Some Old and New Results on Degree Lists of Graphs

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Joint work with
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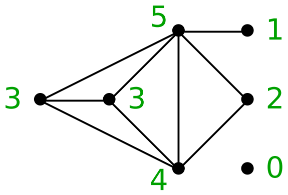

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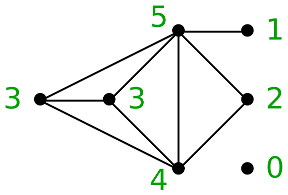

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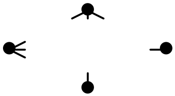
Each edge contributes to degree at two vertices, so $\sum_{v \in V(G)} d(v)$ is even.

Questions to Ask

Not all lists of nonnegative integers are degree lists.



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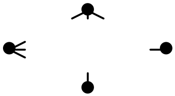

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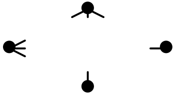

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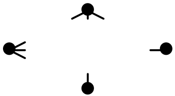

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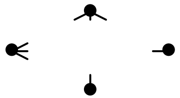

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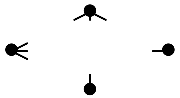

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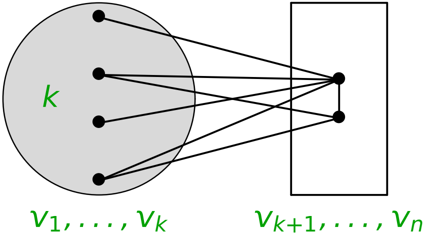
• Degrees are also studied in relation to graph coloring, connectivity, matchings, cycle lengths, etc.

Erdős–Gallai Condition

- Necessary condition



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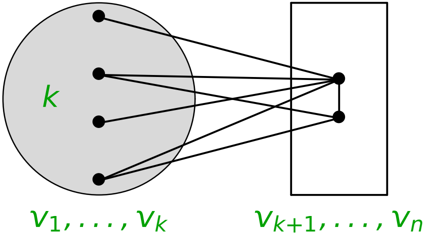

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

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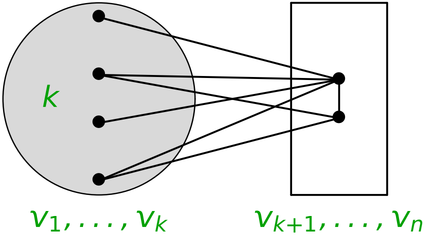

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Thm. (Erdős–Gallai [1960]) A nonincreasing nonneg. integer n-tuple d is graphic if and only if the sum is even and the Erdős–Gallai inequalities hold for all k.

Approach to Sufficiency

Idea: (Tripathi–Venugopalan–West [2010])
A subrealization of a list $d_1, \ldots, d_n$ is a graph with vertices $v_1, \ldots, v_n$ such that $d(v_i) \le d_i$ for $1 \le i \le n$.

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Write $v_i \leftrightarrow v_j$ when $v_i v_j \in E(G)$; otherwise, $v_i \not\leftrightarrow v_j$.

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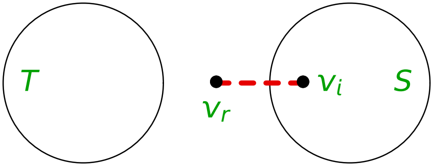
Case 0: $v_r \not\leftrightarrow v_i$ for some v_i with $d(v_i) < d_i$.



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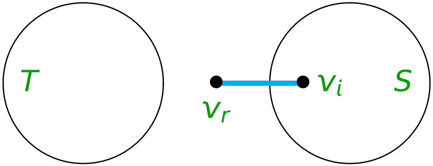
Add $v_r v_i$.



