Acquisition Number of Graphs

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Joint work with
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The Problem

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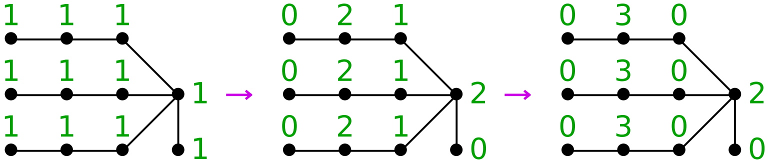
Ex. $a(T) = 4$, 11 vertices.



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Thm. For almost all trees, $\alpha(T) \ge .06n$.

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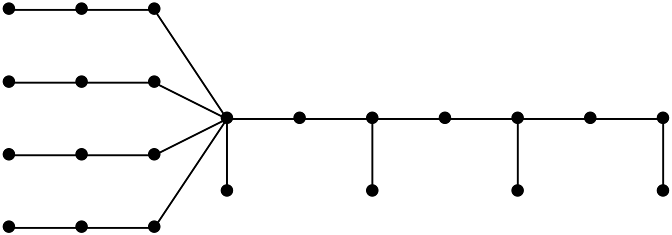
Thm. If $\operatorname{diam} G = 2$, then $\alpha(G) \leq 250 \lg n \lg \lg n$.
If $\operatorname{diam} G = 2$ and $C_4 \not\subseteq G$ and $\Delta(G) \geq 8$, then $\alpha(G) = 1$.

Trees with $\alpha(G)$ Large

Ex. The tree $T_{4,3}$.



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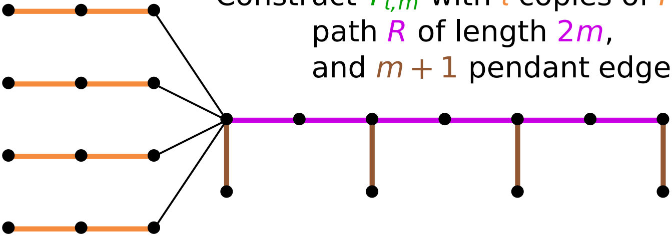

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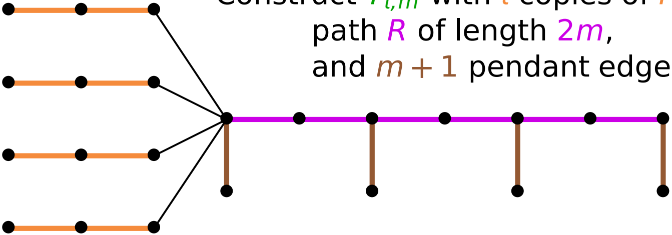

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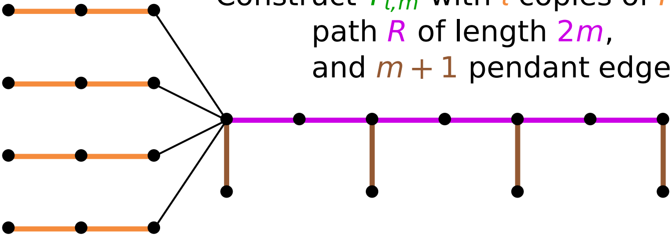

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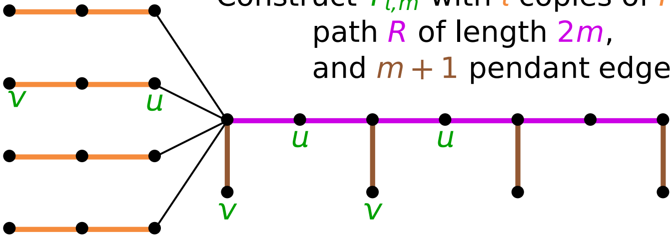

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Prop. $a(T_{l,m}) = l+m+1 = (n+1)/3$.

Pf. Moving wt $v \to u$ requires u to get other wt first.
$\therefore$ can't get wt from two leaves. $\therefore$ $a(T) \geq$ #leaves.

