Gray Code Enumeration of Families of Integer Partitions

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Abstract

In this paper we show that the elements of certain families of integer partitions can be listed in a minimal change, or *Gray code*, order. In particular, we construct Gray code listings for the classes $P_\delta(n,k)$ and $D(n,k)$ of partitions of n into parts of size at most k in which, for $P_\delta(n,k)$, the parts are congruent to one modulo δ and, for $D(n,k)$, the parts are distinct. The change required between successive partitions is the increase of one part by δ (or the addition of δ ones) and the decrease of one part by δ (or the removal of δ ones), where, in the case of $D(n,k)$, $\delta = 1$.

1 Introduction

Recent work in combinatorial enumeration has considered listing special sets so successive elements differ by a small, pre-specified change. Examples include (1) generating permutations by adjacent transpositions [5, 15] (2) generating bit strings by changing one bit [4, 3], (3) generating subsets by changing one element [1, 8, 11], (4) generating binary trees by rotations [7], (5) generating Coxeter group elements by reflection [2], and (6) generating linear extensions of certain posets by transpositions [9, 10, 12, 14, 16]. Such enumeration schemes are called *minimal change algorithms* or *combinatorial Gray codes*, in honor of the reflected binary code of Gray for solving problem (2) above. These schemes may permit "efficient" generation of combinatorial families of exponential size. They may list the elements of a class C in time $O(|C|)$, independent of the size of the objects. The approach has succeeded for the problems listed above.

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A combinatorial Gray code problem is the problem of finding a Hamiltonian path in the associated graph whose vertices are the objects of the class, with vertices adjacent if they differ in the allowed way. Under this view, many Gray code problems are special cases of open problems in graph theory. In cases such as 1, 2, 3, and 5 above, the associated graph is vertex transitive; it is an open question of Lovász whether every connected vertex transitive graph has a Hamiltonian path [6]. The question remains open even if the vertex transitive graph is the Cayley graph of a finite group, as for (1) and (5) above.

Herb Wilf suggested the problem of constructing Gray codes for integer partitions. A partition of an integer n is a string $x_1 x_2 \ldots x_t$ of positive integers whose sum is n, ordered so that $x_1 \geq x_2 \geq \ldots \geq x_t$. Wilf asked whether it is possible, given n, to list the partitions of n so that successive parts differ only in that one part increases by 1 (or a part of size 1 is added) and one part decreases by 1 (or a part of size 1 is removed). In the list of partitions of 6 in lexicographic order, this condition is violated only between the successive partitions 3 1 1 1 and 2 2 2. If the list is reordered in the following way, no violations occur:

$$6, \quad 5\ 1, \quad 4\ 2, \quad 4\ 1\ 1, \quad 3\ 2\ 1, \quad 2\ 2\ 2, \quad 2\ 2\ 1\ 1, \quad 3\ 1\ 1\ 1, \quad 2\ 1\ 1\ 1\ 1, \quad 1\ 1\ 1\ 1\ 1\ 1.$$

By a doubly recursive construction, Savage [13] proved that such a listing always exists. In fact, for any n and k satisfying $n \geq k \geq 1$, the set of partitions of n into parts of size at most k has such a listing. Furthermore, unless $(n, k) = (6, 4)$, the enumeration can be required to start at the lexicographically largest partition (any listing must end at the partition having n parts of size 1, since this partition has only one neighbor). This anomaly for $(6, 4)$ caused considerable complication for the recursive construction.

In this paper, we produce Gray codes for families of integer partitions with two types of restrictions placed on the parts. In the first class, we require the parts to be congruent to 1 modulo δ, for some fixed δ, and we allow changes in which one part increases by δ (or δ parts of size 1 are added) and one part decreases by δ (or δ parts of size 1 are deleted). This generalizes the case of unrestricted partitions, where $\delta = 1$. By generalizing the construction of [13], we prove in Section 3 that for arbitrary δ, Gray codes always exist.

In Section 4, we consider partitions into distinct parts, with adjacencies as above for $\delta = 1$. Due to the sparseness of this class in the set of all partitions, the previous construction fails. By a suitable refinement, we prove nevertheless that (surprisingly to us) there is always a Gray code enumeration of the partitions into distinct parts. Although there is a bijection between partitions into distinct parts and partitions into odd parts ($\delta = 2$), the associated

Gray code graphs are not isomorphic under the specified changes. Hence the Gray code for the first problem above does not provide a Gray code for the second.

In Section 5 we discuss other families of partitions where our basic strategy may yield Gray codes. Basic definitions and notation are presented in Section 2.

2 Definitions and Notation

For any integers n, k, δ with $\delta \geq 1$, let $P_\delta(n, k)$ denote the set of partitions of n into parts of size at most k in which parts are congruent to 1 modulo δ. For example, $P_2(n, n)$ is the set of partitions of n into odd parts, and $P_3(11, 8) = P_3(11, 7)$ is the set

$$\{7\ 4,\ \ 7\ 1\ 1\ 1\ 1,\ \ 4\ 4\ 1\ 1\ 1,\ \ 4\ 1^7,\ \ 1^{11}\},$$

where exponents on part-sizes indicate multiplicities, with x^j denoting j parts of size x.

Let $D(n, k)$ denote the set of partitions of n into distinct parts of size at most k. For example, $D(10, 5) = \{5\ 4\ 1,\ \ 5\ 3\ 2,\ \ 4\ 3\ 2\ 1\ \}$ and $D(11, 4) = \emptyset$.

Let $P_\delta(n, k) = D(n, k) = \emptyset$ if $n \leq 0$ or $k \leq 0$, except that if $n = 0$ and $k \geq 0$, then $P_\delta(n, k) = D(n, k) = \{\epsilon\}$, where ϵ is the empty partition.

We write a partition as $x_1 x_2 \ldots x_t$ or as $(x_1)(x_2)\ldots(x_t)$, using parentheses for clarity or to resolve ambiguity. When j is clear from context, we may denote 1^j or $(1)^j$ by $(1..1)$ to indicate that the "remaining" parts have size 1.

If π is the partition $x_1 x_2 \ldots x_t$ and $k \geq x_1$, then we may write the partition $k x_1 x_2 \ldots x_t$ as $(k)\pi$. If S is a set of partitions and k is at least as large as any part in any $\pi \in S$, then $(k)S$ denotes the set of partitions $\{(k)\pi : \pi \in S\}$. Similarly, if L is a list of partitions, list $(k)L$ is obtained by prepending part k to every element on L. We use $\overline{L}$ to denote the list L in reverse order. If L and M are lists, we use L, M to denote the concatenation of the lists, with L followed by M.

The lexicographic ordering on partitions is defined by $x_1 x_2 \ldots x_s < y_1 y_2 \ldots y_t$ if $x_i < y_i$ for the smallest index i where the partitions differ, taking $x_{s+1} = y_{t+1} = 0$ by convention. Then $\max(S)$ and $\min(S)$ denote the lexicographically maximum and minimum partitions, respectively, in a set S.

By repeated application of the two identities

$$P_\delta(n, k) = (k)P_\delta(n - k, k) \cup P_\delta(n, k - \delta)$$

$$D(n, k) = (k)D(n - k, k - 1) \cup D(n, k - 1),$$

$P_\delta(n,k)$ and $D(n,k)$ can be decomposed in various ways. We usually partition these sets into subsets according to the two largest parts. The subsets will be viewed as boxes, arranged to form the full set in an array in which rows correspond to the largest part and columns to the second largest part. See Figure 1 ("$\circ$" denotes a line break in the description of a set). Some boxes may be empty. Note that $(k)P_\delta(n-k,k-\delta)$ is the union of all boxes in row 1 of Figure 1(a) except the first, and $D(n,k-1)$ is the union of all boxes not in the first row of Figure 1(b). A bold dot in the upper left (lower right) corner of a box represents the lexicographically maximum (minimum) element in the set represented by the box.

Let $G_\delta(n,k)$ $[G(n,k)]$ be the *Gray code graph* of the adjacency relation we have specified for $P_\delta(n,k)$ $[D(n,k)]$. A *Gray code enumeration* (GCE) of $P_\delta(n,k)$ is a Hamiltonian path in $G_\delta(n,k)$. As an example, note that $|P_\delta(4\delta+2,3\delta+1)| = 9$. The Gray code graph $G_\delta(4\delta+2,3\delta+1)$ appears in Fig. 3a, with two GCE's listed in Fig. 3b. This particular graph turns out to be exceptional.

3 A Gray Code Enumeration of $P_\delta(n,k)$

Informally, our strategy for constructing a GCE of $P_\delta(n,k)$ involves recursively constructing GCEs for the boxes in the decomposition of $P_\delta(n,k)$ indicated in Figure 1. The GCEs of a box will be required to start and end at the maximum and minimum elements, respectively, of the box; call this a *max-min* GCE. Boxes will then be linked together via their maximum and minimum elements. Unfortunately, adjacencies between the maximum elements of adjacent boxes depend on the relative values of n and k. This dependence essentially drives the entire Gray code construction, which must be divided into cases by the relative values of n and k. We begin by establishing adjacencies between elements of boxes adjacent in the same row or column or along a diagonal; these are summarized in Figure 3.

Lemma 1 *For $\delta \geq 1$, $m \geq 0$, $t \geq 1$, and $t \equiv 1 \bmod \delta$, let $\alpha = \max(P_\delta(m,t))$ and $\beta = \max(P_\delta(m+\delta,t))$. Then α can obtained from β by deleting δ ones from β or by decreasing one part of β by δ.*

Proof. We use induction on (m,t) in the lexicographic ordering. If $m=0$, then $\alpha=\epsilon$ and $\beta=(1)^\delta$. If $t=1$, then $\alpha=(1)^m$ and $\beta=(1)^{m+\delta}$. Suppose $m>0$, $t>1$, and (m,t) satisfies the hypotheses of the lemma. If $m+\delta<t$, then $P_\delta(m,t)=P_\delta(m,t-\delta)$ and $P_\delta(m+\delta,t)=P_\delta(m+\delta,t-\delta)$. If $m\geq t$, then $\max(P_\delta(m,t))=\max((t)P_\delta(m-t,t))$ and $\max(P_\delta(m+\delta,t))=\max((t)P_\delta(m-t+\delta,t))$. In these cases, the result follows by induction.