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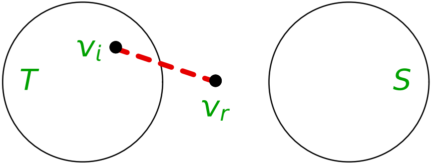

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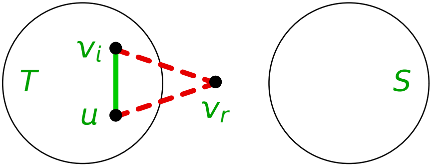

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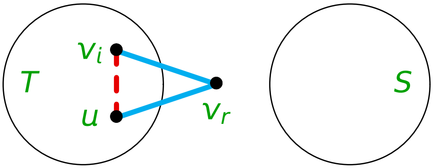

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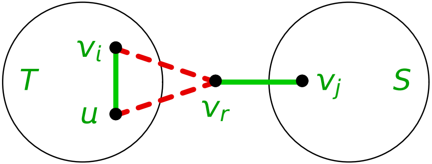
If $d_r - d(v_r) \geq 2$, then switch uv_i to uv_r, v_iv_r.
If $d_r - d(v_r) = 1$, then $\sum d_i - \sum d(v_i)$ even $\Rightarrow$ $\exists\, v_j \in S$
with $d(v_j) < d_j$. Not Case 0 $\Rightarrow$ $v_r \leftrightarrow v_j$.



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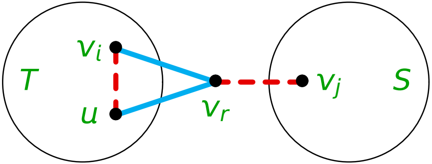
Switch v_rv_j, uv_i to uv_r, v_iv_r.



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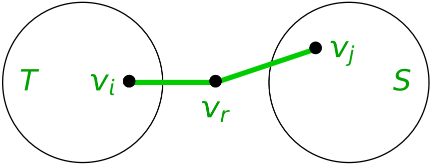

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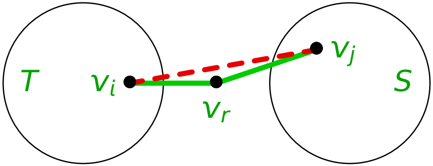
Now $d_j < r \Rightarrow v_j \not\leftrightarrow v_i$ for some $v_i \in T$.



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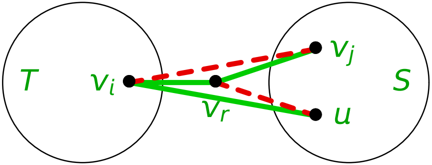

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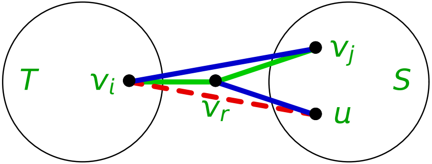

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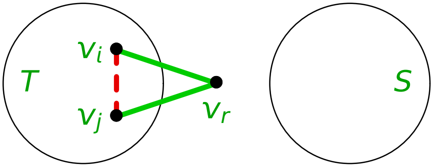

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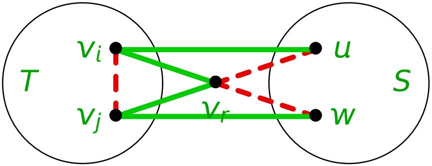

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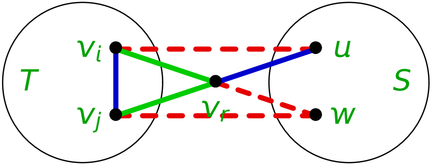
Now $v_r \leftrightarrow T \;\Rightarrow\; u, w \in S$; Switch uv_i, wv_j to uv_r, v_iv_j.



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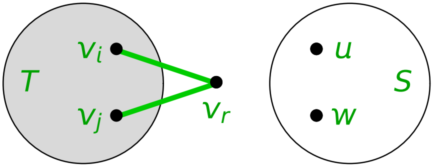

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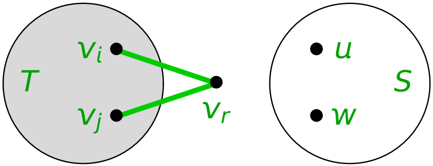

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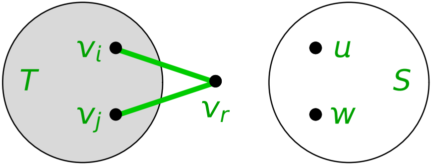
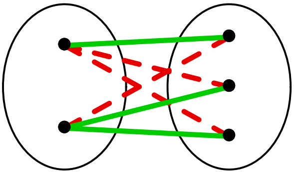
E-G inequality $\Rightarrow$ $d(v_r) = d_r$. Defect is 0, augment r.



