Acquisition Parameters of Graphs

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Results with or by
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Noah Prince, Chris Stocker, Paul S. Wenger,
Leslie Wiglesworth, Pratik Worah

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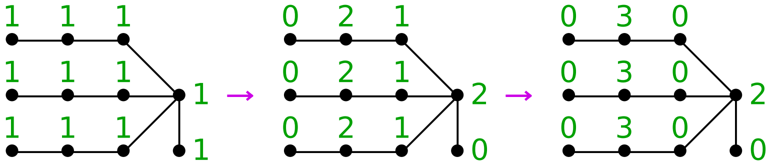
Def. total aquisition number $a_t(G)$ = min size of the final indep. set when each vertex starts with weight 1.

An Example

Ex. $a_t(T) = 4$, 11 vertices.



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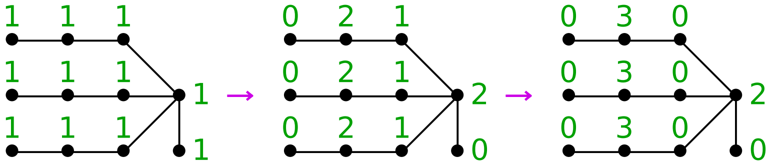

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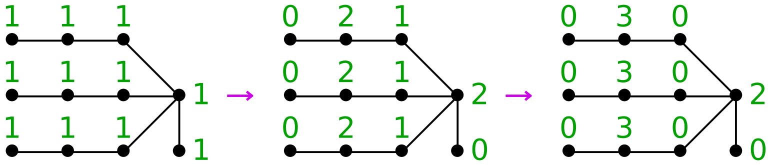

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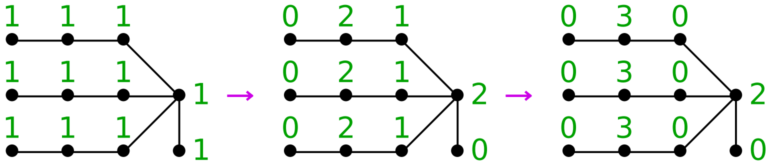

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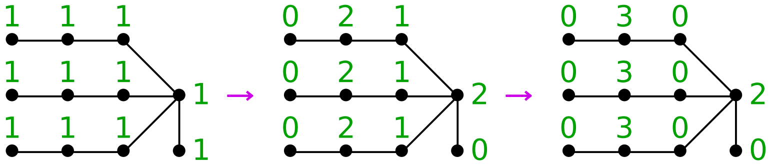

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game acquisition: move all weight, but two players Min and Max alternate moves — $a_g(G)$

Total Acquisition Number — Extremal Problem

All our graphs have n vertices.

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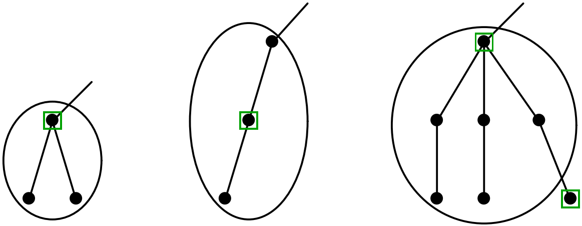

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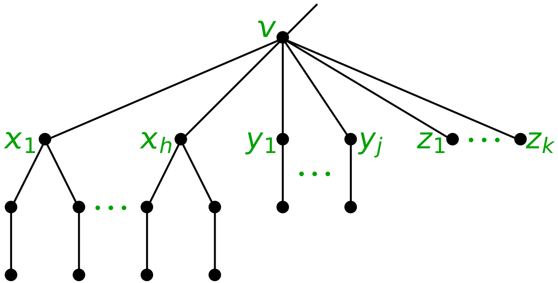

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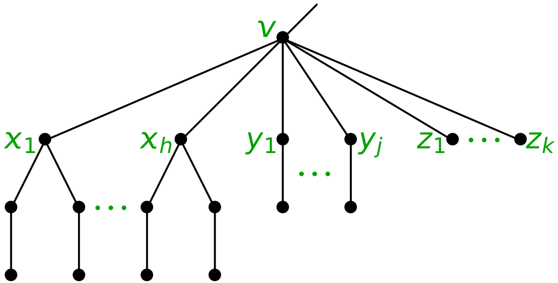
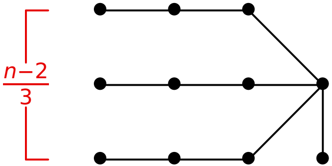


Sharpness:



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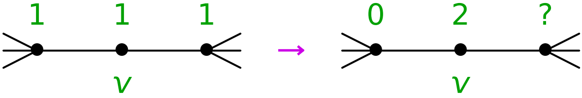

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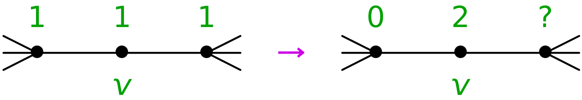

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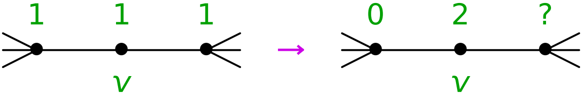