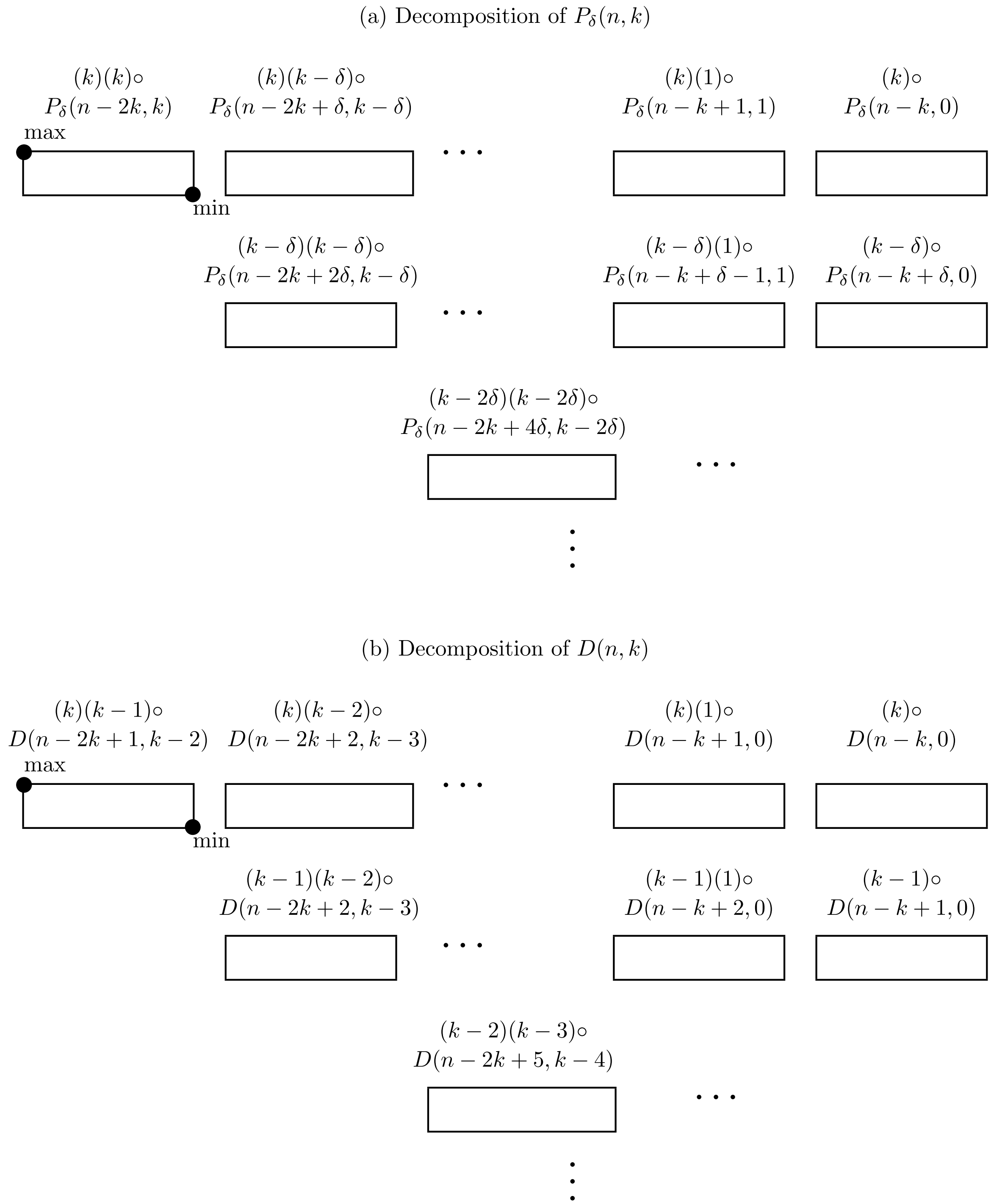
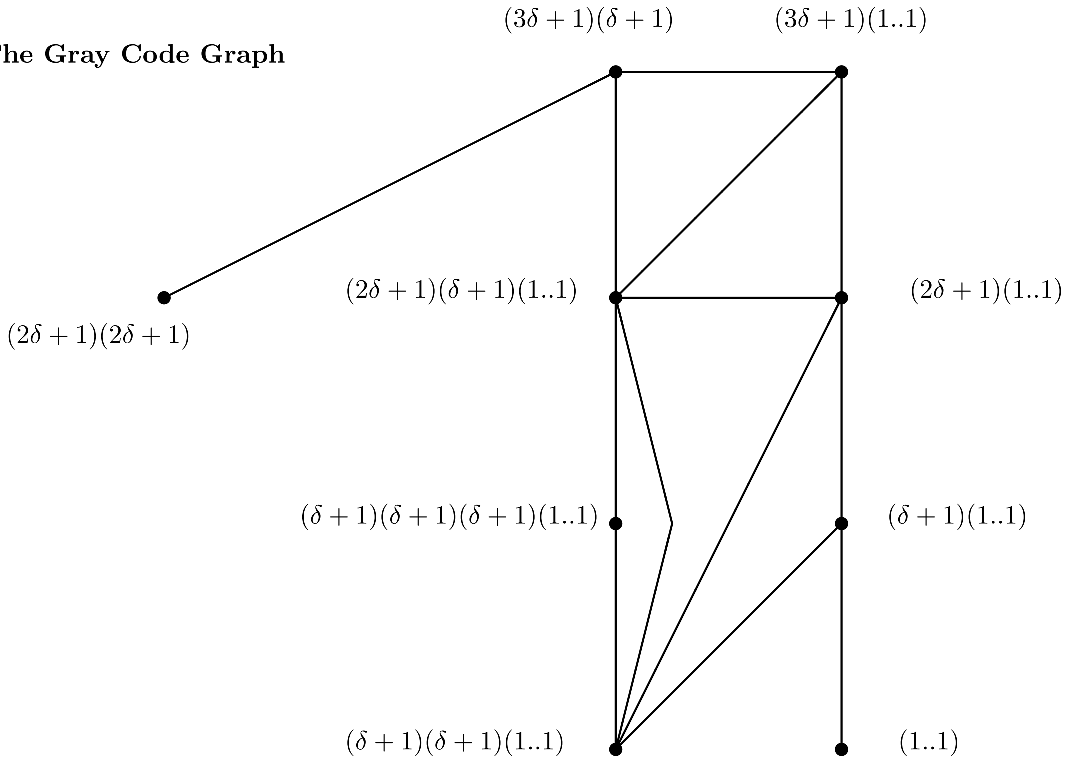


Figure 1: Decomposition of $P_\delta(n,k)$ and $D(n,k)$ according to the two largest parts. (Some boxes may be empty.)

(a) **The Gray Code Graph**



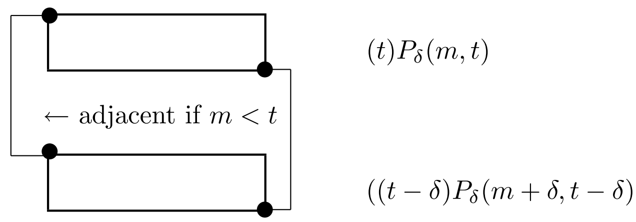
(b) **Two Hamilton Paths (GCEs)**

$A_\delta(4\delta+2, 3\delta+1) =$
$(2\delta+1)^2$,
$(3\delta+1)(\delta+1)$,
$(3\delta+1)(1..1)$,
$(2\delta+1)(1..1)$,
$(2\delta+1)(\delta+1)(1..1)$,
$(\delta+1)^3(1..1)$,
$(\delta+1)^2(1..1)$,
$(\delta+1)(1..1)$,
$(1..1)$

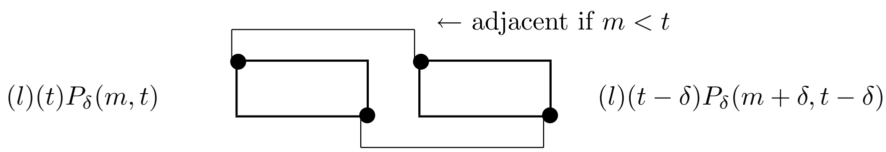
$B_\delta(4\delta+2, 3\delta+1) =$
$(3\delta+1)(1..1)$,
$(3\delta+1)(\delta+1)$,
$(2\delta+1)^2$,
$(2\delta+1)(\delta+1)(1..1)$,
$(\delta+1)^3(1..1)$,
$(\delta+1)^2(1..1)$,
$(2\delta+1)(1..1)$,
$(\delta+1)(1..1)$,
$(1..1)$

Figure 2: The special case $P_\delta(4\delta+2, 3\delta+1)$

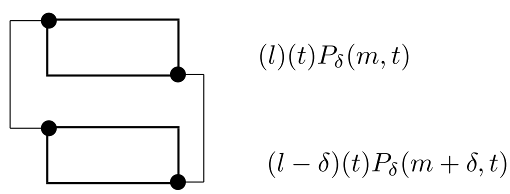
(a) Adjacent rows (Corollary 1, Lemma 3(i)):



(b) Adjacent boxes in same row (Corollary 1, Lemma 3(ii)):



(c) Adjacent boxes in same column (Corollary 2, Lemma 3(iii)):



(d) Diagonally adjacent boxes (Lemma 2, Lemma 3(iv)):

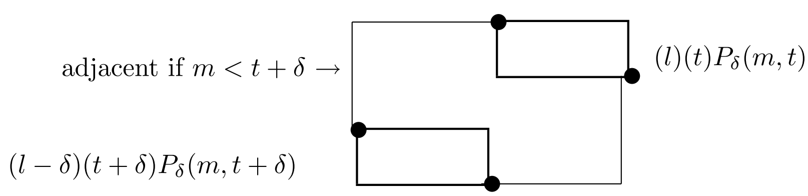


Figure 3: Adjacencies guaranteed by Corollaries 1 and 2 and Lemmas 2 and 3.

In the remaining case, $m < t \leq m + \delta$. Now the largest part in α is $t - \delta$ and the largest part in β is t, and the remainder of each partition is the lexicographically largest element in $P_\delta(m - t + \delta, t)$, so again by induction the claim holds. □.

Corollary 1 *Suppose $\delta \geq 1$, $m \geq 0$, $t \geq \delta + 1$, and $t \equiv 1 \bmod \delta$. Then $\max((t)P_\delta(m,t))$ and $\max((t-\delta)P_\delta(m+\delta, t-\delta))$ are adjacent if $m < t$ (Figure 3(a,b).)*

Proof. If $m < t$, then $P_\delta(m,t) = P_\delta(m, t-\delta)$, and Lemma 1 applies. □

Corollary 2 *Suppose $\delta \geq 1$, $m \geq 0$, $t \geq 1$, $l \geq t + \delta$, and $t, l \equiv 1 \bmod \delta$. Then $\alpha = \max((l)(t)(P_\delta(m,t))$ is adjacent to $\beta = \max((l-\delta)(t)P_\delta(m+\delta, t))$ (Figure 3(c).)*

Proof. This follows from Lemma 1, since the first part of α is δ more than that of β. □

Lemma 2 *Suppose $\delta \geq 1$, $m \geq 0$, $t \geq 1$, $l \geq t + 2\delta$, and $t, l \equiv 1 \bmod \delta$. Then $\alpha = \max((l)(t)(P_\delta(m,t))$ is adjacent to $\beta = \max((l-\delta)(t+\delta)P_\delta(m, t+\delta))$ if $m < t + \delta$ (Figure 3(d).)*

Proof. If $m < t + \delta$, then $P_\delta(m, t+\delta) = P_\delta(m,t)$. Hence α and β differ only in the prefixes $(l)(t)$ and $(l-\delta)(t+\delta)$, which are adjacent. □

Lemma 3 *Suppose $\delta \geq 1$, $m \geq 0$, $t \geq 1$, and $t, l \equiv 1 \bmod \delta$. Then the following pairs are adjacent (when they exist.)*

(i) $\min((t)(P_\delta(m,t))$ *and* $\min((t-\delta)(P_\delta(m+\delta, t-\delta))$ *(if $t \geq \delta + 1$.)*
(ii) $\min((l)(t)(P_\delta(m,t))$ *and* $\min((l)(t-\delta)(P_\delta(m+\delta, t-\delta))$ *(if $l \geq t \geq \delta + 1$.)*
(iii) $\min((l)(t)(P_\delta(m,t))$ *and* $\min((l-\delta)(t)(P_\delta(m+\delta, t))$ *(if $l - \delta \geq t$.)*
(iv) $\min((l)(t)(P_\delta(m,t))$ *and* $\min((l-\delta)(t+\delta)(P_\delta(m, t+\delta))$ *(if $l - \delta \geq t + \delta$.)*

Proof. (See Figure 3.) Each claim follows from the fact that $\min(P_\delta(n,k)) = (1)^n$ for all $k \geq 1$ and $n \geq 0$. □

In Theorem 1 below, we show how to construct a GCE, $L_\delta(n,k)$, of $P_\delta(n,k)$, which will be a max-min GCE except for one exceptional case. The construction will be doubly recursive, requiring definition of an auxiliary list $M_\delta(n,k)$. Define

$$S_\delta(n,k) = P_\delta(n, k-\delta) \cup (k)P_\delta(n-k, k-2\delta).$$

In comparing $S_\delta(n,k)$ and $P_\delta(n,k)$, observe in Figure 1(a) that $S_\delta(n,k)$ consists of everything in $P_\delta(n,k)$ except the first two boxes in the top row. In the lexicographic order on $P_\delta(n,k)$, everything in the first row of Figure 1(a) precedes everything in the second row. In order to construct a max-min GCE of $P_\delta(n,k)$ when $n \geq 2k-\delta$, we will use a GCE, $M_\delta(n,k)$ of $S_\delta(n,k)$ that starts at $\max(P_\delta(n,k-\delta))$ instead of $\max(S_\delta(n,k)))$ and ends at $\min(P_\delta(n,k-\delta)) = \min(S_\delta(n,k))$. We call such a GCE of $S_\delta(n,k)$ a *quasi-max-min* GCE, when it exists.

