Degree Ramsey and On-Line Degree Ramsey Numbers

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Joint work with
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- $\hat{\rho}(G_1, G_2, G_3, \ldots, G_s; s)$ not yet much studied.

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Beck [1990], Pikhurko [2001,2,3], Donadelli–Haxell–Kohay. [2005].

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Pf. Let $k = \chi(G) - 1$.

Any H with $\chi(H) = k^s$ has proper coloring $f\colon V(H) \to [k]^s$.

Give each edge uv in H a color i such that $f(u)_i \neq f(v)_i$.

Color i graph is properly colored by f_i; $\therefore$ no monochr. G.

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Ex. $\chi(G) = 3 \quad \Rightarrow \quad 5 \leq \hat{\chi}(G) \leq 6$.

Equality holds in lower bound $\quad \Leftrightarrow \quad \exists$ hom. $\phi\colon\, G \rightarrow C_5$.

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Lem. Fix $n, k \in \mathbf{N}$. Let $K_k[r]$ be the complete k-partite graph w. parts of size r. When p is large, every red/blue coloring of $E(K_k[p])$ gives some copy of $K_k[n]$ where joining any two parts all edges have the same color.

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Thm. (BEL [1976]) $\hat{\chi}(G) = R(\mathcal{G})$.

Pf. Let $k = R(\mathcal{G})$ and $n = |V(G)|$. Let $H = K_k[p]$ with p as in the lemma. Collapsed coloring of $E(K_k)$ gives a monochr. copy of some G' with $\phi\colon G \to G'$. Each vertex of G' has n copies in $K_k[n]$, so a monochr. G arises with all v such that $\phi(v) = u_i$ placed in the ith part. ■

Chromatic Ramsey Number

Ex. $\hat{\chi}(C_5) = 5$.

Conj. (BEL [1976])
$\min\{\hat{\chi}(G)\colon \chi(G) = k\} = (k-1)^2 + 1$.

The conjecture was proved for $k \le 4$ by BEL.
Zhu [1998] proved it for $k = 5$ and gave nontrivial upper bounds for $k = 6$ and $k = 7$.

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Thm. $\mathrm{dr}(K_{1,m}; s) = \begin{cases} s(m-1) & m \text{ even} \\ s(m-1)+1 & m \text{ odd} \end{cases}$.

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Pf. Upper Bound: $K_{1,s(m-1)+1} \xrightarrow{s} K_{1,m}$.

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Improves when m is even:
When $r > k$ and k is odd, there is an r-regular graph H having no k-factor (Bollobás–Saito–Wormald [1985]).
With $k = m-1$ and $r = s(m-1)$, s-coloring $E(H)$ with no monochromatic $K_{1,m}$ requires a k-factorization.

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Improves when m is odd. When $\Delta(H) \le s(m-1)$,
Petersen's Theorem decomposes $s(m-1)$-regular supergraph H' into 2-factors. Putting $(m-1)/2$ in each color avoids degree m in one color at any vertex. ■

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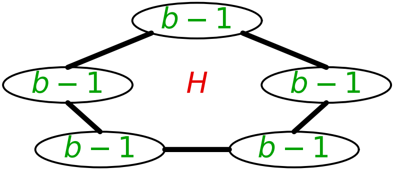
In a red/blue coloring, each vertex is majority red or majority blue. If most of part X is red, then Y must have no red vertex. Hence Y is all blue and X is all red. Now majority of edges are blue and majority are red.

Improved upper bound (b even and $a < b$)

Claim:
$H \to S_{b-1,b}$
for b even



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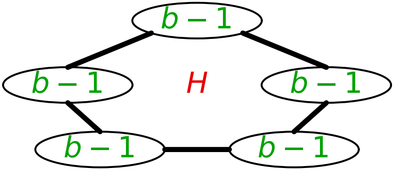

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Pf. Vertices are majority red or majority blue or tied.
Not all are tied (would be odd regular of odd order).

No $S_{b-1,b}$ $\Rightarrow$ all nbrs (via red) of maj red are maj blue.

A maj red vertex forces a maj blue in each direction;
after 5 steps, one set has a maj red and a maj blue.