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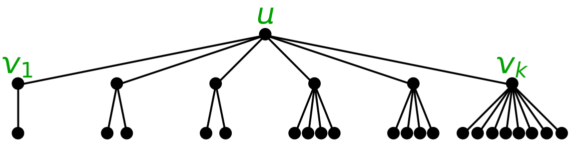

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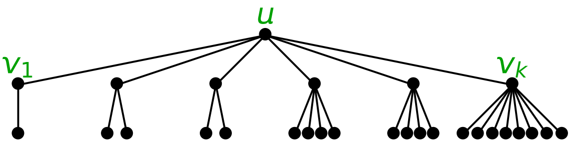

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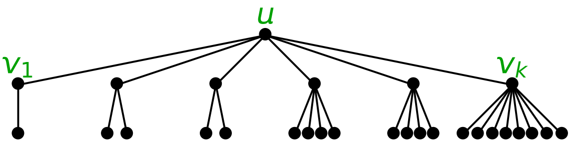

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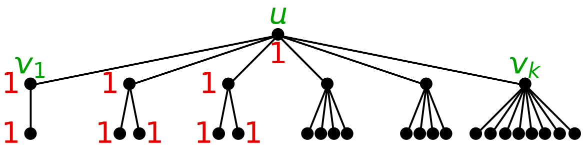

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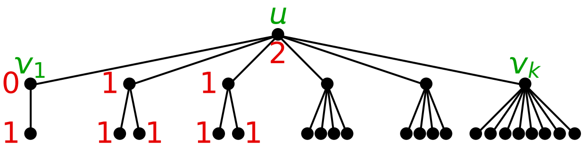

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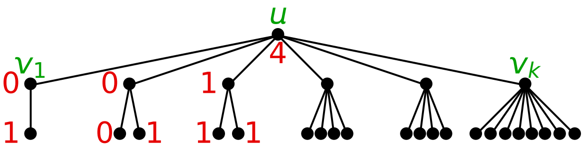

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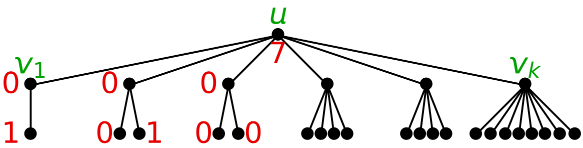

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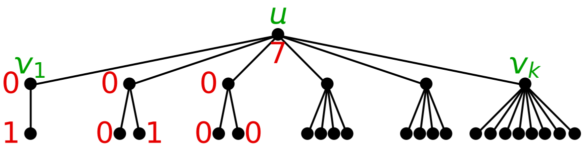

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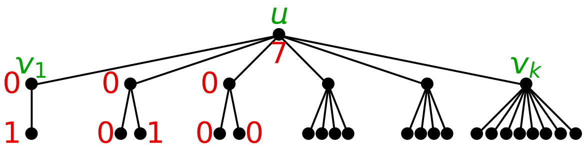

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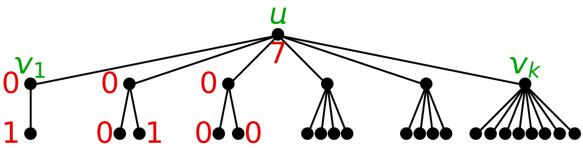

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Hence $a(T) \le 1 + |S|(2n/k) \le 2\sqrt{n \lg 2n}$.

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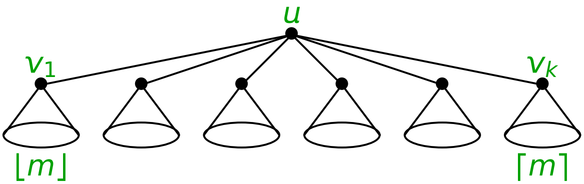

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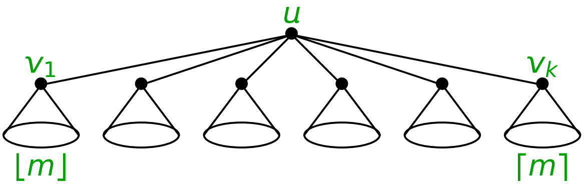

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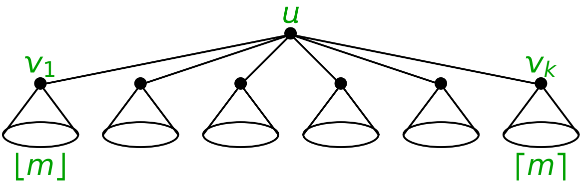

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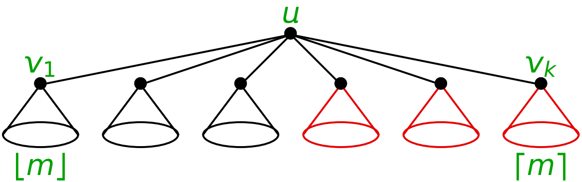

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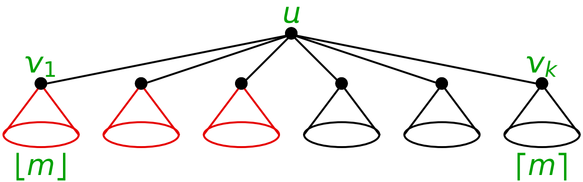

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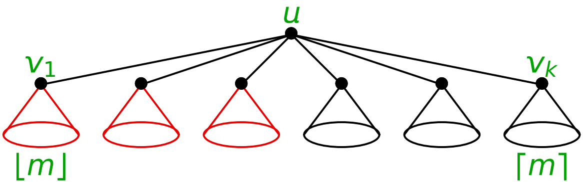

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#leaves stranded $\geq \sum_{i=1}^{q} \max\{d(v_i) - 2^{i-1}, 0\}$

$$\geq m \lg n - 2m = \frac{n-k-1}{k}(\lg n - 2) \geq \frac{1}{20}\sqrt{n \lg n}. \qquad \blacksquare$$

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Thm. (Lampert–Slater [1995]) Testing $\alpha(G) = 1$ on general graphs is NP-complete.