Theorem 1 *For $\delta \geq 1$, $k \equiv 1 \bmod \delta$, and all n, there is a GCE $L_\delta(n,k)$ of $P_\delta(n,k)$. Furthermore, if $P_\delta(n,k) \neq \emptyset$ and $(n,k) \neq (4\delta+2, 3\delta+1)$, then $L_\delta(n,k)$ is a max-min GCE. In addition, if $n \geq 2k-\delta$ and $k \geq 2\delta+1$, there is a quasi-max-min GCE, $M_\delta(n,k)$, of $S_\delta(n,k)$.*

Proof. First consider $P_\delta(4\delta+2, 3\delta+1)$, with adjacencies $G_\delta(4\delta+2,3\delta+1)$ shown in Figure 2. Although this graph has Hamiltonian paths, two of which are given in Figure 2, the vertices $(\delta+1)^3(1..1)$ and $(2\delta+1)(2\delta+1)$ of degree 2 prohibit a Hamiltonian path with origin $\max(P_\delta(4\delta+2,3\delta+1)) = (3\delta+1)(\delta+1)$.

The proof of the theorem is by induction on (n,k) is the lexicographic ordering on ordered pairs. If $n<0$ or $k<0$ or ($k=0$ and $n \geq 1$) then $L_\delta(n,k)$ is the empty list. If $n=0$ and $k \geq 0$, then $L_\delta(n,k)$ contains only the empty string. If $k=1$ and $n \geq 0$, then $L_\delta(n,k)$ contains only the string $(1)^n$.

Assume that $n \geq 1$, $k > 1$, $k \equiv 1 \bmod \delta$, and $(n,k) \neq (4\delta+2, 3\delta+1)$. If $n<k$, then $P_\delta(n,k) = P_\delta(n,k-\delta)$ and the result follows by induction. If $k = \delta+1$, then by induction, $P_\delta(n-\delta-1,k)$ has a max-min GCE $L_\delta(n-\delta-1,k)$, so $L_\delta(n,k) = (\delta+1)L_\delta(n-\delta-1,k), (1)^n$ is a max-min GCE for $P_\delta(n,k)$. Otherwise, $n \geq k \geq 2\delta+1$. We break the proof into three cases by the value of n. M_δ is not defined until $n \geq 2k-\delta$.

(Case L1: $n < 2k-2\delta$, $k \geq 2\delta+1$) Partition $P_\delta(n,k)$ as

$$P_\delta(n,k) = (k)P_\delta(n-k,k) \cup (k-\delta)P_\delta(n-k+\delta,k-\delta) \cup P_\delta(n,k-2\delta).$$

Since $n \geq k \geq 2\delta+1$, each of the sets is nonempty. If none is isomorphic to the exception $P_\delta(4\delta+2,3\delta+1)$, then the induction hypothesis guarantees a max-min GCE for each. By Lemma 3(i), the minima of the first two sets are adjacent. Since $\max(P_\delta(n,k-2\delta)) = \max((k-2\delta)P_\delta(n-k+2\delta,k-2\delta))$ and $n < 2k-2\delta$, the maxima of the second two sets are adjacent by Corollary 1. Therefore,

$$L_\delta(n,k) = (k)L_\delta(n-k,k),\quad \overline{(k-\delta)L_\delta(n-k+\delta,k-\delta)},\quad L_\delta(n,k-2\delta)$$

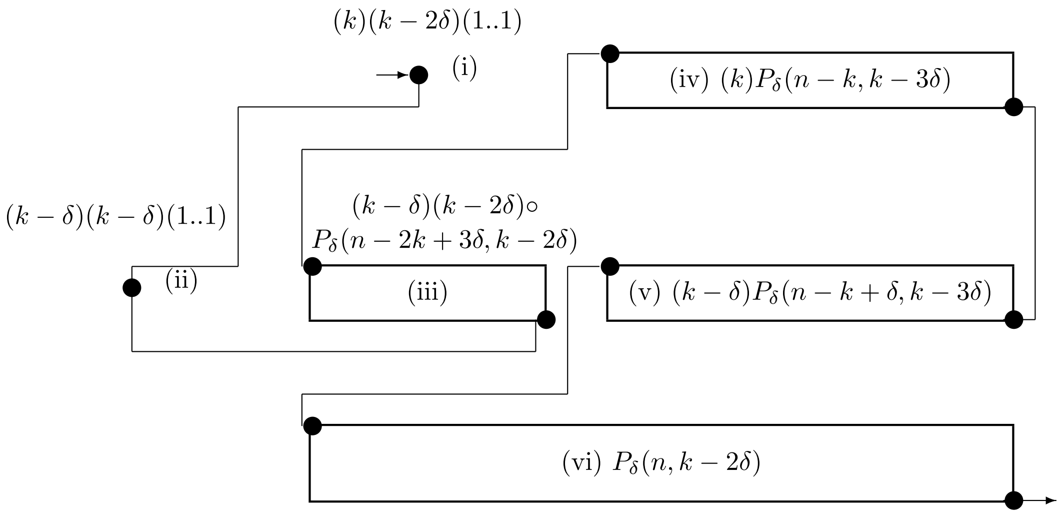


Figure 4: Case L2: max-min GCE of $P_\delta(n,k)$ when $2k-2\delta \le n < 2k-\delta$ and $n \ge 5k+1$.

is a max-min GCE of $P_\delta(n,k)$. To check whether any of the sets could be the exception, suppose

$$(4\delta+2, 3\delta+1) \in \{(n-k,k), (n-k+\delta, k-\delta), (n, k-2\delta)\}.$$

Since $n < 2k-2\delta$, this occurs only if $\delta = 1$ and $(n,k) = (6,6)$. In this case, $P_\delta(n, k-2\delta) = P_1(6,4)$, which has no max-min GCE. However, $B_1(6,4)$ from Figure 2 can be used in the role of $L_1(6,4)$.

(Case L2: $2k-2\delta \le n < 2k-\delta, \quad k \ge 2\delta+1$)

If $k = 2\delta+1$, then

$$L_\delta(n, 2\delta+1) = (2\delta+1)(1..1), (\delta+1)(\delta+1)(1..1), (\delta+1)(1..1), (1..1).$$

If $k = 3\delta+1$ then if $n = 2k-2\delta$, this is the special case in Figure 2. Otherwise,

$$\begin{aligned} L_\delta(n, 3\delta+1) \;=\; & (3\delta+1)(\delta+1)(1..1), (3\delta+1)(1..1), (2\delta+1)(1..1), (2\delta+1)(\delta+1)(1..1), \\ & (2\delta+1)(2\delta+1)(1..1), (2\delta+1)(\delta+1)^2(1..1), L_\delta(n, \delta+1) \end{aligned}$$

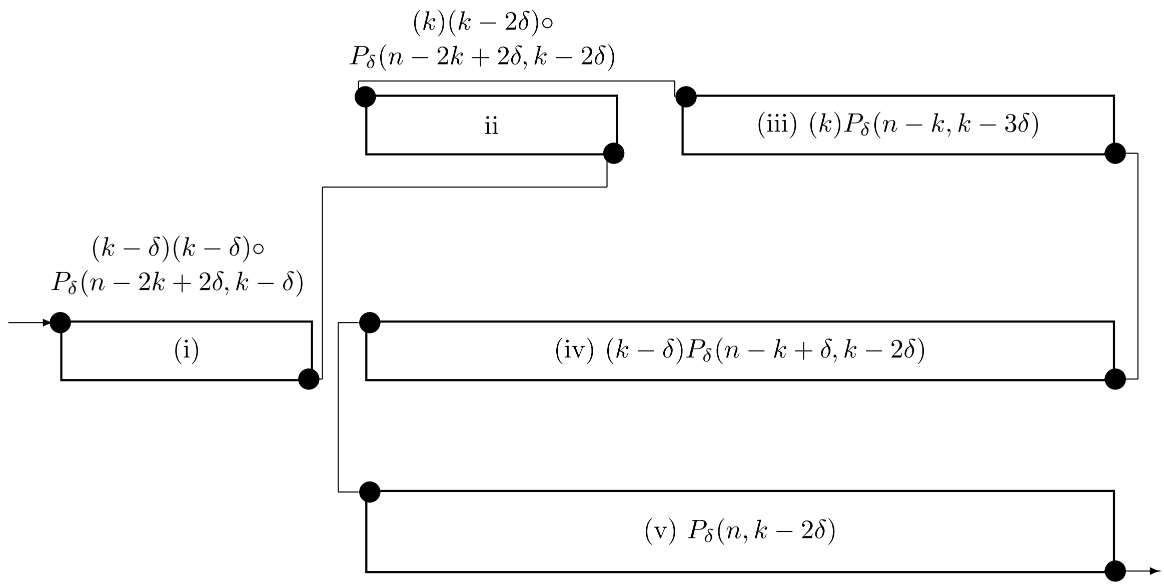


Figure 5: Case M1: quasi-max-min GCE $M_\delta(n,k)$ of $S_\delta(n,k)$, for $2k-\delta \le n < 3k-4\delta$.

is a max-min GCE. If $k = 4\delta+1$, then by induction, the following is a max-min GCE:

$$L_\delta(n,k) = \begin{array}{l} (4\delta+1)(2\delta+1)(1..1), \\ (3\delta+1)^2(1..1), \\ \hline (3\delta+1)(2\delta+1)L_\delta(n-5\delta-2, 2\delta+1), \\ (2\delta+1)^2 L_\delta(n-4\delta-2, 2\delta+1), \\ \hline (3\delta+1)(\delta+1)L_\delta(n-4\delta-2, \delta+1), \\ (4\delta+1)L_\delta(n-4\delta-1, \delta+1), \\ (3\delta+1)(1..1), \\ \hline (2\delta+1)L_\delta(n-2\delta-1, \delta+1), \\ L_\delta(n, \delta+1). \end{array}$$

Otherwise, $k \ge 5\delta+1$, and we partition $P_\delta(n,k)$ as shown in Figure 4. In this range of values for n and k, none of the boxes (i) through (vi) is empty. Using the induction hypothesis for L, we claim we can link the boxes together as indicated in Figure 4 to obtain the following max-min GCE of $P_\delta(n,k)$.

$$L_\delta(n,k) = \begin{array}{ll} (k)(k-2\delta)\ (1..1), & (i) \\ (k-\delta)(k-\delta)\ (1..1), & (ii) \\ \hline (k-\delta)(k-2\delta)\ L_\delta(n-2k+3\delta, k-2\delta), & (iii) \\ (k)\ L_\delta(n-k, k-3\delta), & (iv) \\ \hline (k-\delta)\ L_\delta(n-k+\delta, k-3\delta), & (v) \\ L_\delta(n, k-2\delta) & (vi) \end{array}$$

We must show that elements linked between boxes are adjacent and consider the possibility that some of the boxes may correspond to the exception $(4\delta+2, 3\delta+1)$ for particular values of (n,k). The minimum elements of sets (i) and (ii), sets (ii) and (iii), and sets (iv) and (v) are adjacent by Lemma 3. Since $n \geq 2k-2\delta \geq 2k-3\delta$, the maximum element of set (iv) is equal to $\max((k)(k-3\delta)P_\delta(n-2k+3\delta, k-3\delta))$, which, by Lemma 2 is adjacent to the maximum of set (iii) if $n-2k+3\delta < k-2\delta$. This holds in this case, since $0 \leq k-5\delta-1$ and $n < 2k-\delta$. Finally, let α be the maximum of box (v) and β be the maximum of box (vi). Since $n \geq 2k-2\delta \geq 2k-4\delta$, we have $\alpha = \max((k-\delta)(k-3\delta)P_\delta(n-2k+4\delta, k-3\delta))$ and $\beta = \max((k-2\delta)(k-2\delta)P_\delta(n-2k+4\delta, k-2\delta))$. Now, by Lemma 2, α and β are adjacent if $n-2k+4\delta < k-2\delta$, which is the sum of the given inequalities $n < 2k-\delta$ and $k > 5\delta$.

It remains to consider isomorphism between the boxes of Figure 4 and the exception $P_\delta(4\delta+2, 3\delta+1)$. This occurs if and only if

$$(4\delta+2, 3\delta+1) \in \{(n-2k+3\delta, k-2\delta), (n-k, k-3\delta), (n-k+\delta, k-3\delta)\, (n, k-2\delta)\}.$$

When $2k-2\delta \leq n < 2k-\delta$, this happens only for $(n,k) = (10\delta+3, 6\delta+1)$, in which case box (iv) is $(6\delta+1)P_\delta(4\delta+2, 3\delta+1)$. In this case, $B_\delta(4\delta+1, 3\delta+1)$ from Figure 2 can be used in the role of $L_\delta(4\delta+1, 3\delta+1)$ to give a max-min GCE.

