

# Degree Ramsey numbers for cycles and blowups of trees

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## Abstract

Let  $H \xrightarrow{s} G$  mean that every  $s$ -coloring of  $E(H)$  produces a monochromatic copy of  $G$  in some color class. Let the  $s$ -color degree Ramsey number of a graph  $G$ , written  $R_\Delta(G; s)$ , be  $\min\{\Delta(H) : H \xrightarrow{s} G\}$ . We prove that the 2-color degree Ramsey number is at most 96 for every even cycle and at most 3458 for every odd cycle. For even cycles, the classical Ramsey numbers yield  $R_\Delta(C_{2m}; s) \leq O(s^{m/(m-1)})$ ; as a lower bound, we show  $R_\Delta(C_4; s) \geq .007s^{14/9}$ . The constant upper bound for  $R_\Delta(C_n; 2)$  uses blowups of graphs, where the  $d$ -blowup of a graph  $G$  is the graph  $G'$  obtained by replacing each vertex of  $G$  with an independent set of size  $d$  and each edge  $e$  of  $G$  with a copy of the complete bipartite graph  $K_{d,d}$ . We also prove the existence of a function  $f$  such that if  $G'$  is the  $d$ -blowup of  $G$ , then  $R_\Delta(G'; s) \leq f(R_\Delta(G; s), s, d)$ .

## 1 Introduction

Given a target graph  $G$ , classical graph Ramsey theory seeks a graph  $H$  such that every 2-edge-coloring of  $H$  produces a monochromatic copy of  $G$ . Such a graph  $H$  is a *Ramsey host* for  $G$ ; we then write  $H \rightarrow G$  and say that  $H$  *arrows* or *forces*  $G$ . More generally, we write  $H \xrightarrow{s} G$  when every  $s$ -edge-coloring of  $E(H)$  produces a monochromatic copy of  $G$ .

For any monotone graph parameter  $\rho$ , the  $\rho$ -*Ramsey number* of  $G$ , written  $R_\rho(G; s)$ , is  $\min\{\rho(H) : H \xrightarrow{s} G\}$ . The notion has been studied with  $\rho(G)$  being the clique number [15, 24], the chromatic number [6, 30, 31], the number of edges (called the *size Ramsey number*) [2, 3, 9, 11, 16, 26], and the maximum degree [6, 20].

The *degree Ramsey number* is  $R_\Delta(G; s)$ , where  $\Delta(G)$  denotes the maximum degree of  $G$ . Burr, Erdős, and Lovász [6] began its study (for  $s = 2$ ). Kinnersley, Milans, and West [20] generalized some results of [6] and computed values and bounds on  $R_\Delta(G; s)$  for various classes of graphs. Our focus is on the degree Ramsey numbers of cycles.

Note that  $R_\Delta(G; s) \leq R(G; s) - 1$ , where  $R(G; s)$  is the classical Ramsey number, because  $K_{R(G; s)} \xrightarrow{s} G$ , and this Ramsey host is regular with degree  $R(G; s) - 1$ . Upper bounds on

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Ramsey numbers thus provide upper bounds on degree Ramsey numbers. For even cycles, such upper bounds are well known. Bondy and Simonovits [5] proved that every  $n$ -vertex graph with more than  $cn^{1+1/m}$  edges contains a  $2m$ -cycle. When  $\binom{n}{2} > scn^{1+1/m}$ , we thus have  $K_n \xrightarrow{s} C_{2m}$ . This yields  $R_\Delta(C_{2m}; s) < R(C_{2m}; s) < c_m s^{m/(m-1)}$ , where  $c_m$  is a constant depending on  $m$ . For particular  $m$ , many authors have studied these Ramsey numbers; Li and Lih [23] discussed the history and showed that  $R(C_{2m}; s) \in \Theta(s^{m/(m-1)})$  for  $m \in \{2, 3, 5\}$ .

For even cycles, the upper bounds from the classical Ramsey numbers can be improved for the degree Ramsey numbers. Our main result on cycles is a small constant upper bound on the 2-color degree Ramsey numbers for all cycles; we prove  $R_\Delta(C_n; 2) \leq 96$  when  $n$  is even (Section 2) and  $R_\Delta(C_n; 2) \leq 3458$  when  $n$  is odd (Section 4). (For exact results, it is known that  $R_\Delta(C_3; 2) = R_\Delta(C_4; 2) = 5$  [20].) In contrast, the fixed-color ordinary Ramsey numbers of cycles grow linearly with  $n$  (when  $m$  is sufficiently large,  $R(C_{2m}; 2) = 3m - 1$  and  $R(C_{2m+1}; 2) = 4m + 1$  [14, 27], while  $R(C_{2m}; 3) = 4m$  [4] and  $R(C_{2m+1}; 3) = 8m + 1$  [21]).