Now its neighboring sets together need b maj blue and
b maj red vertices, but they have only $2b-2$ total. ■

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Thm. $$\mathrm{dr}(P_n) = \begin{cases} n-1 & n \le 4 \\ 3 & n \in \{4,5\} \\ 3 \text{ or } 4 & n = 6 \quad \leftarrow \textit{Open} \\ 4 & n \ge 7 \end{cases}.$$

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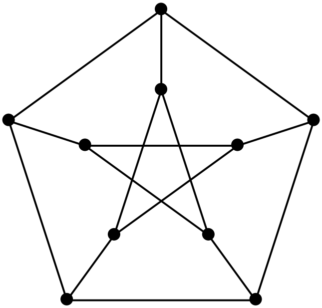

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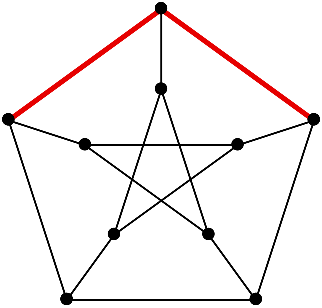

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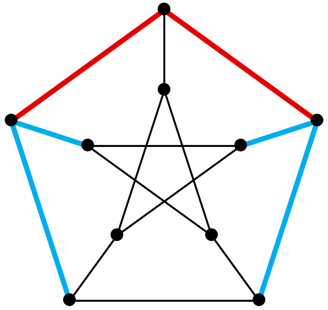

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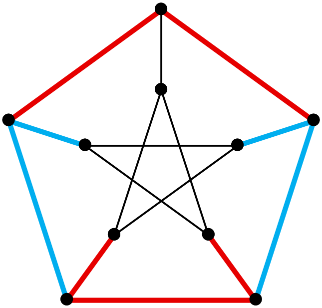

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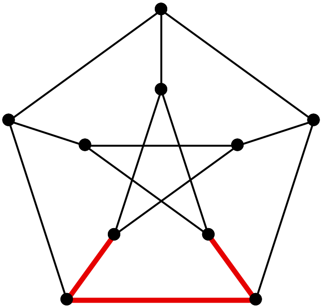

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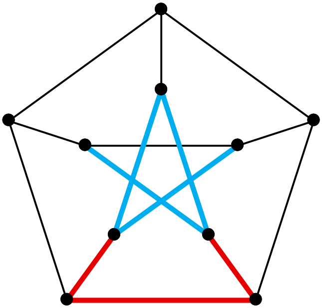
Thm. $\mathrm{dr}(P_n) = \begin{cases} n-1 & n \le 4 \\ 3 & n \in \{4,5\} \\ 3 \text{ or } 4 & n = 6 \quad \leftarrow \textit{Open} \\ 4 & n \ge 7 \end{cases}$.

$\mathrm{dr}(P_n) \ge 3$: Alternate along paths & cycles to avoid P_4.

$\mathrm{dr}(P_5) \le 3$: Petersen $\to P_5$



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s colors on sm edges puts $\ge m$ in some color class.
Since $|V(H)| = m$, this subgraph has a cycle.
Since girth$(H) \ge n$, this color class contains P_n. ∎

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For which parameters ρ is $\hat{\rho}(G)$ bounded by a function of $\rho(G)$? - True for clique number, chromatic number, ...

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Thm. (Haxell–Kohayakawa–Łuczak [1995])
s-color induced size Ramsey # of C_n is linear in n.

- The proof shows that $\mathrm{dr}(C_n) \leq c$ (where c is huge).

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And Now For Something Sort Of

Completely Different

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This defines the on-line Ramsey game $(G, \mathcal{H})$.
Can Builder playing on $\mathcal{H}$ force a monochromatic G?

On-Line Ramsey Parameters

Def. For a monotone graph parameter ρ,
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Def. on-line degree Ramsey number $\mathrm{odr}(G) = \mathring{\Delta}(G) = \min\{k\colon \text{Builder wins } (G,\mathcal{S}_k)\}$, where $\mathcal{S}_k = \{H\colon \Delta(H) \le k\}$.

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Thm. $\mathrm{odr}(G) \le 8$ if $\Delta(G) \le 2$ (maybe less).

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Cor. $\mathrm{odr}(P_n) \ge 3$; $\mathrm{odr}(K_{1,m}) \ge m$; $\mathrm{odr}(S_{a,b}) \ge a + b - 1$; $\mathrm{odr}(G) \ge 2\Delta(G) - 1$ if $\exists$ adjacent maxdegree vertices.

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- Lower bound for $\mathrm{odr}(C_n) \ge 4$ comes from charzn of $\mathrm{odr}(G) \le 3$, which uses greedy linear-forest Painter .

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- Prove upper bounds on odr for trees and cycles by algorithms for Builder to defeat a consistent Painter .

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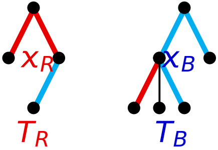
An active vertex becomes
satisfied if it has k children via its own color.
dangerous if it has k incident edges of the other color.

Builder Strategy

Builder plays pendant edges at active vertices (in T_R or T_B) until Painter makes one satisfied or dangerous.