(Case L3: $n \geq 2k-\delta, \quad k \geq 2\delta+1$) $M_\delta(n,k)$ as well as $L_\delta(n,k)$ is defined in this range.

(Subcase M1: $2k-\delta \leq n < 3k-4\delta, \quad k \geq 2\delta+1$)

In this case, we may assume $k \geq 3\delta+1$, since if $k = 2\delta+1$ the inequalities for n are impossible to satisfy. We decompose $S_\delta(n,k)$ as in Figure 5. Since $k \geq 3\delta+1$, no box is empty. By induction, if no box corresponds to the exception, each has a max-min GCE. We link these together as shown in Figure 5 to get the following quasi-max-min GCE of $S_\delta(n,k)$.

$$\begin{array}{rlr}
M_\delta(n,k) = & \underline{(k-\delta)(k-\delta)L_\delta(n-2k+2\delta, k-\delta),} & (i) \\
& \underline{(k)(k-2\delta)L_\delta(n-2k+2\delta, k-2\delta),} & (ii) \\
& \underline{(k)\ L_\delta(n-k, k-3\delta),} & (iii) \\
& \underline{(k-\delta)\ L_\delta(n-k+\delta, k-2\delta),} & (iv) \\
& L_\delta(n, k-2\delta) & (v)
\end{array}$$

The minimum elements of boxes (i) and (ii) are adjacent by Lemma 3, as are the minimum elements of boxes (iii) and (iv). By Corollary 1, the maximum elements of box (ii) and box (iii) are adjacent if $(n-2k+2\delta) < (k-2\delta)$, which is exactly the range of Subcase M1.

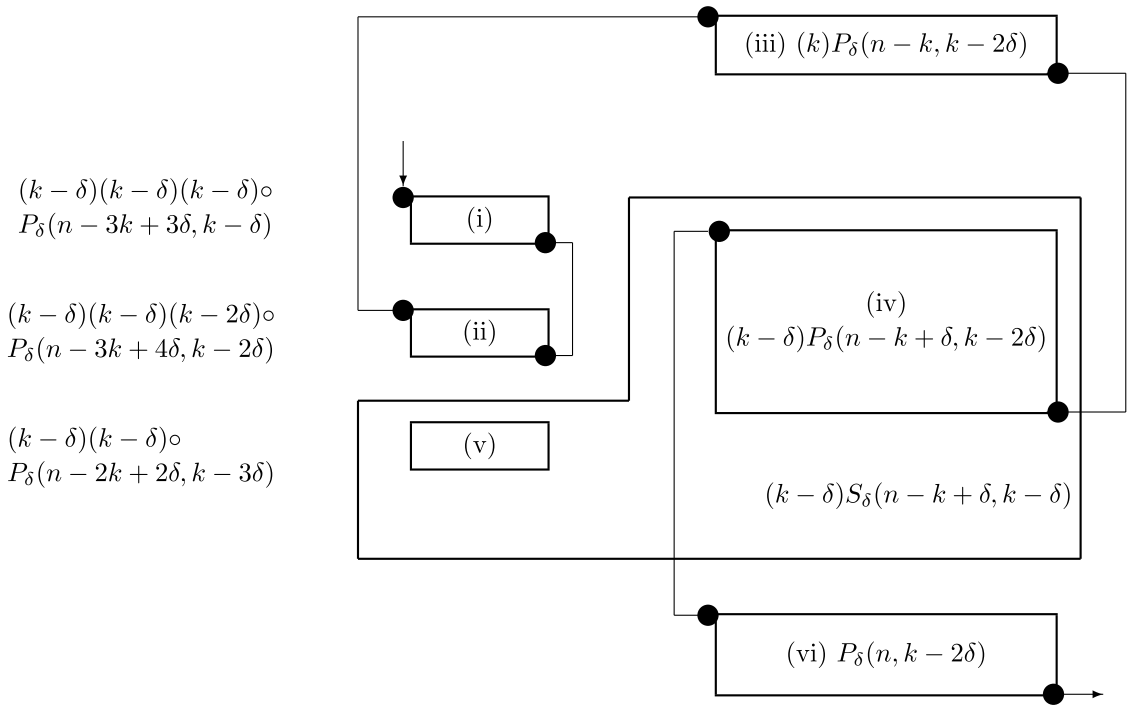


Figure 6: Case M2: quasi-max-min GCE $M_\delta(n,k)$ of $S_\delta(n,k)$, for $n \geq 3k-4\delta$.

The maximum elements of boxes (iv) and (v) are $\max((k-\delta)P_\delta(n-k+\delta,k-2\delta))$ and $\max((k-2\delta)P_\delta(n-k+2\delta,k-2\delta))$, which are adjacent by Lemma 1.

To complete Subcase M1, we must check whether any box is isomorphic to the exception $P_\delta(4\delta+2,3\delta+1)$. This occurs if and only if $(4\delta+2,3\delta+1)$ is in the set

$$\{(n-2k+2\delta,k-\delta),(n-2k+2\delta,k-2\delta),(n-k,k-3\delta),(n-k+\delta,k-2\delta),(n,k-2\delta)\}.$$

For $2k-\delta \leq n < 3k-4\delta$, this happens only if $(n,k,\delta)=(13,7,1)$ or $(n,k,\delta)=(11,6,1)$. In the first case, box (iii) becomes the exception; the GCE $B_1(6,4)$ of Figure 2 can be used in place of $L_1(6,4)$. In the second case, box (iv) becomes the exception, but $A_1(6,4)$ can be used in place of $L_1(6,4)$.

(Subcase M2: $n \geq 3k - 4\delta, \quad k \geq 2\delta + 1$)

If $k = 2\delta + 1$, the following is a quasi-max-min GCE of $S_\delta(n, 2\delta + 1)$:

$$L_\delta(n, 2\delta+1) = (\delta+1)^2 L_\delta(n - 2\delta - 2, \delta + 1), \quad (2\delta+1)(1\ldots1), \quad (\delta+1)(1\ldots1), \quad (1\ldots1).$$

Otherwise, $k \geq 3k + 1$, so decompose $S_\delta(n, k)$ as in Figure 6. Note that the union of boxes (iv) and (v) is $(k - \delta)S_\delta(n - k + \delta, k - \delta)$, where in the usual positioning of Figure 1 for $P_\delta(n - k + \delta, k - \delta)$, box (v) would appear above box (iv) and flush right with it. Since $k - \delta \geq 2\delta + 1$ and $n - k + \delta \geq 2(k - \delta) - \delta$, by induction $S_\delta(n - k + \delta, k - \delta)$ has a quasi-max-min GCE $M_\delta(n - k + \delta, k - \delta)$ and, if none is the exception, each nonempty box (i), (ii), (iii), (vi) has a max-min GCE. The only box which could be empty is box (i), when $n < 3k - 3\delta$. In this case, box (ii) is a singleton which is the maximum element. We show that linking these boxes as in Figure 6 gives the quasi-max-min GCE of $S_\delta(n, k)$ described below.

$$\begin{array}{rll} M_\delta(n,k) = & \underline{(k-\delta)(k-\delta)(k-\delta)L_\delta(n-3k+3\delta, k-\delta),} & (i) \\ & \underline{(k-\delta)(k-\delta)(k-2\delta)L_\delta(n-3k+4\delta, k-2\delta),} & (ii) \\ & \underline{(k)L_\delta(n-k, k-2\delta),} & (iii) \\ & (k-\delta)M_\delta(n-k+\delta, k-\delta), & (iv, v) \\ & L_\delta(n, k-2\delta) & (vi) \end{array}$$

By Lemma 3, the minima of (i) and (ii) and of (iii) and (iv) are adjacent. The maxima of boxes (ii) and (iii) are $\max((k - \delta)(k - \delta)P_\delta(n - 2k + 2\delta, k - 2\delta))$ and $\max((k)(k - 2\delta)P_\delta(n - 2k + 2\delta, k - 2\delta))$, which are adjacent since they differ only in their adjacent prefixes. The maxima of boxes (iv) and (vi) are $\max((k - \delta)P_\delta(n - k + \delta, k - 2\delta))$ and $\max((k - 2\delta)P_\delta(n - k + 2\delta, k - 2\delta))$, which are adjacent by Lemma 1.

To complete subcase M2, we need only consider when boxes (i), (ii), (iii), or (vi) could be the exception. This occurs if and only if

$$(4\delta + 2, 3\delta + 1) \in \{(n - 3k + 3\delta, k - \delta), (n - 3k + 4\delta, k - 2\delta), (n - k, k - 2\delta), (n, k - 2\delta)\}.$$

For $n \geq 3k - 4\delta$, this happens only when $(n, k) = (13\delta + 5, 4\delta + 1)$, in which case box (i) is $(3\delta + 1)^3 P_\delta(4\delta + 2, 3\delta + 1)$, or when $(n, k) = (15\delta + 5, 5\delta + 1)$, in which case box (ii) is $(4\delta + 1)^2(3\delta + 1)P_\delta(4\delta + 2, 3\delta + 1)$. For each of these cases of (n, k), we construct a special

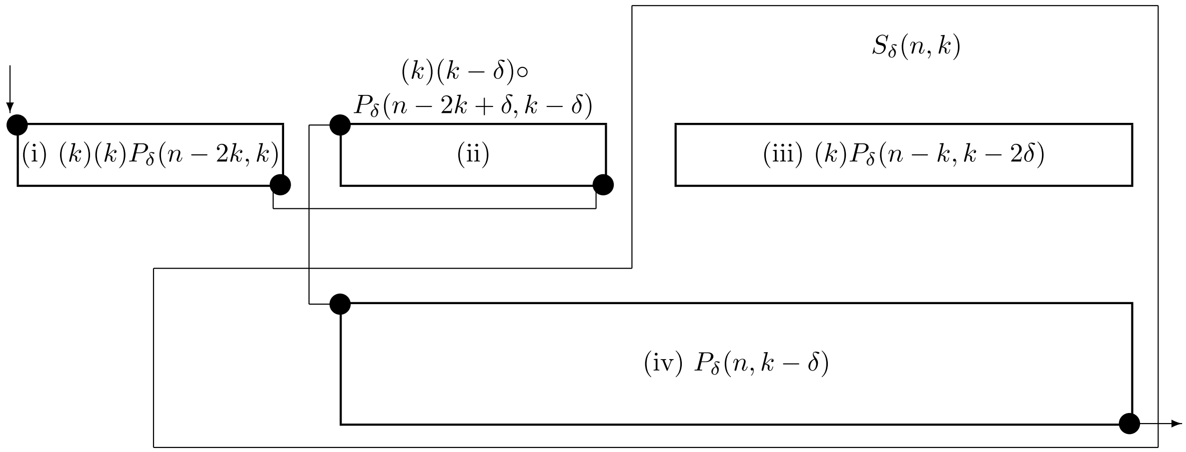


Figure 7: Case L3: max-min GCE of $P_\delta(n,k)$ when $n \geq 2k-\delta$.

quasi-max-min GCE of $S_\delta(n,k)$ as follows:

$M_\delta(13\delta+5, 4\delta+1) =$
$(3\delta+1)^4\ (\delta+1),$
$(3\delta+1)^4\ (1..1),$
$(3\delta+1)^3\ (2\delta+1)(1..1),$
$(3\delta+1)^2(2\delta+1)^2(1..1),$
$(3\delta+1)^2\ (2\delta+1)^2(\delta+1)(1..1),$
$(3\delta+1)^2\ (2\delta+1)^2(\delta+1)^2(1..1),$
$(3\delta+1)^2(2\delta+1)\ L_\delta(5\delta+2, \delta+1),$
$(3\delta+1)^3\ L_\delta(4\delta+2, \delta+1),$
$(3\delta+1)^3\ (2\delta+1)\ (\delta+1)(1..1),$
$(3\delta+1)^3\ (2\delta+1)^2,$
$(3\delta+1)^2\ (2\delta+1)^3(1..1),$
$(4\delta+1)\ L_\delta(9\delta+4, 2\delta+1),$
$(3\delta+1)\ M_\delta(10\delta+4, 3\delta+1),$
$L_\delta(13\delta+5, 2\delta+1)$