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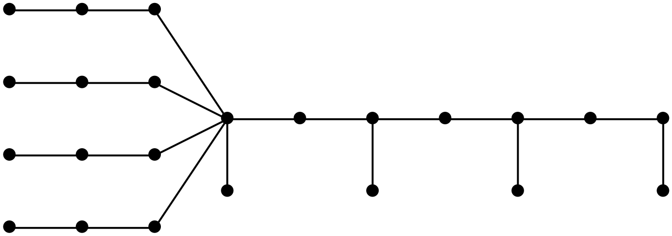
Thm. For $d \geq 3$ and $k \geq 6$, there is a tree T with $\Delta(T) = d$, $\mathrm{diam} T \geq k$, and $a_t(T) = (n+1)/3$.

Trees with $a_t(G)$ Large

Ex. The tree $T_{4,3}$.



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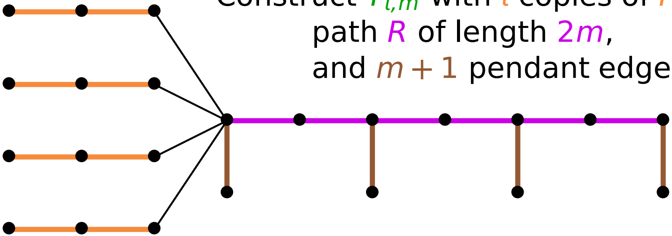

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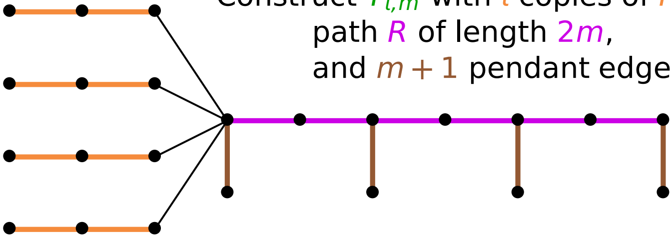

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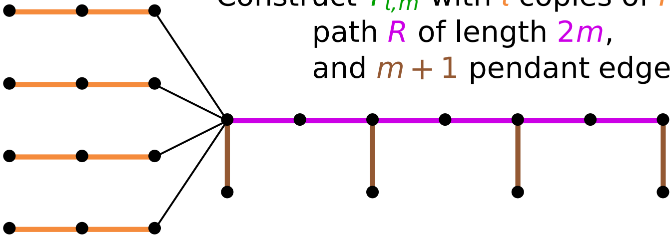

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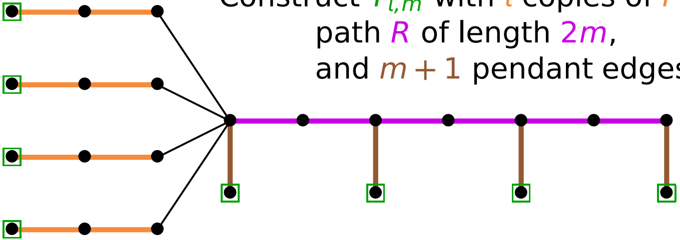

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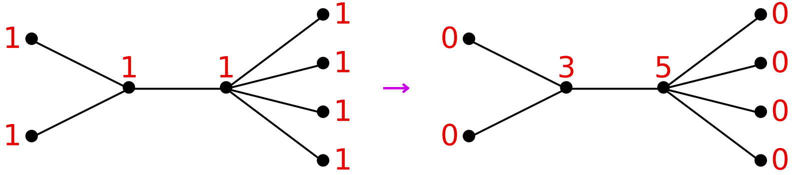

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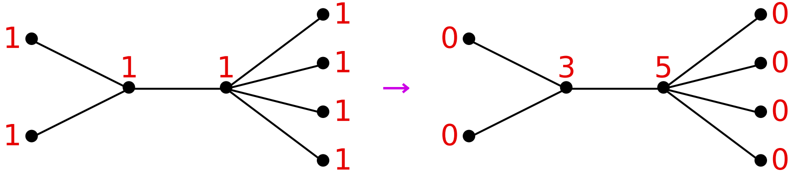

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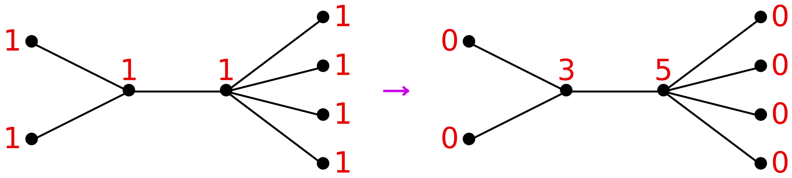

What about diameter 4 and diameter 5?

Thm. For diameter 4 or 5, $\max a_t(T) = \Theta(\sqrt{n \lg n})$.