For 4-cycles, asymptotic results about the multicolor Ramsey numbers result from the work of Chung and Graham [7] and Irving [19]: always  $R(C_4; s) \leq s^2 + s + 1$ , and if  $s - 1$  is an prime power, then  $R(C_4; s) > s^2 - s + 1$ . More generally, Chung and Graham [7] gave an upper bound for  $R(K_{p,q}; s)$ . Using that as an upper bound and the Local Lemma to prove a lower bound, we obtain  $(\frac{1}{epq})^{1/(p+q-2)} s^{(pq-1)/(p+q-2)} \leq R_\Delta(K_{p,q}; s) < \lfloor (q-1)(s + s^{1/p})^p \rfloor$ . For  $p = q = 2$ , the lower bound is  $\frac{1}{\sqrt{4e}} s^{3/2}$ . By a more delicate argument, we improve that special case to  $R_\Delta(C_4; s) \geq .007s^{14/9}$  (Section 3). In contrast to the quadratic upper bound for 4-cycles, [20] showed  $R_\Delta(C_{2m+1}; s) \geq 2^s + 1$  for all  $m$ .

For each  $s$ , an upper bound on the  $s$ -color degree Ramsey number of cycles follows from earlier work of Haxell, Kohayakawa, and Łuczak [17]. Because their bound applies also to odd cycles, it necessarily grows at least exponentially in  $s$ . Since their work used the Regularity Lemma, it yields a very large upper bound even when  $s = 2$ .

We obtain constructions for upper bounds on  $R_\Delta(C_n; 2)$  by using the “blowup” operation. The  $d$ -blowup of a graph  $G$  is the graph  $G'$  obtained by replacing each vertex of  $G$  with an independent set of size  $d$  (called a *cluster*) and replacing each edge  $e$  of  $G$  with a copy of the complete bipartite graph  $K_{d,d}$  whose partite sets are the clusters arising from the endpoints of  $e$ . The resulting graph has also been called the “ $d$ -multiplication” of  $G$ .

Blowups also shed some light on a fundamental question in degree Ramsey theory.

**Question 1.1.** Is  $R_\Delta(G; s)$  bounded by some function of  $\Delta(G)$  and  $s$ ?

By analogy with Gyárfás’ concept of  $\chi$ -bounded graph families, we say that a family  $\mathcal{G}$  of graphs is  $R_\Delta$ -bounded if there is a function  $f$  such that  $R_\Delta(G; s) \leq f(\Delta(G), s)$  whenever  $G \in \mathcal{G}$  and  $s \geq 2$ . Question 1.1 asks whether the family of all graphs is  $R_\Delta$ -bounded.

In Section 5, we prove the existence of a function  $h$  such that if  $G'$  is the  $d$ -blowup of  $G$ , then  $R_\Delta(G'; s) \leq h(R_\Delta(G; s), s, d)$ . As a result, if  $\mathcal{G}$  is an  $R_\Delta$ -bounded family of graphs, then the family of  $d$ -blowups of graphs in  $\mathcal{G}$  is also  $R_\Delta$ -bounded.

A *closed  $k$ -blowup* of a graph  $G$  is obtained from  $G$  by replacing each vertex with a complete graph on  $k$  vertices and each edge with a complete bipartite graph. Horn, Milans, and Rödl [18] proved that the family of closed blowups of trees is  $R_\Delta$ -bounded. In particular, let  $k$  and  $s$  be integers with  $k \geq 2$  and  $s \geq 1$ , and let  $r = R(K_{2k}; s)$ . If  $T$  is a tree with  $|V(T)| \geq 3$  and  $G$  is the closed  $k$ -blowup of  $T$ , then  $R_\Delta(G; s) \leq (r - 1) \left(2 \binom{r}{2k} (\Delta(T) - 1)\right)^{\binom{r-1}{2k-1}}$ . This bound is quite a bit larger than the bound we obtain for the ordinary  $k$ -blowup in Section 5.

The degree Ramsey problem is much simpler for paths than for cycles. Alon, Ding, Oporowski, and Vertigan [1] gave a short proof that  $R_\Delta(P_n; s) \leq 2s$  for all  $n$ . By a more difficult argument, they showed that equality holds when  $n$  is sufficiently large in terms of  $s$ . For  $s = 2$ , Thomassen [29] showed that  $R_\Delta(P_n; 2) = 4$  for  $n \geq 7$ , but it remains open whether  $R_\Delta(P_6; 2)$  is 3 or 4. For short paths, an old result of Egawa, Urabe, Fukuda, and Nagoya [10] yields  $R_\Delta(P_4; s) \leq 2s - 3$  for  $s \geq 4$ , and [20] shows  $R_\Delta(P_4; s) \geq s + 1$ .