$M_\delta(15\delta+5, 5\delta+1) =$
$[(4\delta+1)^4]$ {occurs iff $\delta = 1$,}
$(4\delta+1)^3\ (3\delta+1)(1..1),$
$(4\delta+1)^2\ (3\delta+1)^2(1..1),$
$(4\delta+1)^3\ (2\delta+1)(1..1),$
$(4\delta+1)^3\ (2\delta+1)(\delta+1),$
$(4\delta+1)^3 L_\delta(3\delta+2, \delta+1),$
$(4\delta+1)^2(3\delta+1)L_\delta(4\delta+2, 2\delta+1),$
$(4\delta+1)^2\ (3\delta+1)^2(\delta+1),$
$(5\delta+1)\ L_\delta(10\delta+4, 3\delta+1),$
$(4\delta+1)\ M_\delta(11\delta+4, 4\delta+1),$
$L_\delta(15\delta+5, 3\delta+1)$

Completion of Case L3

We now construct $L_\delta(n,k)$ for $n \geq 2k-\delta$. Decompose $P_\delta(n,k)$ as in Figure 7. Note that the union of boxes (iii) and (iv) is $S_\delta(n,k)$, which has just been shown to have a quasi-max-min GCE $M_\delta(n,k)$. By induction, if neither box(i) nor (ii) is the exception, each has a min-max GCE. By Lemma 3, the minima of boxes (i) and (ii) are adjacent. The maximum elements of boxes (ii) and (iv) are adjacent by Corollary 2. Hence we have the following max-min GCE of $P_\delta(n,k)$:

$$L_\delta(n,k) = \begin{array}{ll} (k)(k)L_\delta(n-2k,k), & (i) \\ \hline (k)(k-\delta)L_\delta(n-2k+\delta,k-\delta), & (ii) \\ M_\delta(n,k) & (iii,iv) \end{array}$$

Since $n \geq 2k-\delta$, box (i) can be the exception only if $(n,k) = (10\delta+4, 3\delta+1)$, in which case we use $A_\delta(4\delta+1, 3\delta+1)$ from Figure 2. Box (ii) can be the exception only if $(n,k) = (11\delta+4, 4\delta+1)$. For this case, we construct a special max-min GCE:

$$L_\delta(10\delta+3, 6\delta+1) = \begin{array}{l} (3\delta+1)^3(\delta+1), \quad (3\delta+1)^2(2\delta+1)^2, \\ (3\delta+1)(2\delta+1)^3(1..1), \quad (3\delta+1)(2\delta+1)^2(\delta+1)(1..1), \\ (3\delta+1)(2\delta+1)^2(1..1), \quad (3\delta+1)^2(2\delta+1)(1..1), \\ (3\delta+1)^3(1..1), \quad (3\delta+1)^2(2\delta+1)(\delta+1)(1..1), \\ (3\delta+1)^2 L_\delta(4\delta+2, \delta+1), \\ \hline (3\delta+1)(2\delta+1)L_\delta(5\delta+2, \delta+1), \\ (3\delta+1)(2\delta+1)^2(\delta+1)^2(1..1), \\ M_\delta(10\delta+4, 3+1) \end{array}$$

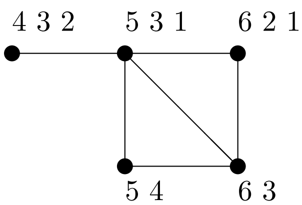
□

4 A Gray Code Enumeration of $D(n,k)$

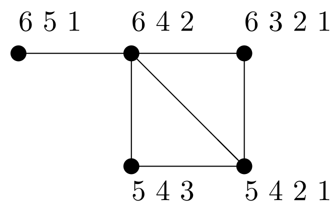
In this section we construct a GCE for $D(n,k)$, the set of partitions of n into distinct parts of size at most k. The adjacency relation is that of the previous section with $\delta = 1$. Again, we seek a Hamiltonian path in the Gray code graph $G(n,k)$ of this relation. Figure 8 shows the Gray code graphs for $D(9,6)$ and $D(12,6)$, along with two GCE's for each. Like $P_\delta(4\delta+2, 3\delta+1)$, these will turn out to be exceptional cases.

The strategy for constructing a GCE of $D(n,k)$ is like that for $P_\delta(n,k)$ (inductively construct a max-min GCE), but additional complications arise. Most seriously, boxes which contained the key elements for linking in the previous section may now be empty because they represent a set $D(n,k)$ for which n is larger than the sum of the elements 1 through k. To handle this, we must devise alternative linking strategies with additional dependencies on the values of n and k. Also, there are now two anomalous cases: $D(9,6)$, in which no GCE starts at the maximum element, and $D(12,6)$, in which no GCE ends at the minimum element. As in $P_\delta(n,k)$, this is handled by alternative constructions for the (finite number of) cases which would otherwise depend recursively on $D(9,6)$ and $D(12,6)$. Finally, unless $n=1$, $\min(D(n,k)) \neq 1^n$, which means that the adjacencies between minimum elements of adjacent boxes, although they still exist, require more work to verify.

(a) **The Gray Code Graphs**



$D(9,6)$



$D(12,6)$

(b) **Hamiltonian Paths (GCEs)**

$A(9,6) =$	$B(9,6) =$	$A(12,6) =$	$B(12,6) =$
6 2 1	5 4	6 5 1	6 5 1
6 3	6 3	6 4 2	6 4 2
5 4	6 2 1	6 3 2 1	5 4 3
5 3 1	5 3 1	5 4 2 1	5 4 2 1
4 3 2	4 3 2	5 4 3	6 3 2 1

Figure 8: The exceptional cases $D(9,6)$ and $D(12,6)$.

We begin by establishing conditions under which adjacencies exist between maximum and minimum elements of adjacent and diagonal boxes. These are summarized in Figure 9. To simplify expressions, we use S_t to represent $\sum_{i=1}^{t} i$.

Lemma 4 *For $m \geq 0$, $t \geq 1$, and $m+1 \leq S_t$, let $\alpha = \max(D(m,t))$ and $\beta = \max(D(m+1,t))$. Then α can obtained from β by deleting a 1 from β or by decreasing one part of β by 1.*

Proof. We use induction on (m,t). If $m = 0$, then $\alpha = \epsilon$ and $\beta = (1)$, so the claim holds. If $t = 1$, then $m = 0$. Now suppose $m > 0$ and $t > 1$ for the inductive step; the conditions on m and t imply that both α and β exist.

If $m+1 < t$, then $D(m,t) = D(m,t-1)$ and $D(m+1,t) = D(m+1,t-1)$. If $m \geq t$, then $\alpha = \max((t)D(m-t,t-1))$ and $\beta = (t)\max(D(m-t+1,t-1))$. In both cases, the claim follows by induction. Otherwise, $t = m+1$, in which case $\alpha = (t-1)$ and $\beta = (t)$. □.

Corollary 3 *Suppose $t \geq 3$ and $1 \leq m+1 \leq S_{t-2}$. Then $\max((t)D(m,t-1))$ and $\max((t-1)D(m+1,t-2))$ are adjacent if $m < t-1$ (Figure 9(a,b)).*

Proof. If $m < t-1$, then $D(m,t-1) = D(m,t-2)$ and Lemma 4 applies. □

Corollary 4 *If $l > t+1 > 2$ and $1 \leq m+1 \leq S_{t-1}$, then $\max((l)(t)(D(m,t-1))$ is adjacent to $\max((l-1)(t)D(m+1,t-1))$ (Figure 9(c).)*

Proof. This follows from Lemma 4. □

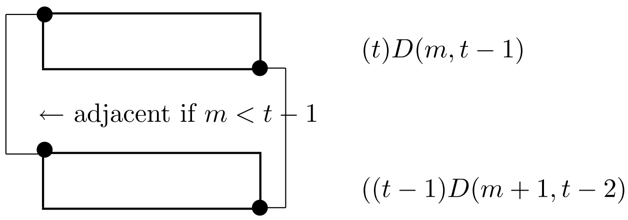
Lemma 5 *Suppose $l > t+2 \geq 3$ and $0 \leq m \leq S_{t-1}$. Then $\alpha = \max((l)(t)(D(m,t-1))$ is adjacent to $\beta = \max((l-1)(t+1)D(m,t)$, if $m < t$ (Figure 9(d).)*

Proof. If $m < t$, then $D(m,t) = D(m,t-1)$, so α and β differ only in the prefixes $(l)(t)$ and $(l-1)(t+1)$, which are adjacent. □

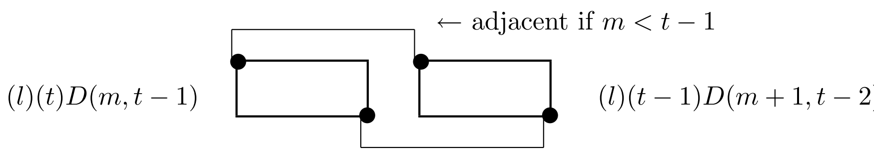
Lemma 6 *(a) If $t \geq 0$ and $0 \leq m \leq S_t$ then $\min(D(m,t) = \min(D(m,t+1))$.*
(b) If $t \geq 1$ and $1 \leq m+1 \leq S_t$ then $\alpha = \min(D(m,t))$ can be obtained from $\beta = \min(D(m+1,t))$ by deleting a 1 from β or by decreasing one part of β by 1.

Proof. Result (a) holds because $D(m,t) \neq \emptyset$, so the elements of $D(m,t+1) \setminus D(m,t)$ cannot be minimal in $D(m,t)$. We prove (b) by induction on (m,t). If $m = 0$, then

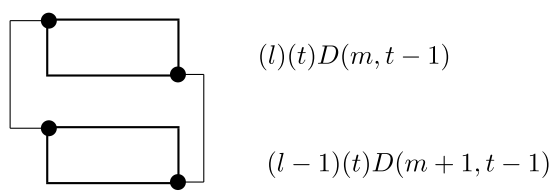
(a) Adjacent rows (Corollary 3, 5(i)):



(b) Adjacent boxes in same row (Corollary 3, 5(ii)):



(c) Adjacent boxes in same column (Corollary 4,5(iii)):



(d) Diagonally adjacent boxes (Lemma 5, Corollary 5(iv)):

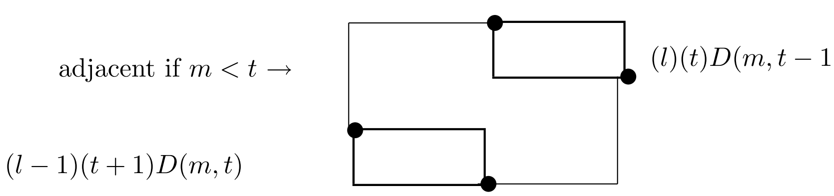


Figure 9: Adjacencies guaranteed by Corollaries 3-5 and Lemmas 4-6.

$\alpha = \epsilon$ and $\beta = (1)$, so the lemma is true. If $t = 1$, then $m = 0$. For the inductive step, suppose $m > 0$ and $t > 1$. If $m + 1 \leq S_{t-1}$, then $\alpha = \min(D(m, t-1))$ and $\beta = \min(D(m+1, t-1))$. If $m+1 > 1+S_{t-1}$, then $m > S_{t-1}$ implies $\alpha = \min((t)D(m-t, t-1))$ and $\beta = \min((t)D(m-t+1, t-1))$. In this cases, the claim follows by induction. In the remaining case, $m + 1 = S_{t-1} + 1$, we have $\alpha = \min((t-1)D(m-t+1, t-2))$ and $\beta = \min((t)(D(m-t+1, t-1))$. By part (a), α and β differ only in their first part. $\square$

Corollary 5 *The following pairs of elements are adjacent (when they exist).*

(i) $\min((t)D(m,t-1))$ *and* $\min((t-1)D(m+1,t-2))$*, when* $t \geq 3$ *and* $1 \leq m+1 \leq S_{t-2}$.

(ii) $\min((l)(t)D(m, t-1))$ *and* $\min((l)(t-1)D(m+1, t-2))$ *when* $l \geq t \geq 3$ *and* $1 \leq m+1 \leq S_{t-2}$.

(iii) $\min((l)(t)D(m, t-1))$ *and* $\min((l-1)(t)D(m+1, t-1))$ *when* $l > t+1 \geq 3$ *and* $1 \leq m+1 \leq S_{t-1}$.