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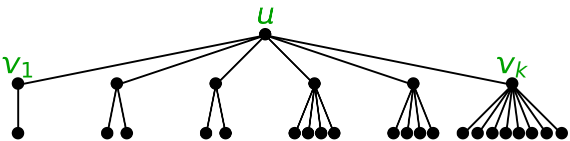

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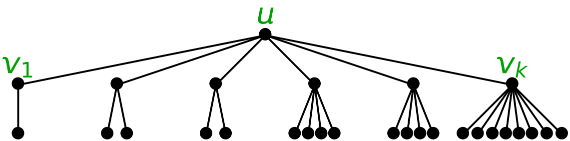

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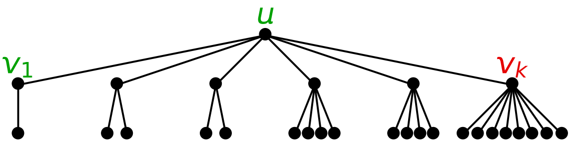

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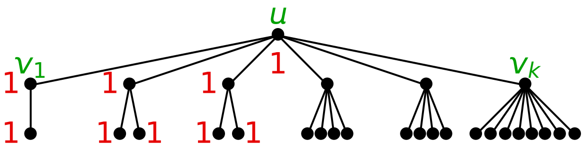

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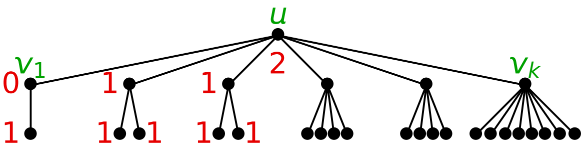

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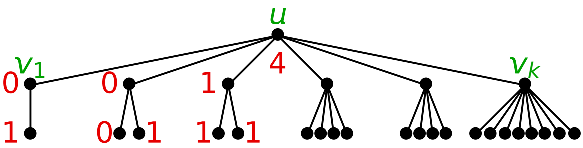

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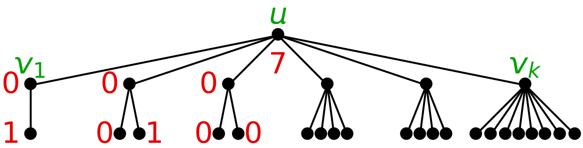

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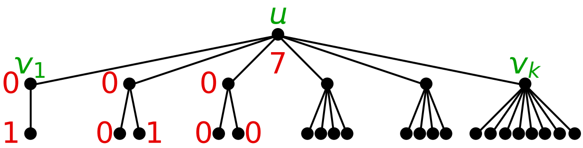

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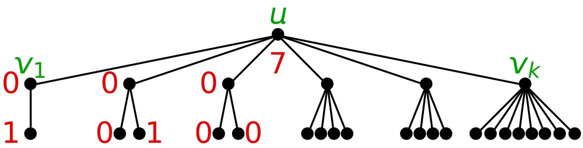

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$$m \le w_m < d(v_m) \le d(v_k) < 2\sqrt{n} < k/2.$$

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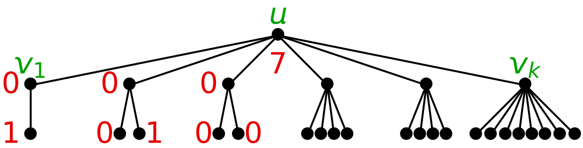

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$d(v_m)(k - m) < n$, so $d(v_m) < 2n/k$ (using $m < k/2$).

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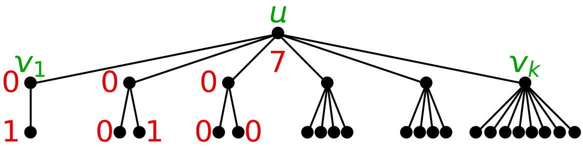

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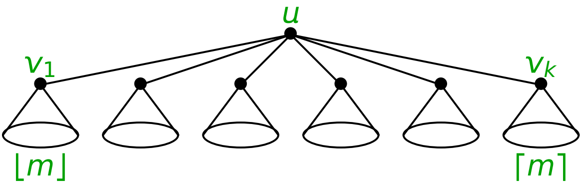
$d(v_m)(k-m) < n$, so $d(v_m) < 2n/k$ (using $m < k/2$). Hence $a_t(T) < 1 + |S|\,(2n/k) < 2\sqrt{n \lg(2n)}$. ■

Trees with Diameter 4, Lower Bound



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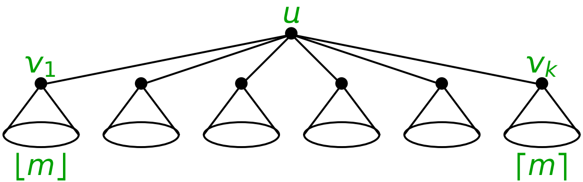

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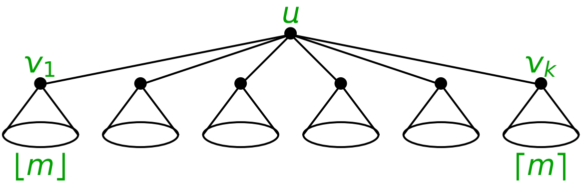