Koren's Condition

For any vertex partition, every cross edge is in G or $\overline{G}$.



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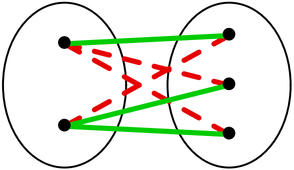

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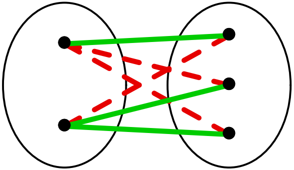

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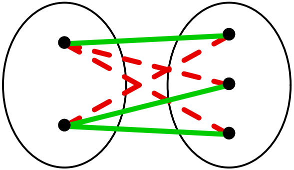

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• The result extends to loopless multigraphs with edge multiplicity $\le p$ via $\sum_{i\in I} d_i + \sum_{j\in J}[p(n-1)-d_j] \ge p\,|I|\,|J|$.

Tools for Sufficiency

Lem. (Fulkerson–Hoffman–McAndrew [1965])
If d is a nonincreasing graphic list, and $d_j > 0$ for some j with $j > 1$, then $v_j \leftrightarrow v_1$ in some realization.

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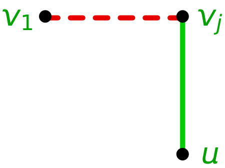

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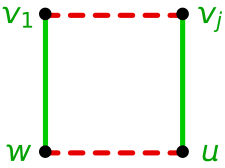

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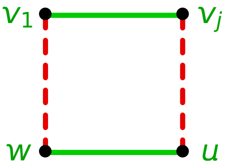

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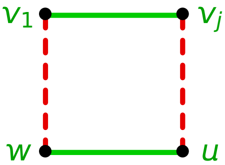

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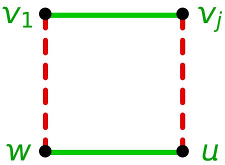

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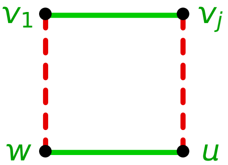

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Basis: $n + \sum d_i = 1$. One vertex, no edges.

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Other reductions leave only $d = (n-2, \ldots, n-2, 1)$, which has odd sum. ∎

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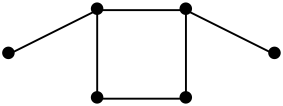
Although $(3, 3, 1, 1)$ is not graphic, $(3, 3, 2, 2, 1, 1)$ is.



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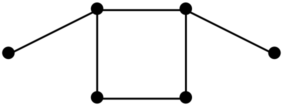

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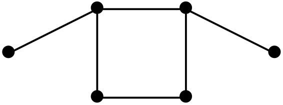

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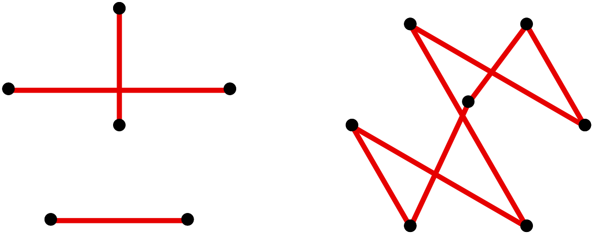

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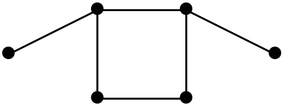

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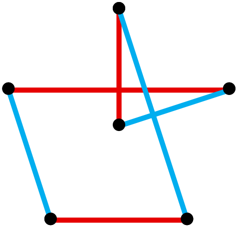