(iv) $\min((l)(t)D(m, t-1))$ *and* $\min((l-1)(t+1)D(m, t))$ *when* $l > t+1 \geq 3$ *and* $1 \leq m+1 \leq S_{t-1}$.

Proof. (See Figure 9.) Part (iii) follows directly from Lemma 6(b) and part (iv) from Lemma 6(a). For parts (i) and (ii), since $m + 1 \leq S_{t-2}$, $\min(D(m, t-1)) = \min(D(m, t-2))$ and the result follows from Lemma 6(b). $\square$

In Theorem 2 below, we show how to construct a GCE $LD(n, k)$ of $D(n, k)$. As in the previous section, the construction will be doubly recursive, requiring definition of an auxiliary list $MD(n, k)$. Define

$$SD(n, k) = D(n, k-1) \cup (k)D(n-k, k-3).$$

In Figure 1(b), $SD(n, k)$ consists of everything in $D(n, k)$ excepts the first two boxes in the top row. In order to construct a GCE of $P_\delta(n, k)$, we will make use of a GCE, $MD(n, k)$ of $SD(n, k)$ which starts at $\max(D(n, k-1))$ (instead of $\max(SD(n, k))$) and ends at $\min(D(n, k-1)) = \min(SD(n, k))$. We call such a GCE of $SD(n, k)$ a *quasi-max-min* Gray code, when it exists.

Theorem 2 *For all integers* n, k*,* $D(n, k)$ *has a GCE,* $LD(n, k)$*. Furthermore, if* $D(n, k) \neq \emptyset$ *and* $(n, k) \notin \{(9, 6),\ (12, 6)\}$*, then* $LD(n, k)$ *is a max-min GCE. In addition, if* $k \geq 6$ *and* $2k-2 \leq n \leq 1 + S_{k-2}$*, a quasi-max-min Gray code,* $MD(n, k)$*, for* $SD(n, k)$ *always exists.*

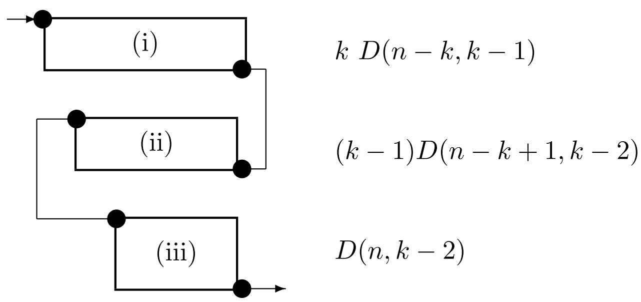


Figure 10: Case LD1: max-min GCE of $D(n, k)$ when $n < 2k-3$.

Proof. The graph $G(9, 6)$ has no Hamiltonian path starting at $\max(D(9, 6)) = 6\ 3$, and the graph $G(12, 6)$ has no Hamiltonian path ending at $\min(D(12, 6)) = 5\ 4\ 2\ 1$. Alternative Hamiltonian paths are given in Figure 8(b).

The proof is by induction on (n, k). If $n < 0$ or $k < 0$ or $n > S_k$, then $LD(n, k)$ is the empty list. If $n = 0$ and $k \geq 0$, then $LD(n, k)$ contains only the empty string (ϵ). If $n = 1$ and $k \geq 1$, then $LD(n, k) = (1)$. Otherwise, for $k = 2$, $LD(2, 2) = 2$ and $LD(3, 2) = 2\ 1$. $MD(n, k)$ is undefined when $n < 2$ or $k < 3$.

Assume indent that $n \geq 2$, $k \geq 3$, and $n \leq S_k$. If $n < k$, let $LD(n, k) = LD(n, k-1)$; $MD(n, k)$ is undefined. Otherwise, $n \geq k \geq 3$.

(<u>Case LD1: $3 \leq k \leq n < 2k-3$</u>)

If also $n > S_{k-2}$, then (n, k) must be $(4, 4)$; in this case, let $D(n, k) = 4,\ 3\ 1$. Otherwise, $n \leq S_{k-2}$; in this case $D(n, k)$ can be partitioned as in Figure 10 with each box nonempty. By Corollary 5(i), the minimum elements of the first two boxes are adjacent. Since $n-k+1 < k-2$ and the maximum element of box (iii) is $\max((k-2)D(n-k+2, k-3))$, the maximum elements of boxes (ii) and (iii) are adjacent by Corollary 3. Using the induction hypotheses, we obtain the following max-min GCE of $D(n, k)$ if none of the boxes is isomorphic to $D(9, 6)$ or $D(12, 6)$.

$$
\begin{aligned}
LD(n, k) = \quad & (k)LD(n-k, k-1), && (i) \\
& \overline{(k-1)LD(n-k+1, k-2)}, && (ii) \\
& LD(n, k-2). && (iii)
\end{aligned}
$$

The exceptions arise only if $(n, k) = (9, 8)$, in which case the third box is isomorphic to $D(9, 6)$, or if $(n, k) = (12, 8)$, in which case the third box is isomorphic to $D(12, 6)$. For

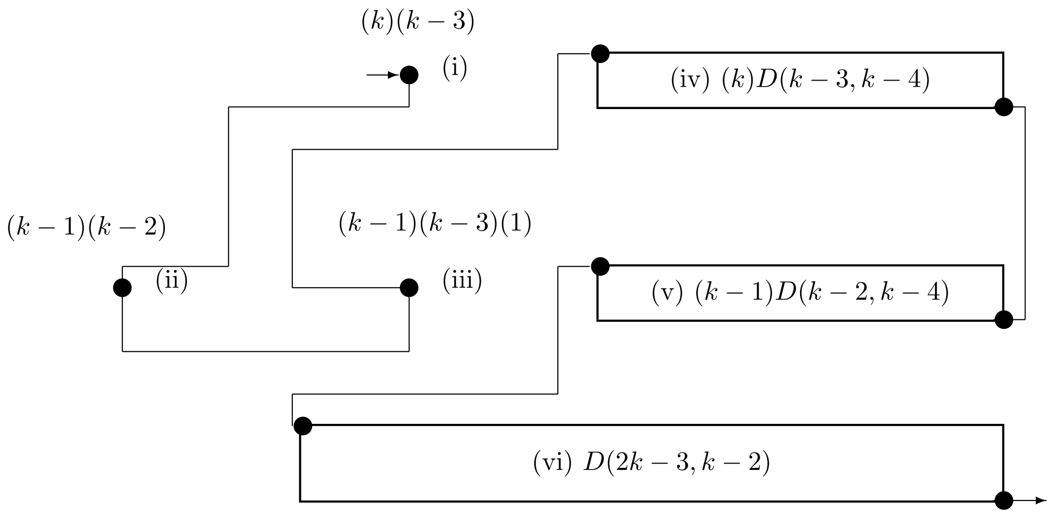


Figure 11: Case LD2: max-min GCE of $D(n,k)$ when $n = 2k-3$ and $k > 6$.

$D(9,8)$, use $A(9,6)$ from Figure 8 in place of $LD(n,k-2)$. For $D(12,8)$, a max-min GCE is:

$$\begin{aligned} LD(12,8) &= (8)(4),\ (7)(5),\ (6)(5)(1),\ (7)(4)(1),\ (8)(3)(1), (7)(3)(2), \\ &\quad (6)(3)(2)(1),\ (6)(4)(2),\ (5)(4)(3),\ (5)(4)(2)(1). \end{aligned}$$

(Case LD2: $n = 2k-3$, $k \geq 3$)

For $3 \leq k \leq 5$, $LD(2k-3,k)$ is given explicitly as

$$LD(3,3) = 3,\ \ 2\ 1; \quad LD(5,4) = 4\ 1,\ \ 3\ 2; \quad LD(7,5) = 5\ 2,\ \ 4\ 3,\ \ 4\ 2\ 1.$$

For $k = 6$, (n,k) is the exceptional case $(9,6)$ in Figure 8. If $k > 6$, $D(n,k)$ can be partitioned as in Figure 11 so that none of the boxes (i) - (iv) is empty. The single elements of boxes (i), (ii), (iii) form a path in order. The maximum element of box (iv) is $(k)(k-4)(2)$, which is adjacent to the element of box (iii). The minimum elements of boxes (iv) and (v) are adjacent by Corollary 5(i). The maximum elements of boxes (v) and (vi) are adjacent by Lemma 5. Finally, none of (iv), (v), (vi) is a copy of $D(9,6)$ or $D(12,6)$. Therefore, the induction hypothesis guarantees the following max-min GCE of $D(2k-3,k)$ when $k > 6$.

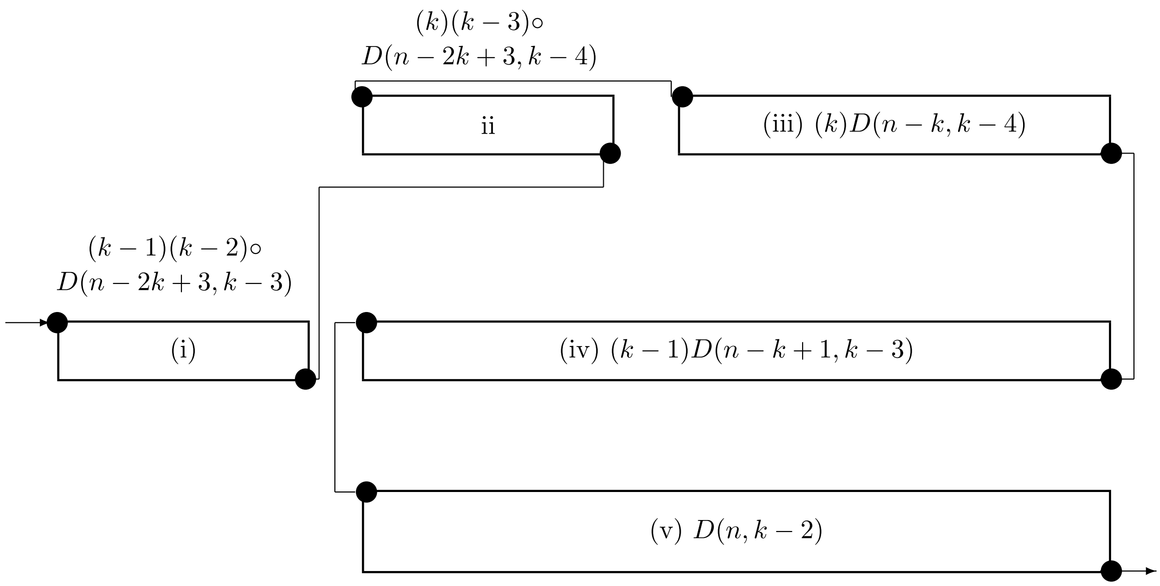


Figure 12: Case MD1: quasi-max-min GCE $MD(n,k)$ of $SD(n,k)$, for $2k-2 \leq n < 3k-7$.

$$
\begin{array}{rlr}
LD(2k-3,k) = & (k)(k-3), & (i)\\
& (k-1)(k-2), & (ii)\\
& (k-1)(k-3)(1), & (iii)\\
& (k)LD(k-3,k-4), & (iv)\\
\hline
& (k-1)LD(k-2,k-4), & (v)\\
& LD(2k-3,k-2). & (vi)
\end{array}
$$

(Case LD3: $2k-2 \leq n \leq S_k,\quad k \geq 3$) $MD(n,k)$ and $LD(n,k)$ are both defined here.

(Subcase MD: $2k-2 \leq n \leq 1+S_{k-2},\quad k \geq 3$) This range is empty unless $k \geq 6$.

(Subcase MD1: $2k-2 \leq n < 3k-7,\quad k \geq 6$)

Decompose $SD(n,k)$ as in Figure 12. Since $2k-2 \leq n \leq 3k-7$, no box is empty, unless $k=6$, in which case box (iii) is empty. If $k=6$, then $n=10$, and $MD(10,6) =$ 5 4 1, 6 3 1, 5 3 2, 4 3 2 1 is a quasi-max-min GCE for $D(10,6)$.

