrating at first (and time-consuming!), but it gives them a better chance to improve. Eventually most begin to make connections on their own and appreciate the instructor's efforts.

Similar techniques can also be used in lectures, pausing to ask the students what to do next in a proof or example. This keeps control the pace of the lecture, enlivens the class, and helps students remain engaged and thinking. It also gives students the opportunity to ask questions of their own when they would hesitate to interrupt a lecture.

It has been suggested that frequent unannounced quizzes be used to improve attendance. It is far more important to improve attendance by making the classes interesting and worthwhile. Quizzes waste valuable class time. Attendance by students who are only there for the quizzes and otherwise stare out the window deadens the class for everyone else. Treating the students like high school students will not help them grow up.

My grading system has evolved over many years and seems fairly robust; I offer it as an example. Weekly homework requires five from a choice of six problems, worth three points each. Flexibility to accommodate students entering the course late or getting sick or having occasional heavy assignments in other courses is achieved by keeping only the twelve highest of 14 weekly homework grades, yielding 180 possible points. The three tests are worth 100 points each, and the final is worth 150 points (630 points total). Since tests allow only limited time for discovery of proofs, the tests emphasize assessment of students' understanding and their skill in applying the material to somewhat easier problems.

My grading scale makes it easy to pass the course but not easy to get an *A*. Collaborative study sessions allow students to pass by working hard and understanding enough to solve a few test problems. An *A* requires doing well on the homework and solving a substantial fraction of test problems. My median is usually in the *B*– range. This takes into account the course drops that occur due to requiring steady work from the beginning of the semester.

Many instructors have successful techniques for courses like this. For the future, we should collect such pointers for the instructors of Math 247, presenting a portfolio of accumulated wisdom. For example, one can say many things about effective lecture style. A lecture can emphasize the thought process in developing a proof and why the technique is natural. It keeps the interest of students better by introducing a problem and then developing the technique to solve it instead of simply making definitions and proving theorems. Etc., etc.