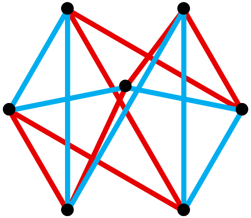

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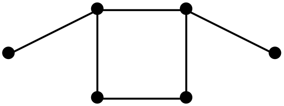

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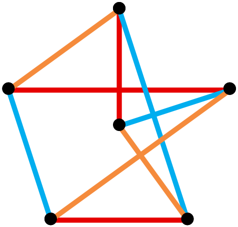

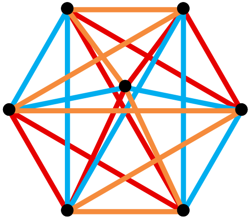

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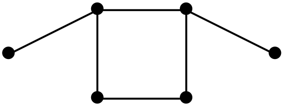

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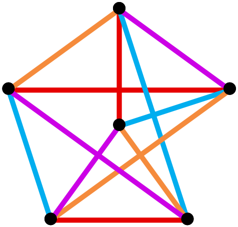

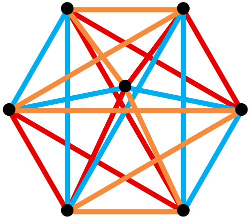

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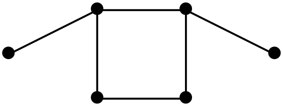

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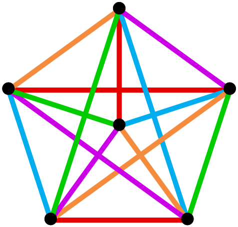

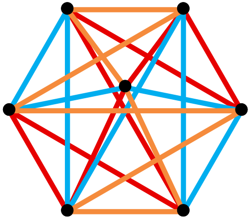

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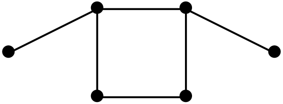

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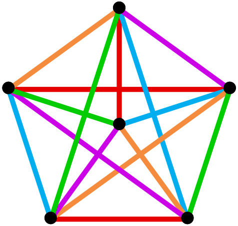

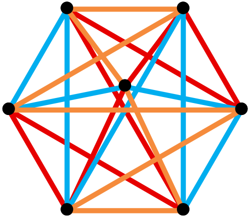

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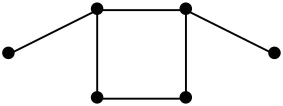

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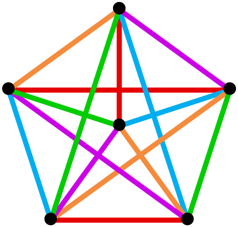

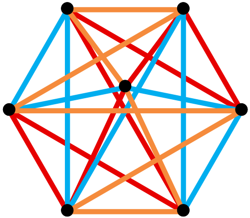

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Thm. (B–H–J–W) When consecutive entries in d differ by at most g, let $q = (r-s)/g$. The list d (with even sum) is graphic if $n \geq \frac{r+s+q}{2} + \frac{(r+s)(r-s-q+2)+\epsilon}{4s}$, and this is sharp. (Here $\epsilon \in \{0, g, g+1\}$.)

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Thm. (B–H–J–W) When consecutive entries in d differ by at most g, let $q = (r-s)/g$. The list d (with even sum) is graphic if $n \geq \frac{r+s+q}{2} + \frac{(r+s)(r-s-q+2)+\epsilon}{4s}$, and this is sharp. (Here $\epsilon \in \{0, g, g+1\}$.)

Set $g = 1 = \epsilon - 1$ and $g = r - s = \epsilon$ for the special cases.