For $n \geq 7$, no box is empty. The minimum elements of boxes (i) and (ii) are adjacent by Corollary 5(iv); the maximum elements of boxes (ii) and (iii) are adjacent by Corollary 3, since $n < 3k-7$; the minimum elements of boxes (iii) and (iv) are adjacent by Corollary 5(i); and the maximum elements of boxes (iv) and (v) are adjacent by Corollary 4, since

$n < 3k-7$. Thus, by the induction hypothesis for LD, if none of the boxes (i) - (v) is one of the exceptional cases, the following is a quasi-max-min GCE of $SD(n,k)$:

$$\begin{array}{rll} MD(n,k) = & (k-1)(k-2)LD(n-2k+3,k-3), & (i) \\ & (k)(k-3)LD(n-2k+3,k-4), & (ii) \\ & (k)LD(n-k,k-4), & (iii) \\ & (k-1)LD(n-k+1,k-3), & (iv) \\ & LD(n,k-2). & (v) \end{array}$$

The exceptional cases can only occur as follows. Box (iii) is (10) $D(9,6)$ if $n = (19,10)$ and is (10) $D(12,6)$ if $(n,k) = (22,10)$. The GCE's, $A(9,6)$ and $A(12,6)$ of Figure 8 can be used in the role of $LD(9,6)$ and $LD(12,6)$, respectively. Since $2k-2 \le n \le 3k-7$, the only other way a box can correspond to an exception is if $(n,k) = (17,9)$, when box (iv) is (8) $D(9,6)$. Here, the GCE, $B(9,6)$, of Figure 8 can be used for $LD(9,6)$.

(Subcase MD2: $3k-7 \le n \le 1+S_{k-2}, \ \ k \ge 6$) Decompose $SD(n,k)$ as in Figure 13.

(Subcase MD2.1: $3k-7 \le n \le 3+S_{k-3} \ \ k \ge 6$)

If $3k-7 \le 3+S_{k-3}$, then $k > 7$. For $k > 7$, all boxes in Figure 13 are nonempty. The union of boxes (iv) and (v) is $(k-1)SD(n-k+1,k-2)$ which, by induction has a quasi-max-min GCE, $MD(n-k+1,k-2)$. Note that this requires that $2k'-2 \le n' \le S_{k'-2}+1$, where $n' = n-k+1$ and $k' = k-2$. But this is satisfied since $3k-7 \le n \le +S_{k-3}$.

The minimum elements of boxes (i) and (ii) are adjacent by Lemma 6(ii). The maximum elements of boxes (ii) and (iii) are the maxima of the sets $(k-1)(k-2)D(n-2k+3,k-4)$ and $(k)(k-2)D(n-2k+2,k-3)$, which are adjacent by Lemma 3.

The minimum elements of boxes (iii) and (iv) are adjacent by Corollary 5(i) and the maximum elements of boxes (iv) and (vi) are adjacent by Corollary 3.

If none of the boxes (i), (ii), (iii), (vi) corresponds to one of the two exceptions, by induction, each has a max-min GCE and the following is a quasi-max-min GCE of $SD(n,k)$. (see Figure 13.)

$$\begin{array}{rll} MD(n,k) = & (k-1)(k-2)(k-3)LD(n-3k+6,k-4), & (i) \\ & (k-1)(k-2)(k-4)LD(n-3k+7,k-5), & (ii) \\ & (k)LD(n-k,k-3), & (iii) \\ & (k-1)MD(n-k+1,k-2), & (iv,v) \\ & LD(n,k-2). & (vi) \end{array}$$

One of the boxes (i), (ii), (iii), or (vi) corresponds to one of the exceptions $D(9,6)$ or $D(12,6)$ if and only if $(n,k) \in \{(35,11),\ (38,11),\ (21,9)\}$. If $(n,k) = (35,11)$, box (ii)

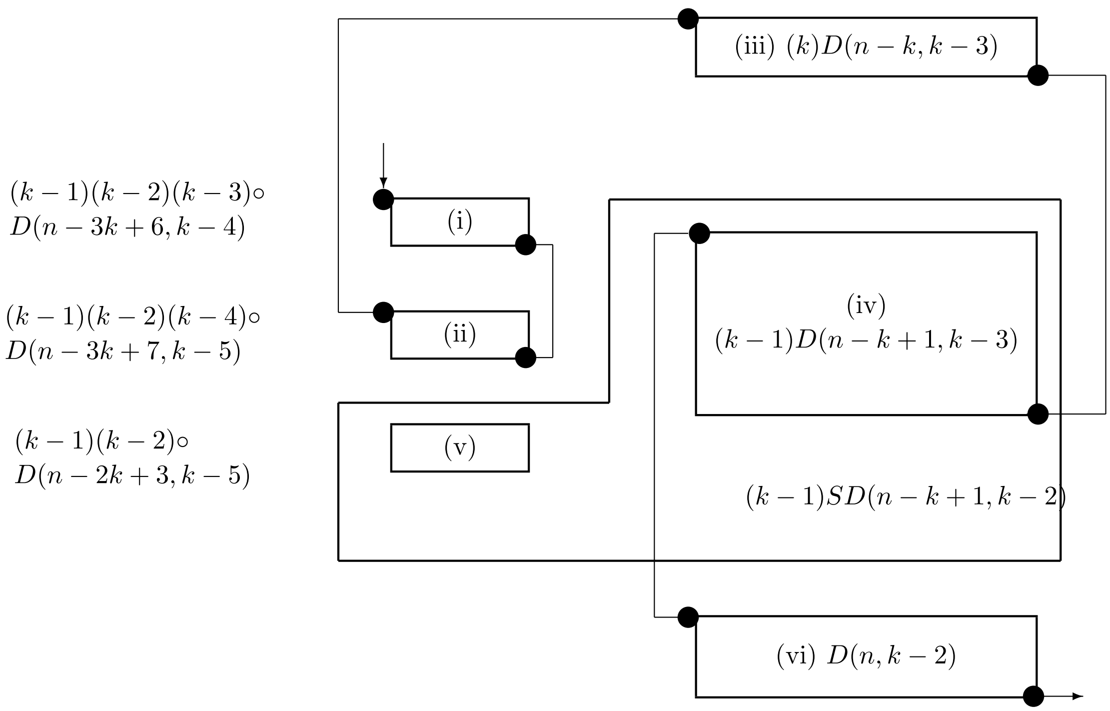
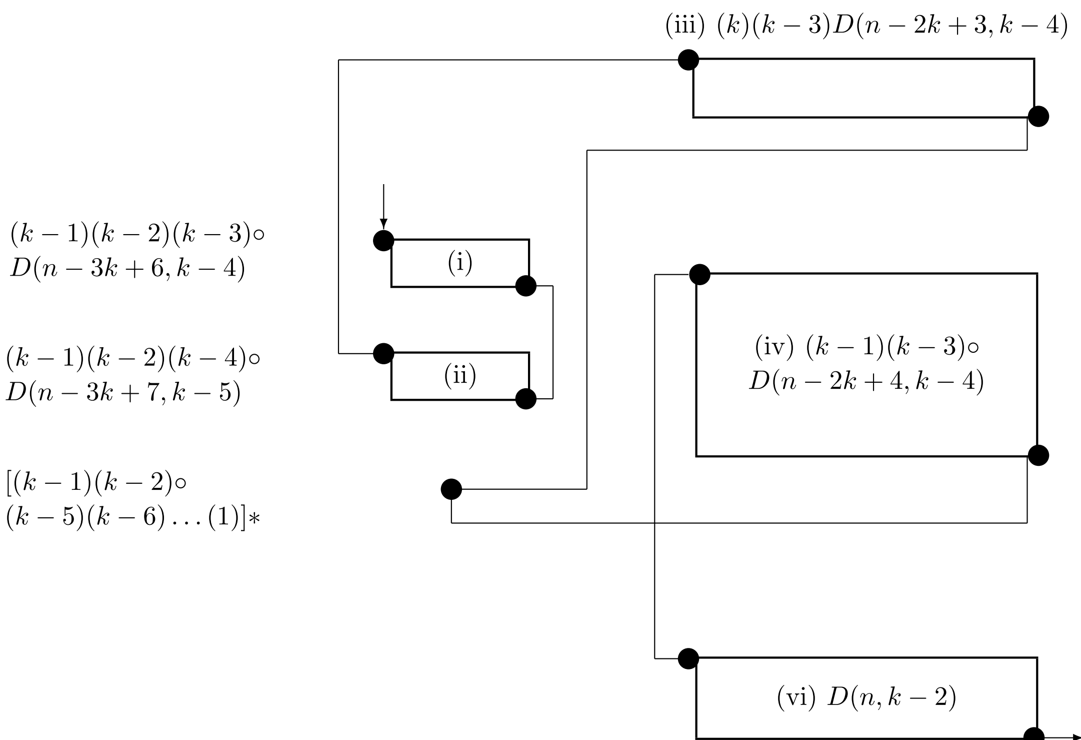


Figure 13: Case MD2.1: quasi-max-min GCE $MD(n,k)$ of $SD(n,k)$, for $3k-7 \leq n \leq S_{k-3}+3$.



(*) occurs only if $n = S_{k-3} + 4$

Figure 14: Subcase MD2.2: quasi-max-min GCE $MD(n,k)$ of $SD(n,k)$, for $S_{k-3}+4 \le n \le S_{k-2}+1$.

becomes $(10)(9)(7)D(9,6)$. A max-min GCE for this case is:

$$\begin{aligned} MD(35,11) = \quad & (10)(9)(8)LD(8,7),\ \overline{(10)(9)(7)A(9,6)},\ (11)(8)(7)(6)(2)(1), \\ & (11)(8)(7)(6)(3),\ (11)(8)(7)(5)(4),\ (11)(8)(7)(5)(3)(1), \\ & (11)(8)(7)(4)(3)(2),\ (11)(8)(6)(4)(3)(2)(1),\ (11)(8)(6)(5)(3)(2), \\ & (11)(8)(6)(5)(4)(1),\ (11)(7)(6)(5)(4)(2),\ (11)(7)(6)(5)(3)(2)(1), \\ & \overline{(10)MD(25,9)},\ LD(35,9). \end{aligned}$$

If $(n,k) = (38,11)$, box (ii) will be $(10)(9)(7)D(12,6)$ and $A(12,6)$ of Figure 8 can be used in place of $LD(12,6)$. If $(n,k) = (21,9)$, box (iii) is $(9)D(12,6)$, but then $A(12,6)$ can be used in place of $LD(12,6)$.

(Subcase MD2.2: $4 + S_{k-3} \le n \le 1 + S_{k-2}, \quad k \ge 6$)

This case is illustrated in Figure 14 which is derived from Figure 13 by noting the following. (1.) Box (iii) is nonempty and is equal to $(k)(k-3)D(n-2k+3,k-4)$. (2.) Box (iv) is nonempty and is equal to $(k-1)(k-3)D(n-2k+1,k-4)$. (3.) Boxes (i)and (ii) are nonempty. (4.) Box (vi) is nonempty unless $n = 1 + S_{k-2}$, in which case box (iv) is a singleton. (5.) Box (v) is either empty or has a single element which is adjacent both to the minimum element of box (iii) (by (1) and Corollary 5(i)) and to the minimum of box(iv) (by (2) and Corollary 5(ii).)

If none of boxes (i) - (vi) corresponds to $D(9,6)$ or $D(12,6)$, then by the induction hypothesis for LD, each has a GCE from maximum to minimum and the following is a GCE of $D(n,k)$ (see Figure 14):

$$\begin{array}{lll} MD(n,k) = & \underline{(k-1)(k-2)(k-3)LD(n-3k+6,k-4)}, & (i) \\ & \overline{(k-1)(k-2)(k-4)LD(n-3k+7,k-5)}, & (ii) \\ & (k)LD(n-k,k-3), & (iii) \\ & \underline{(k-1)(k-2)LD(n-2k+3,k-5)}, & (v) \\ & \overline{(k-1)LD(n-k+1,k-3)}, & (iv) \\ & LD(n,k-2). & (vi) \end{array}$$

Note that is a GCE even if box (v) is empty, since the minimum elements of boxes (iii) and (iv) are adjacent by (1), (2) and Corollary 5(iii).