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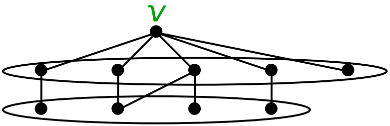
Lem. If the neighborhood of some vertex v with degree at least $n/2$ dominates G, then $\alpha(G) = 1$.



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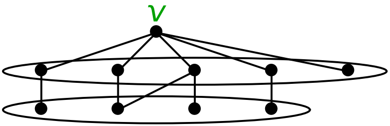

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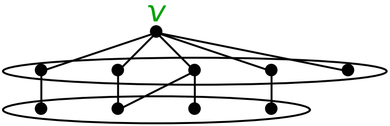

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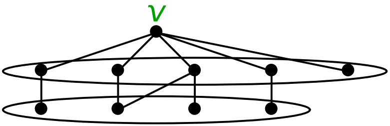

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Hence v can absorb the weight of neighbors with weight 1 and then absorb all the weight.

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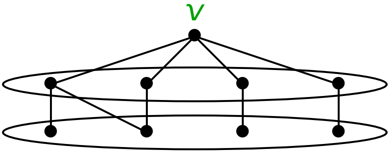
Lem. If G is $\frac{n-1}{2}$-regular and not C_5, then $\alpha(G) = 1$.



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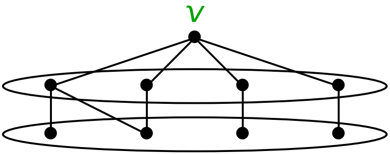

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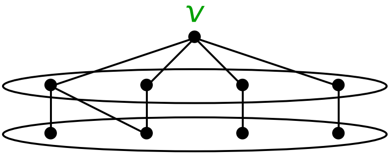

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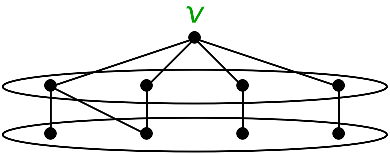

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$$\begin{array}{ll} \text{Case 1}: & 3\ 3\ 2\ \cdots\ 2\ 1\ 1 \\ \text{Case 2}: & 3\ 2\ 2\ \cdots\ 2\ 2\ 1 \\ \text{Case 3}: & 2\ 2\ 2\ \cdots\ 2\ 2\ 2 \end{array}$$

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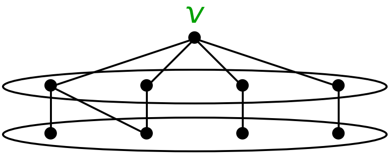

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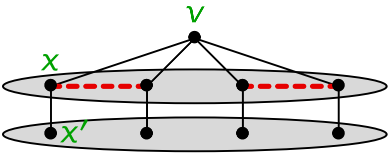

Lemma $\Rightarrow$ move weight from $N[v]$ to x, rest to x'.

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Thm. If $G \neq C_5$ and $d(u)+d(v) \geq n-1$ when $uv \notin E(G)$, then $\alpha(G) = 1$.

Pf. The hypothesis gives a common nbr to nonadjacent vertices, so every neighborhood dominates.

Either $\Delta(G) \geq n/2$, or G is $(n-1)/2$-regular. ■

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Diameter 2

Thm. If $\operatorname{diam} G = 2$, then $\alpha(G) \leq 250 \lg n \lg \lg n$.
If $\operatorname{diam} G = 2$ and $C_4 \not\subseteq G$ and $\Delta(G) \geq 8$, then $\alpha(G) = 1$.

Open Problems

Conj. There is a constant c such that, if $\mathrm{diam} G = 2$, then $\alpha(G) \leq c$.

Perhaps $c = 2$. This suffices for Moore graphs, polarity graphs, and G with $\Delta(G) \leq 7$ and $C_4 \notin G$.

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Ques. game acquisition number (Slater–Wang [2004]) Minimizer and Maximizer alternate acquisition moves.

S–W computed it for paths. M–S–W–W provide bounds for trees and show the value is usually 2 for $K_{r,s}$.