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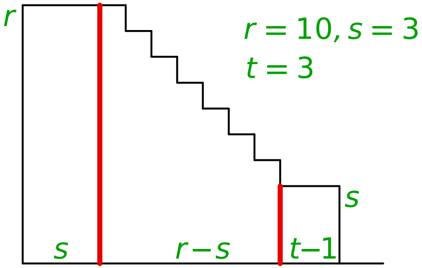

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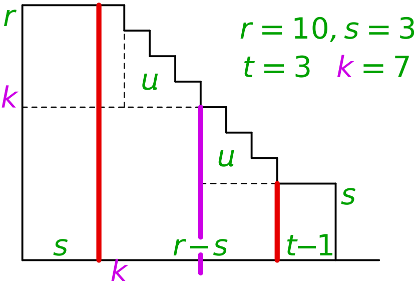

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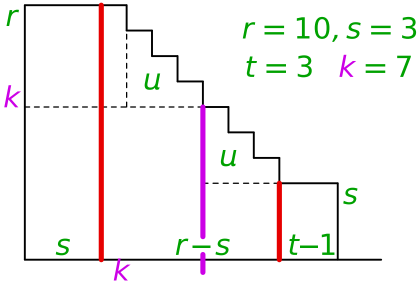

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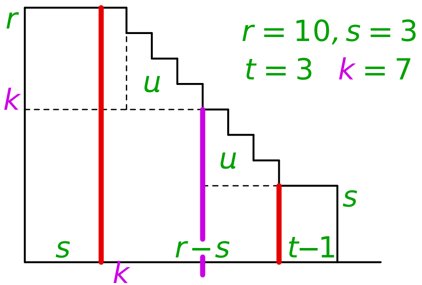

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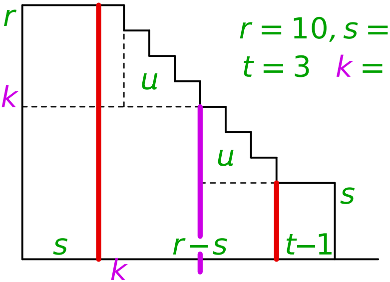

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Compare $s+r-k+1 = \left\lfloor \frac{r+s}{2} \right\rfloor + 1$ with $s(t-1) = s\left\lfloor \frac{r-s+1}{2s} \right\rfloor$.

Without Gap Constraint

Sufficiency threshold $= h = \left\lceil \frac{(r+s+1)^2}{4s} \right\rceil$

Let $x = \left\lfloor \frac{r+s}{2} \right\rfloor$. Let $y = h - x - 2$.

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• The constructions may produce lists with odd sum.
We actually obtain the sharp threshold such that every "(r, s, g)-list" satisfies all the Erdős–Gallai inequalities.

The Aigner–Triesch Method

Idea: (Aigner–Triesch [1994]) To prove that a condition Q on lists is sufficient for a property R:

1: Define a partial order P on the lists and show that if d satisfies R and $d' < d$ in P, then d' satisfies R.

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Def. A list d dominates a list d' if $\sum_{i=1}^{k} d_i \geq \sum_{i=1}^{k} d'_i$ for all k (trailing 0s added as needed).

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In any $P_{m,r,s,g,n}$, all or none satisfy the length threshold.

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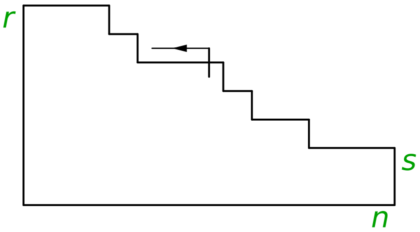

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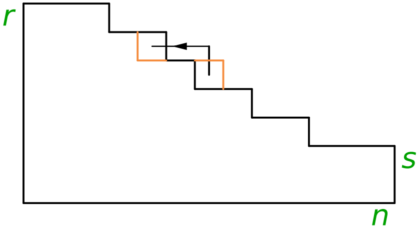

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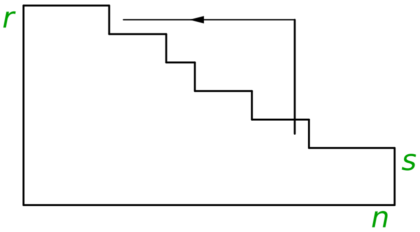