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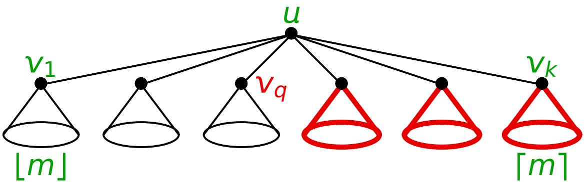

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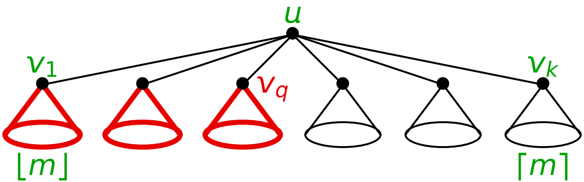

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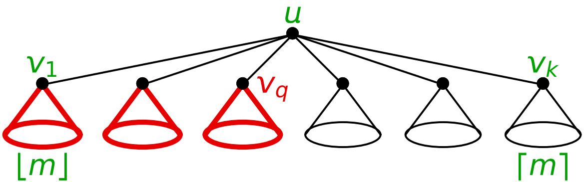

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Thus #leaves stranded $\geq \frac{n}{2\sqrt{n \lg n}} \lg n = \frac{1}{2}\sqrt{n \lg n}$. ∎

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Thm. (Lampert–Slater [1995]) Testing $a_t(G) = 1$ on general graphs is NP-complete.

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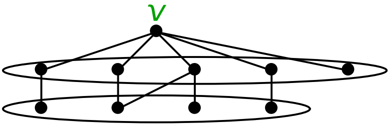

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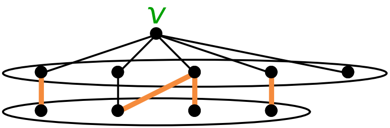

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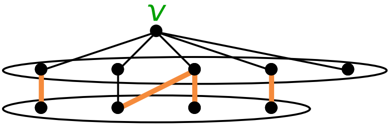

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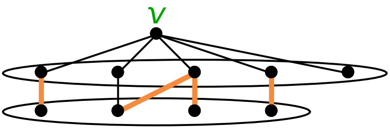

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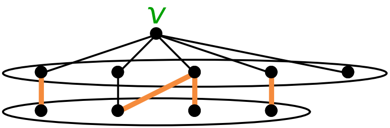

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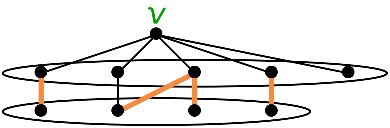

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Pf. Since nonadjacent vertices have a common nbr, $N(v)$ dominates. Previous argument almost works. ■

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Pf. The hypothesis gives a common nbr to nonadjacent vertices, so every neighborhood dominates.

Either $\Delta(G) \geq n/2$, or G is $(n-1)/2$-regular. ■

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Diameter 2

Thm. If $\mathrm{diam} G = 2$, then $a_t(G) \leq 250 \lg n \lg \lg n$.
If $\mathrm{diam} G = 2$ and $C_4 \not\subseteq G$ and $\Delta(G) \geq 8$, then $a_t(G) = 1$.

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Conj. There is a constant c such that, if $\mathrm{diam}\, G = 2$, then $a_t(G) \le c$.

Perhaps $c = 2$. This suffices for Moore graphs, polarity graphs, and graphs without 4-cycles.

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For $k = 2$, note that $a_t(C_n) = \lceil n/4 \rceil$,
but also $a_t(G) > (\frac{1}{4} + \frac{1}{1024})n$ is possible
(binary trees with triangles at the leaves).

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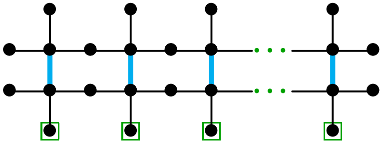
Ex. $a_u(G) \le a_t(G) \le M$, where M is the least number of double-stars that cover $V(G)$, and this may be sharp.



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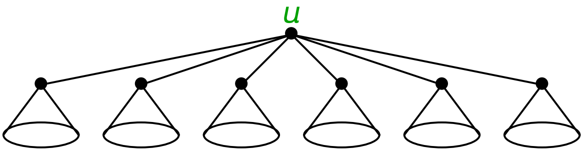

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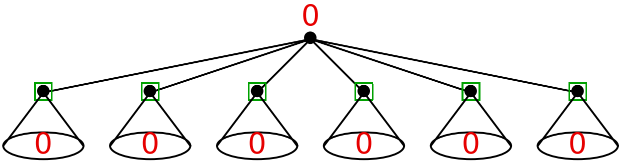

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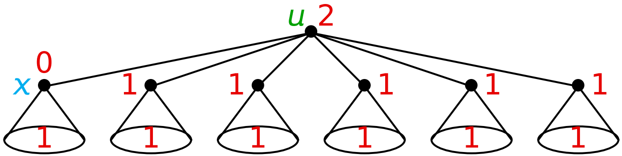

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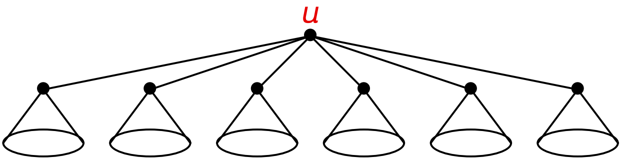

Construction: Make tree with $d(u) = m$ and $d(x) = m$ for $x \in N(u)$, so $n = m^2 + 1$. The first move involving u creates m components with positive weight.