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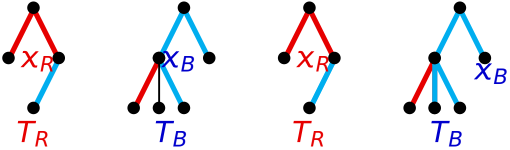

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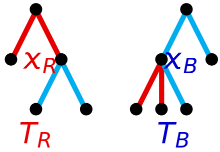

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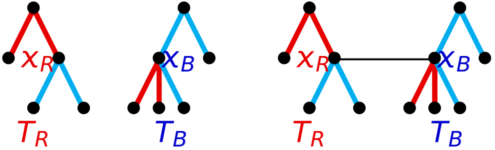

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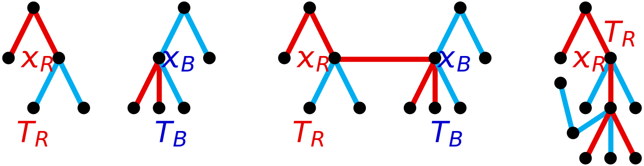

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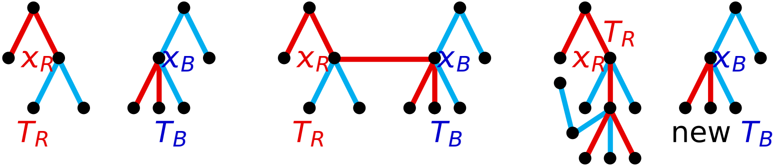
Then Builder regenerates the other tree.



Even Cycles

Assume Builder plays on $\mathcal{S}_k$ and Painter is consistent.
(Weight = bound on total red + blue at a vertex.)

Lem. Let F_1, F_2 be weighted graphs Builder can force in red, with vertices u_1, u_2. Form F from $F_1 + F_2$ by adding $u_1 u_2$ and increasing weights on u_1 and u_2 by 2. If q is even, then Builder can force a red F or a blue C_q.

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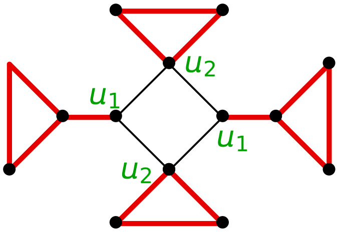
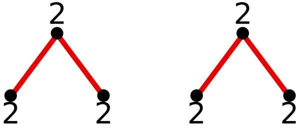
Pf. Builder forces $q/2$ copies of F_1 and F_2 and then adds a cycle alternating between the copies of u_1 and u_2. ■



Trees for Even Cycles

Consistent Painter makes the same monochr. P_3 (with weights 2) in any isolated triangle; we may assume it is red. Painter wants to avoid a monochromatic C_q.



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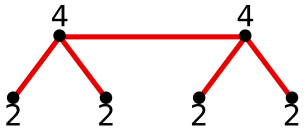

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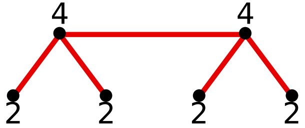
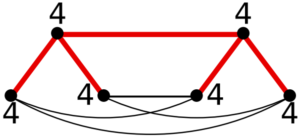


$q = 4$



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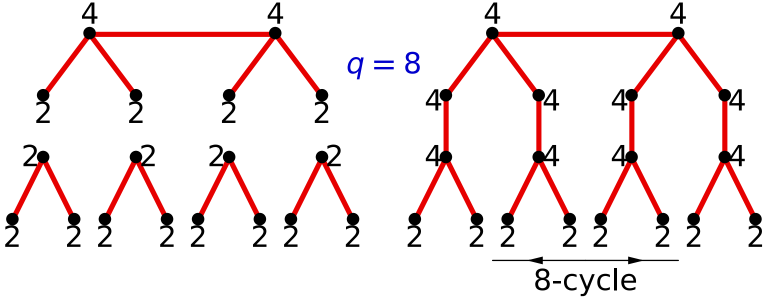

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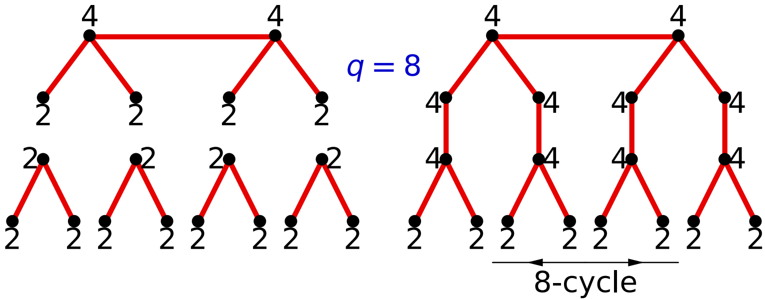
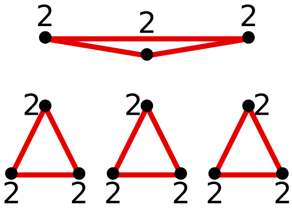


Further extensions of the tree force any even cycle C_q (just extend one half if $q \equiv 2 \mod 4$), but C_6 and C_{10} are special.