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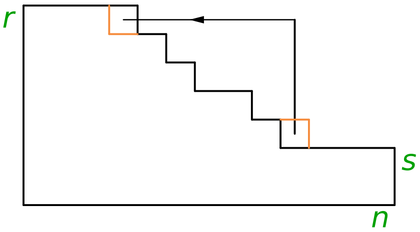

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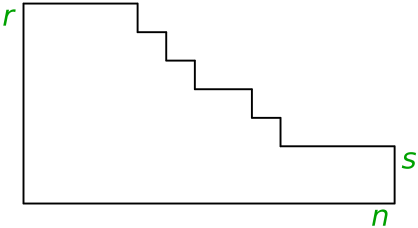

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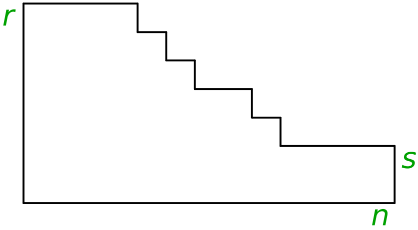

The unique maximal list has at most one repeated entry not in $\{r, s\}$.

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Pf. From $k-1$ to k, left side adds d_k, right adds $2k-2-d_k$ (given that $\min\{k, d_k\} = d_k$). ∎

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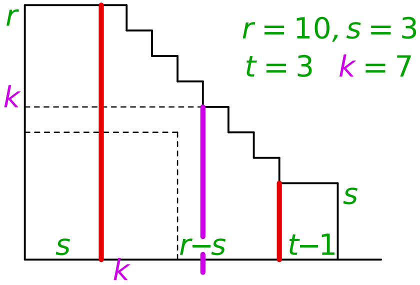
Pf. Reducing k, left side loses more than right side. ∎



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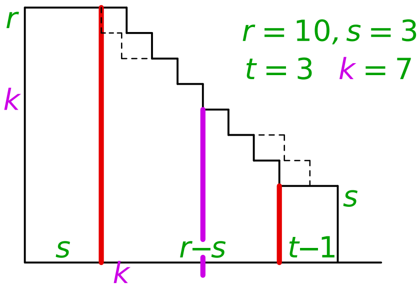

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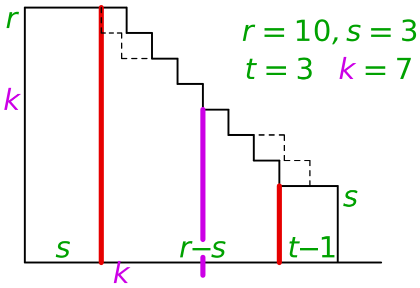

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Ex. Let $d = (3, 1, 2, 2, 0, 0)$ and $d' = (1, 3, 0, 0, 2, 2)$. The sum $(4, 4, 2, 2, 2, 2)$ is realized by $K_{2,4}$, and d and d' are graphic. In every realization of d or d', the vertex of degree 3 is adjacent to all other nonisolated vertices. Thus v_1 and v_2 are adjacent in every realization of d or d'; the lists do not pack.

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- Sauer–Spencer implies that G and G' pack when $\Delta(G) + \Delta(G') < \sqrt{2n}$, but that allows reordering vertices, so the result on packing graphic lists is stronger.

Sharpness Construction

For $m > 1$, let $n = 2m^2$. Let

$$d = (m, m, (2m)^{(m-1)}, 0^{(m-1)}, 1^{(m^2-m)}, 0^{(m^2-m)}),$$

$$d' = (m, m, 0^{(m-1)}, (2m)^{(m-1)}, 0^{(m^2-m)}, 1^{(m^2-m)}).$$

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In a realization G, let S be the $m+1$ vertices with degree > 1. Their degree-sum is $2m^2$, which equals $2\binom{m+1}{2} + (m^2 - m)$. To reach this, S must be a clique, and all other edges must join S to leaves. Thus $v_1 \leftrightarrow v_2$ in every realization of d or d', so they don't pack.