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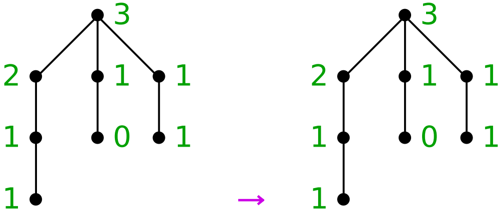

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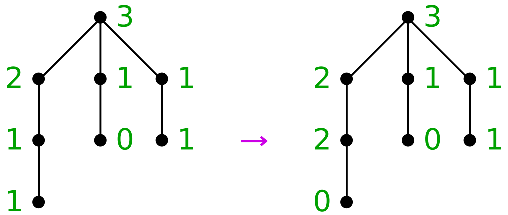

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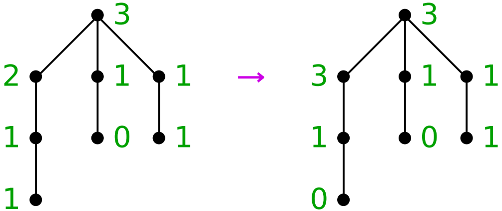

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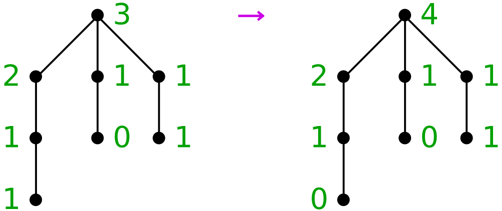

A Tool for Unit Acquisition

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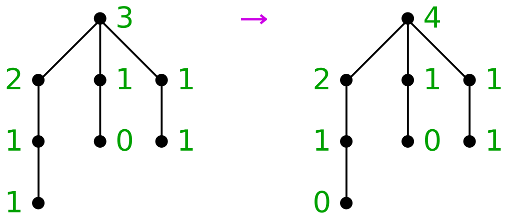
Lem. All weight in an ascending tree can be acquired to the root by unit acquisition moves.

Pf. One unit can be moved from any leaf to the root. ■



Thm. $a_u(T)=1 \Leftrightarrow$ an ascending tree can be produced.

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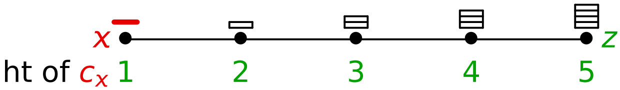

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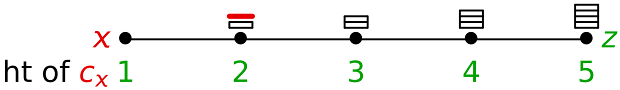

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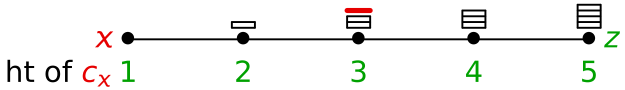

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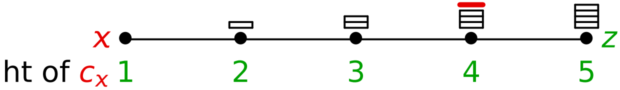

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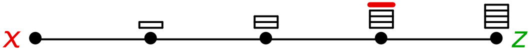
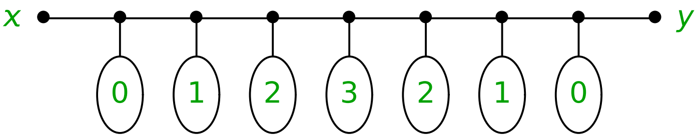


ht of c_x 1 2 3 4 5

- The chips under c_x can't be on the next vertex when c_x takes its ith step. Also, earlier chips supporting c_x can't be among the bottom i there.

Merging Chips

Lem. If chips from leaves at distance $k+4$ reach a common vertex between them, then the subtrees at the central k vertices of the path together have at least $\sum_{i=1}^{k} \lceil i/2 \rceil$ vertices.



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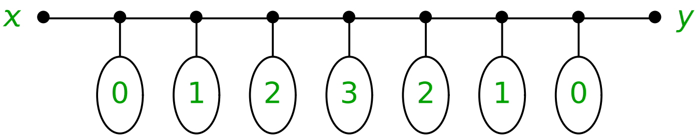

In trees, this can be used to establish lower bounds on a_u (like the degree-2 vertices earlier) or to seek a characterization of when $a_u(T) = 1$.