Special Case: C_6

Consistent Painter makes consistent triangles.

Case 1: monochromatic



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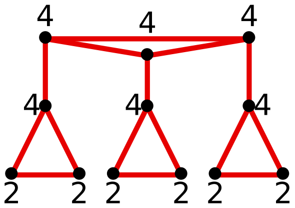

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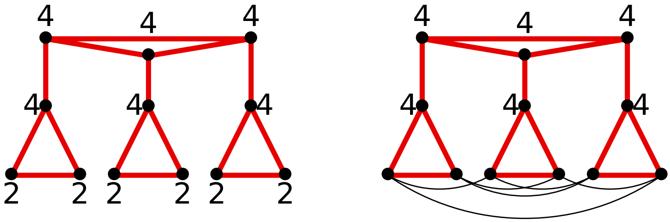

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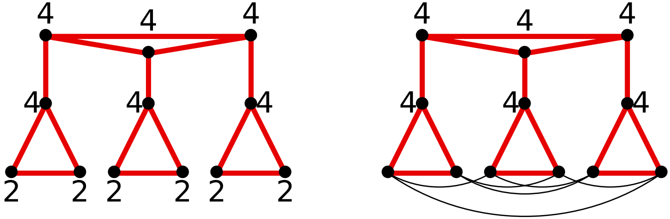
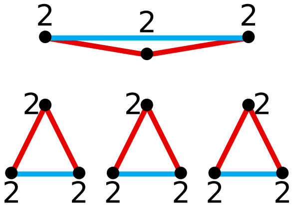


Case 2: not monochromatic



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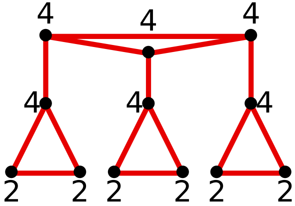

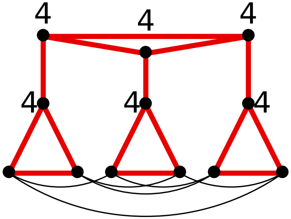

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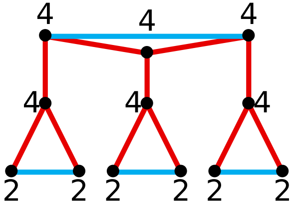

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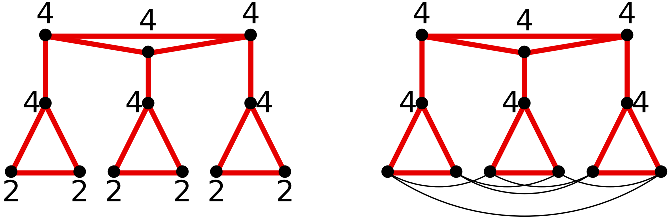

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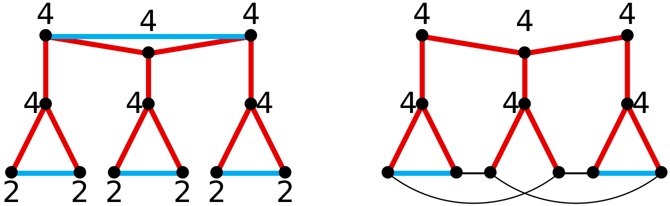

Summary for Cycles; Open Questions

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Ques. What is $\mathrm{odr}(C_5)$? (4 or 5)
What is $\mathrm{odr}(K_{1,3} + e)$? (4 or 5)
What is $\mathrm{odr}(C_4 + e)$? (5 or 6 or 7.)

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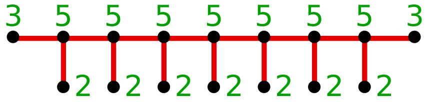

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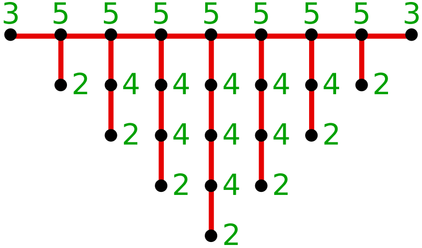

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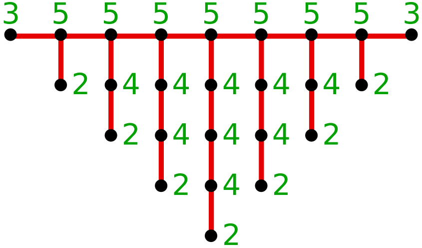

Leaf distances $q-1$ (opposite halves or to middle).
Cycle through the leaves is all blue or some red.