The only way one of the boxes can be an exception is if (n,k) is $(33,10)$ or $(36,10)$. Box (i) is $(9)(8)(7)D(9,6)$ in the first case and $(9)(8)(7)D(12,6)$ in the second. A max-min GCE for $MD(33,10)$ is given explicitly as:

$$\begin{array}{ll} MD(33,10) = & (9)(8)(7)(6)(3),\ (9)(8)(7)(6)(2)(1),\ (9)(8)(7)(5)(3)(1), \\ & (9)(8)(7)(5)(4),\ (9)(8)(6)(5)(4)(1),\ (9)(8)(6)(5)(3)(2), \\ & (9)(8)(7)(4)(3)(2),\ (9)(8)(6)(4)(3)(2)(1),\ (10)(7)(6)(4)(3)(2)(1), \\ & (9)(7)(6)(5)(3)(2)(1),\ (10)(7)(6)(5)(3)(2),\ (10)(7)(6)(5)(4)(1), \\ & (9)(7)(6)(5)(4)(2),\ (8)(7)(6)(5)(4)(3),\ (8)(7)(6)(5)(4)(2)(1) \end{array}$$

For $MD(36,10)$, the list $A(12,6)$ can be used in place of $LD(12,6)$.

(Continuation of Case LD3)

(Subcase LD3.1: $2k-2 \le n \le 1+S_{k-2},\quad k \ge 3$)

If $2k-2 \le n \le 1+S_{k-2}$ then $k \ge 6$. Decompose $D(n,k)$ as in Figure 15. Since $2k-2 \le n \le 1+S_{k-2}$, none of the boxes (i) - (iv) is empty and the union of boxes (iii) and (iv) is $SD(n,k)$ which, by induction, has a quasi-max-min GCE, $MD(n,k)$. The minimum elements of boxes (i) and (ii) are adjacent by Corollary 5(ii) and the maximum elements of boxes (ii) and (iv) are adjacent by Corollary 4. Unless box (i) or (ii) corresponds to an

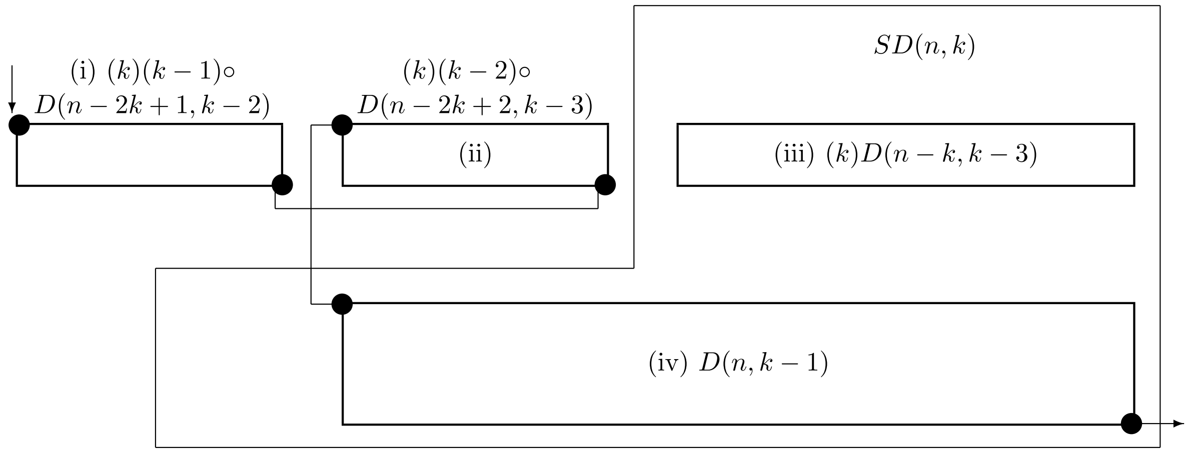


Figure 15: Case LD3.1: max-min GCE of $D(n,k)$ when $2k-1 \leq n \leq 1+S_{k-2}$.

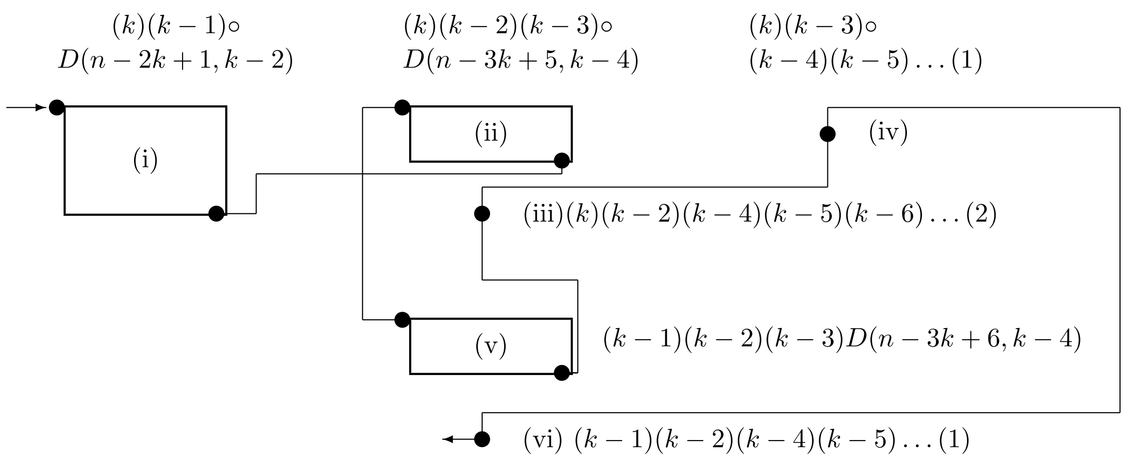


Figure 16: Case LD3.2: max-min GCE of $D(n,k)$ when $n = 2+S_{k-2}$ and $k \geq 3$.

exception, by induction, each has a max-min GCE and the following is a max-min GCE of $D(n,k)$:

$$\begin{array}{rll} LD(n,k) = & \underline{(k)(k-1)LD(n-2k+1,k-2)}, & (i) \\ & \underline{(k)(k-2)LD(n-2k+2,k-3)}, & (ii) \\ & MD(n,k). & (iii,iv) \end{array}$$

Box (i) is never an exception in this range, but box (ii) becomes $(9)(7)D(9,6)$ when $(n,k) = (25,9)$ and becomes $(9)(7)D(12,6)$ when $(n,k) = (28,9)$. In these cases, $B(9,6)$ and $B(12,6)$ can be used in place of $LD(9,6)$ and $LD(12,6)$.

(Subcase LD3.2: $n = 2+S_{k-2}$, $k \geq 3$)

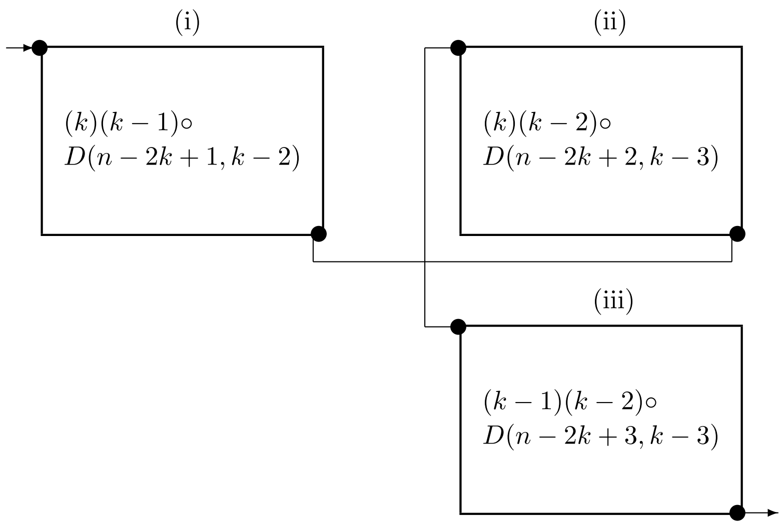


Figure 17: Case LD3.3: max-min GCE of $D(n,k)$ when $2+S_{k-2}<n\le S_k$ and $k\ge 3$.

Define $LD(n,k)$ for $k \le 6$ by $LD(3,3) = 3, \quad 2\ 1$; $LD(5,4) = 4\ 1, \quad 3\ 2$; $LD(8,5) = 5\ 3, \quad 5\ 2\ 1, \quad 4\ 3\ 1$. When $k=6$, then $n=12$ and this is the exception $D(12,6)$ of Figure 8. For $n \ge 7$, the only nonempty boxes in Figure 1(b) are the first three boxes of row 1 and the first box of row 2. Decompose $D(n,k)$ as in Figure 16. None of the boxes (i), (ii), (v) is empty and boxes (iii), (iv), and (vi) are singletons. The minima of boxes (i) and (ii) are adjacent by Lemma 4 since the minimum of box (i) is $\min((k)(k-1)(k-3)D(n-3k+4,k-4))$. The maxima of boxes (ii) and (v) are adjacent by Corollary 4. The minima of boxes (v) and (iii) are adjacent since the minimum of box (v) is

$$\min((k-1)(k-2)(k-3)(k-5)D(n-4k+11,k-6))$$

and (iii) can be written as

$$\min((k)(k-2)(k-4)(k-5)D(n-4k+11,k-6)).$$

Finally, it can be checked that none of the boxes corresponds to an exception. Thus, by induction, the following is a max-min GCE of $D(n,k)$:

$$
\begin{array}{lll}
LD(n,k) = & \underline{(k)(k-1)LD(n-2k+1,k-2),} & (i)\\
& (k)(k-2)(k-3)LD(n-3k+5,k-4), & (ii)\\
& (k-1)(k-2)(k-3)LD(n-3k+6,k-4), & (v)\\
& (k)(k-2)(k-4)(k-5)(k-6)\ldots(2), & (iii)\\
& (k)(k-3)(k-4)(k-5)(k-6)\ldots(1), & (iv)\\
& (k-1)(k-2)(k-4)(k-5)(k-6)\ldots(1). & (vi)
\end{array}
$$

(Subcase LD3.3: $S_{k-2}+2 < n \leq S_k$ and $k \geq 3$)

In this case, the only boxes of Figure 1(b) which could be nonempty are the first two of row 1 and the first two of row 2. So, decompose $D(n,k)$ as in Figure 17. Then at least one of boxes (i) and (ii) is nonempty. If box (ii) is empty, so is box (iii). Also, if box (iii) is empty, box (ii) is either empty or a singleton. If the relevant boxes are nonempty, then the minima of boxes (i) and (ii) are adjacent by Corollary 3 and the maxima of boxes (ii) and (iii) are adjacent by Corollary 4. If none of the boxes corresponds to an exception, then by induction, the following is a max-min GCE of $D(n,k)$:

$$
\begin{array}{lll}
LD(n,k) = & \underline{(k)(k-1)LD(n-2k+1,k-2),} & (i)\\
& (k)(k-2)LD(n-2k+2,k-3), & (ii)\\
& (k-1)(k-2)LD(n-2k+3,k-3). & (iii)
\end{array}
$$

In this subrange of values for n and k, only box (i) could ever correspond to a special case: when $(n,k) = (24,8)$ or when $(n,k) = (27,8)$. For these cases, max-min GCE's are:

$$
\begin{array}{lll}
LD(24,8) & = & (8)(7)(6)(3),\ (8)(7)(6)(2)(1),\ (8)(7)(5)(3)(1),\ (8)(7)(5)(4),\\
& & (8)(6)(5)(4)(1),\ (7)(6)(5)(4)(2),\ (8)(6)(5)(3)(2),\\
& & (8)(7)(4)(3)(2),\ (8)(6)(4)(3)(2)(1),\ (7)(6)(5)(3)(2)(1);\\
LD(27,8) & = & A(12,6),\ (8)(6)(5)(4)(3)(1),\ (7)(6)(5)(4)(3)(2).
\end{array}
$$

This completes the proof of Theorem 2. □

5 Concluding Remarks

Similar techniques can be used to investigate Gray codes in other families of integer partitions. For example, we suspect that all of the families below have Gray code enumerations, for arbitrary values of the parameters n, k, $\delta \geq 1$, $t \geq 1$, $d \geq 1$:

a. distinct odd parts

b. distinct parts congruent to 1 modulo δ

c. at most t copies of each part

d. parts congruent to 1 modulo δ, at most t copies of each part.

e. exactly d distinct parts

f. partitions whose Ferrers graph lies inside a k by n rectangle. In this case, we would call two partitions adjacent if they differ in that one part increases by 1 (or a part '1' appears) **and/or** one part decreases by 1 (or a part '1' disappears).

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