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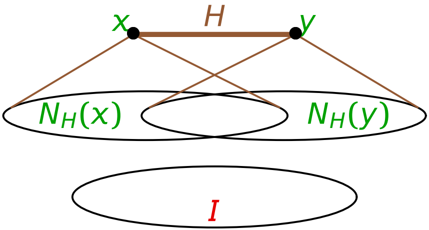

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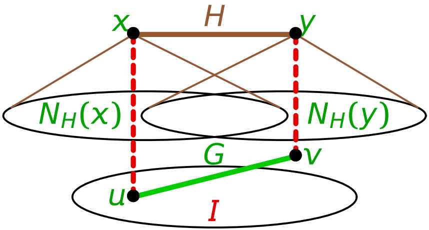

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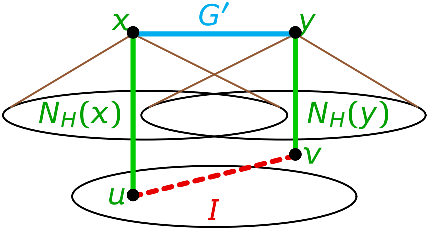

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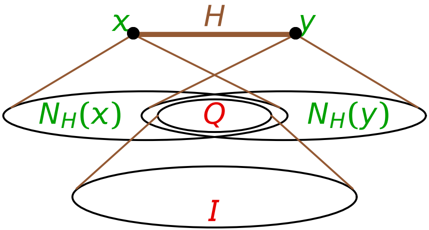
Thus $Q = N_H(I) \subseteq N_G(x) \cap N_G(y)$.



More Structure

Claim: $N_G(I)$ is a clique in H (similarly for $N_{G'}(I)$).

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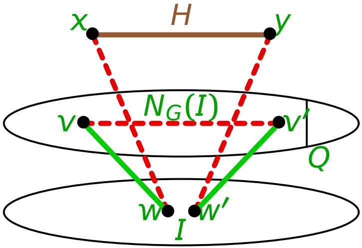

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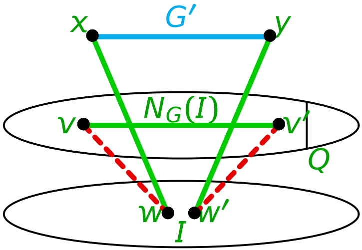

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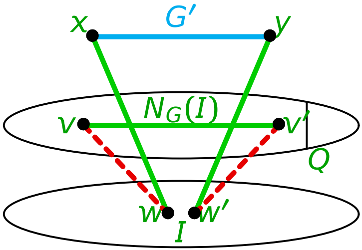

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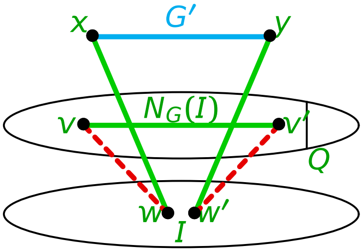

$\therefore$ Q is covered by two cliques in H. With $r = |E(H[Q])|$,

$$r \geq \binom{|Q|}{2} - \frac{|Q|^2}{4} = \frac{|Q|(|Q|-2)}{4}, \qquad (1)$$

since the complement $\overline{H}[Q]$ is bipartite.

Counting Argument

Since xy is a shared edge, $|N_H(x)|, |N_H(y)| \leq D - 1$.
Since also $Q \subseteq N_H(x) \cap N_H(y))$,

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■

Conj. Graphic n-tuples d and d' pack if always $d_i d'_i < n/2$. This stronger statement would be a more direct analogue of the Sauer–Spencer Theorem.

References

These slides can be found at
http://www.math.uiuc.edu/~west/pubs/degreet.pdf.

The proofs and results are from the papers below, found at http://www.math.uiuc.edu/~west/pubs/publink.html.

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G. Isaak and D.B. West, The edge-count criterion for degree lists, (submitted).

M.D. Barrus, S.G. Hartke, K.F. Jao, and D.B. West, Length thresholds for graphic lists with fixed largest and smallest entries and bounded gaps, (in preparation).

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