Necessary Condition for $a_u(T) = 1$

Cor. If $a_u(T) = 1$, then for any k-vertex path P in T that extends at least two steps farther at each end, the subtrees attached to vertices of P together have at least $\sum_{i=1}^{k} \lceil i/2 \rceil$ vertices.

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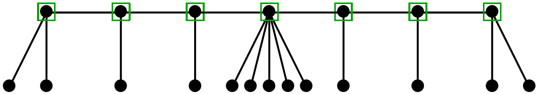

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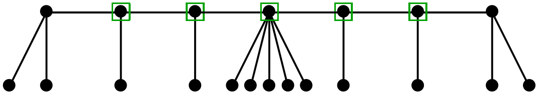

Conj. If T is a caterpillar, then $a_u(T) = 1$ iff if every set S of k consecutive internal vertices along the spine has together at least $\sum_{i=1}^{k} \lceil i/2 \rceil$ leaf neighbors.

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Pf. (Idea) Complicated inductive construction. ■

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Open Problems:

- Find general lower bounds for $a_u(G)$ in terms of other parameters of G.

- Characterize the trees with $a_u(T) = 1$.

- What is the complexity of recognizing $a_u(G) = 1$?

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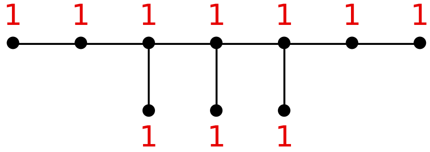

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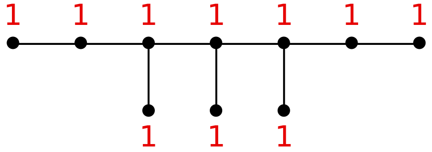
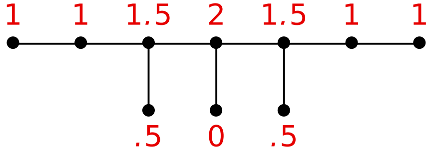
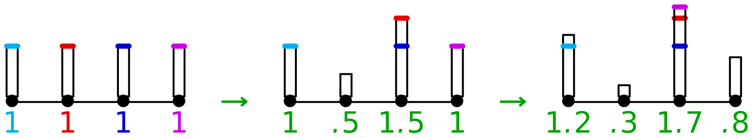


Move .5 twice to create an ascending tree with min difference .5: $a_f(G) = 1$.



Tool for Lower Bounds

Def. Represent the weight at a vertex by a tower of that height. The tops of the initial intervals are tips. When weight α is moved from u to v, the amount α is removed from the top of the tower at u and placed on top of the tower at v, along with any tips it contains.



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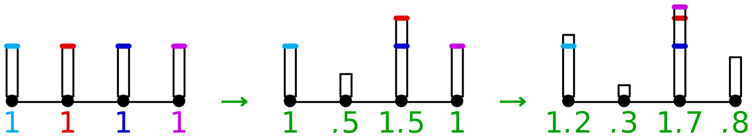

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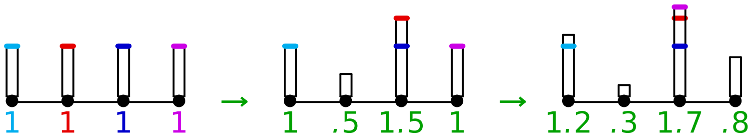

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To make $\alpha_f(G)$ small, tips must reach common towers.

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- Similarly, $a_f(C_n) = \lceil n/4 \rceil$.

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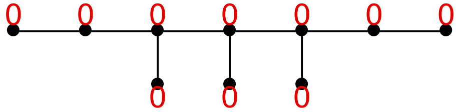

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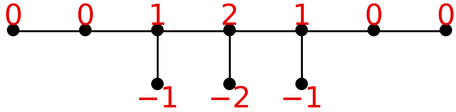

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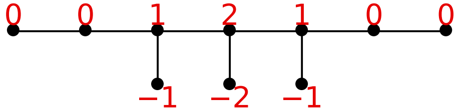

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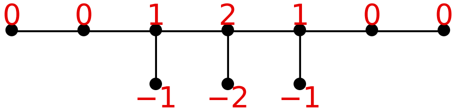

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3) Inductively produce an ascending tree in this model.

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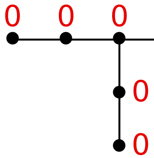

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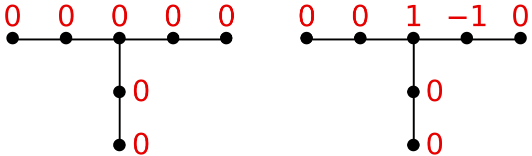

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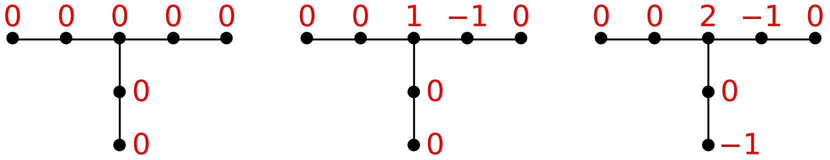

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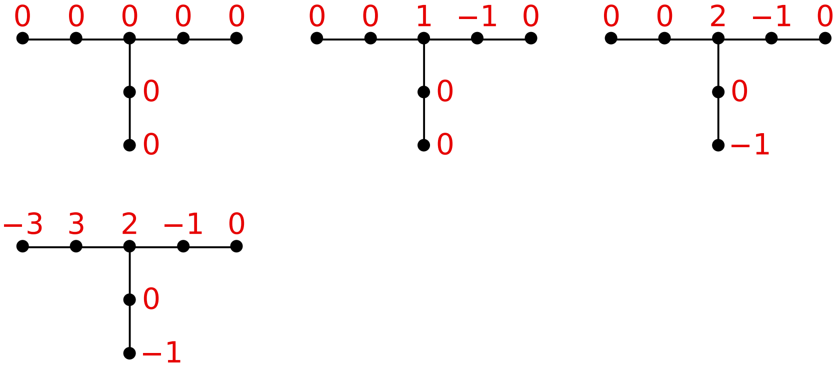

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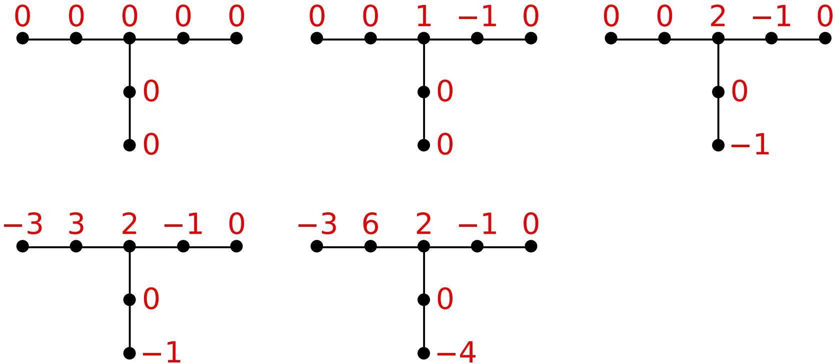

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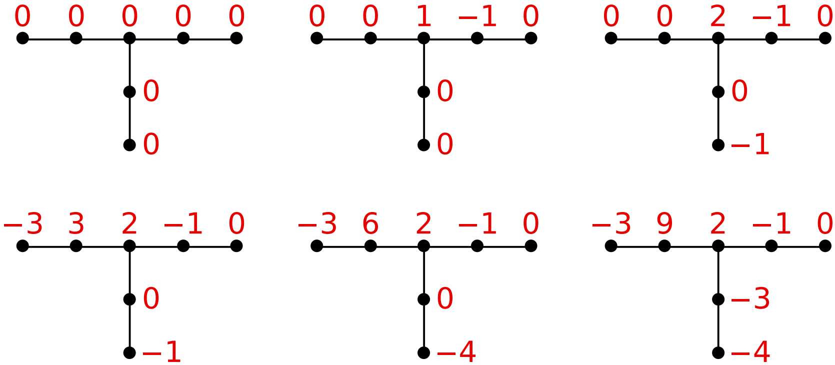

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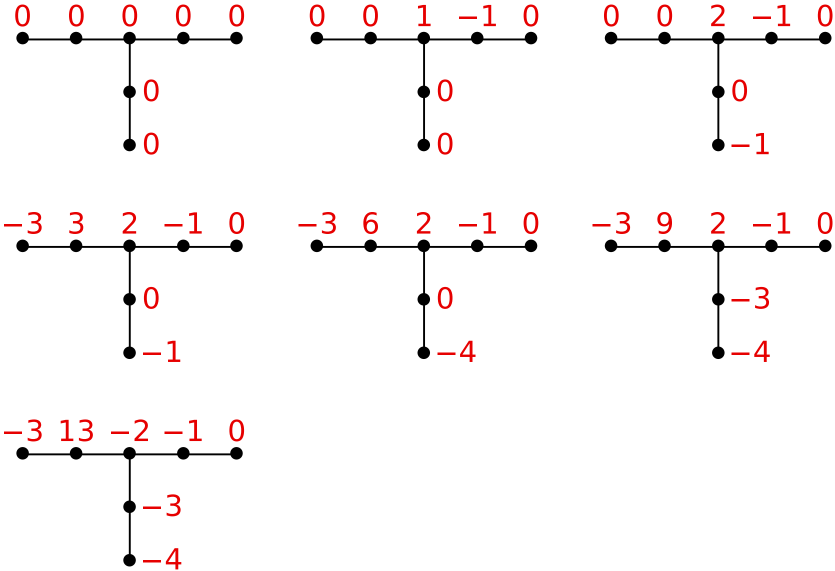

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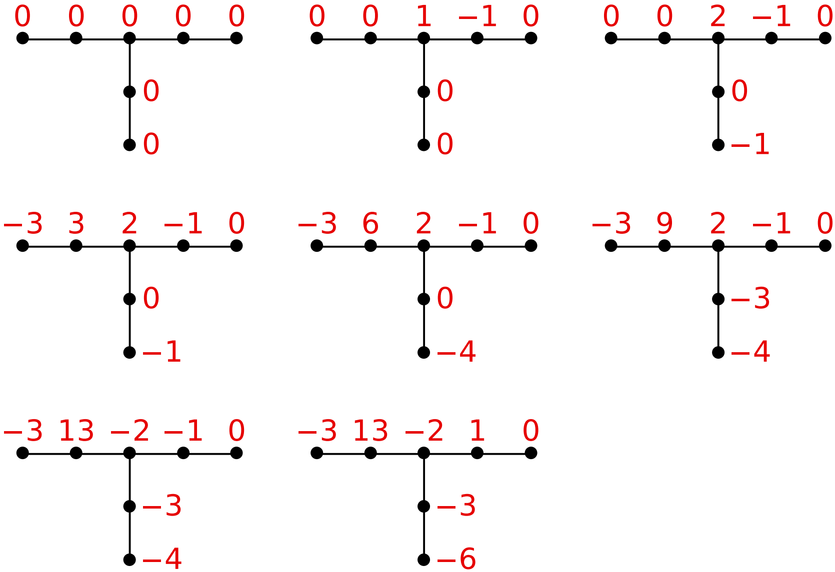
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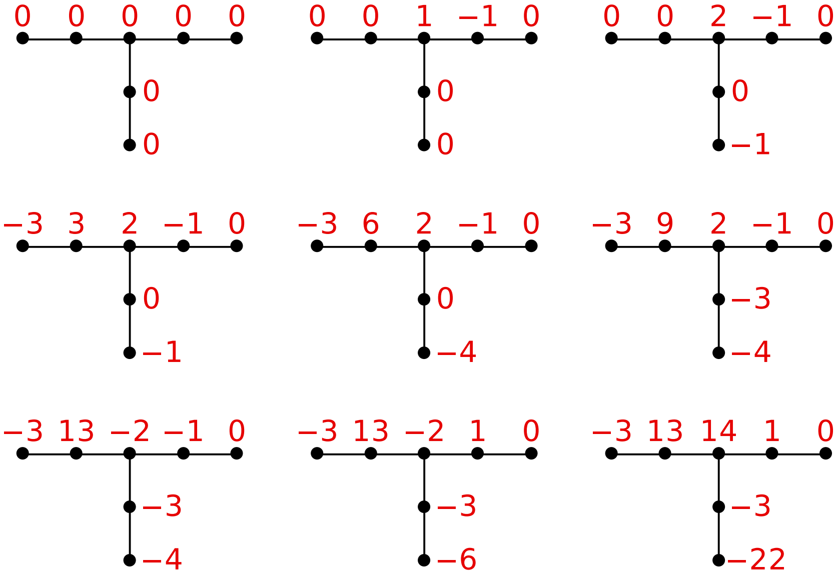
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−3 9 2 −1 0 −3 −4
−3 13 −2 −1 0 −3 −4
−3 13 −2 1 0 −3 −6
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Game Acquisition

Def. (Slater–Wang [2004]) Minimizer and Maximizer alternate total acquisition moves, aiming to min or max the final independent set. The game acquisition number(s) $a_g(G)$ and $\hat{a}_g(G)$ are the result of best play when Min or Max plays first, respectively.

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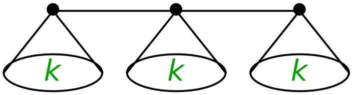
Ex. For the tree T below, $a_g(T) \approx 2n/3$. Max kills one end of the spine in the first round and combines the two remaining spine vertices in the second round.



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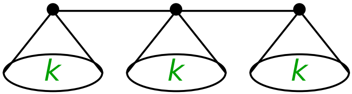

Ques. What is $\max a_g(T)$ among n-vertex trees?

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Ques. Note that $a_g(K_{n,n}) = 2$ but $\hat{a}_g(K_{n,n}) = 1$.
When does $a_g(G) > \hat{a}_g(G)$ hold? How much difference?

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In any such situation, if Min moves next, then Max restores these conditions: still at least $n - m$ excess serfs and one king in Y, no kings in X.

If Min makes a new X-king, Max eats it with a Y-king. Otherwise, Max eats an X-serf with a Y-king (if X has no serf, the game is over). Since X has no king, Min has no other way to disturb the conditions. ∎

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If Max eats a serf with a king, Min eats a serf with the other king; excess unchanged. If Max makes a new king, Min eats it with the opposite king; both sides lose one serf. If Max eats a king, then Min replaces it (the game is over if this can't be done). ∎

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If Max makes a king, then Min eats it, unless Max left two kings in X and none in Y; Min then makes a king in Y (if impossible, then the two kings in X eat the serfs in Y to end with 2). If Max eats a king, then Min restores it. Otherwise, Max eats a serf with a king; Min maintains the imbalance by eating a serf on the other side unless no king is available. In that case, Max ate a serf from Y and reduced the imbalance; Min makes a king in Y. ∎

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The conjecture includes the statement that this bound holds for all n and is optimal when $n > 5m/2$.

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