The argument of [1] generalizes to give upper bounds for trees. We will use the ideas of Theorem 1.2 in our upper bounds on  $R_\Delta(C_n; 2)$ . The subsequent Corollary 1.4 is relevant to our comments on  $R_\Delta$ -bounded families.

**Theorem 1.2.** *If  $T$  is a tree and  $H$  is a graph with average degree at least  $2s(\Delta(T) - 1)$  and girth larger than  $2|V(T)|$ , then  $H \xrightarrow{s} T$ .*

*Proof.* Let  $r = \Delta(T)$ , and consider an  $s$ -edge-coloring of  $H$ . By the pigeonhole principle, some color class has average degree at least  $2(r - 1)$ . It is well known that when  $r$  is an integer greater than 1, any graph with average degree at least  $2(r - 1)$  contains a subgraph with minimum degree at least  $r$ .

Thus  $H$  contains a monochromatic subgraph  $H'$  with minimum degree at least  $r$ . Let  $u$  be a vertex in  $H'$ . Because  $H'$  has minimum degree at least  $r$  and girth larger than  $2|V(T)|$ , the subgraph of  $H'$  induced by  $\{v : \text{dist}_{H'}(v, u) \leq |V(T)|\}$  contains  $T$ .  $\square$

The argument of Erdős and Sachs [13] for regular graphs with large degree and girth can be modified easily to yield a bipartite analogue.

**Lemma 1.3.** *For each  $g$  and  $k$ , there is a  $k$ -regular bipartite graph with girth at least  $g$ .*

A bound for trees follows immediately from Theorem 1.2.

**Corollary 1.4.** *If  $T$  is a tree, then  $R_\Delta(T; s) \leq 2s(\Delta(T) - 1)$ .*

The lower bound of Alon *et al.* [1] that  $R_\Delta(P_n; s) \geq 2s$  shows that Corollary 1.4 is sharp when  $T$  is a path. In [20], it is shown that for each  $\varepsilon > 0$ , there is an  $s$  such that for all sufficiently large  $k$ , there exists a tree  $T$  with  $\Delta(T) = k$  such that  $R_\Delta(T; s) \geq (2 - \varepsilon)s(k - 1)$ .

By Corollary 1.4, within the family of trees, the degree Ramsey number is bounded by a function of the number of colors and the maximum degree. Section 5 extends this statement to larger families.

## 2 Upper Bound for Even Cycles

As noted earlier, the result of Haxell, Kohayakawa, and Łuczak [17] yields a very large constant  $c$  such that  $R_{\Delta}(C_n; 2) \leq c$  for all  $n$ . In this section, we prove that  $R_{\Delta}(C_{2n}; 2) \leq 96$ .

For a connected bipartite graph  $F$  with partite sets of size  $\ell$ , an  $F$ -blowup of a graph  $G$  is a graph  $H$  obtained from  $G$  by replacing each vertex in  $G$  with an independent set of size  $\ell$  and making each edge in  $G$  expand into a copy of  $F$ . By allowing the vertices of the cluster from a vertex  $u$  in  $G$  to represent each vertex of  $F$  equally often as the edges at  $u$  expand into copies of  $F$ , we can keep the maximum degree of the blowup small.

**Lemma 2.1.** *Let  $F$  be a connected bipartite graph with partite sets of size  $\ell$  and average degree  $d$ . Let  $G$  be a  $k$ -regular graph, where  $k$  is a multiple of  $\ell$ . If  $G$  is bipartite and  $kd$  is an integer, then there is a  $kd$ -regular  $F$ -blowup of  $G$ .*

*Proof.* We construct a blowup  $G'$  of  $G$ , expanding each vertex of  $G$  into a cluster of size  $\ell$  and each edge of  $G$  into a copy of  $F$ . Each edge of  $G$  becomes  $\ell d$  edges in  $G'$ . Since  $G'$  has  $\ell|V(G)|$  vertices and  $\ell dk|V(G)|/2$  edges, it suffices to ensure that  $G'$  is regular.

Since  $G$  is bipartite and regular, it decomposes into 1-factors, providing a  $k$ -coloring of  $E(G)$  that uses each color once at each vertex. Let  $A$  and  $B$  be the partite sets of  $F$ . If the partite sets of  $G$  are  $X$  and  $Y$ , then for each edge of  $G$  the endpoint in  $X$  expands into a cluster that represents  $A$ , and the other endpoint represents  $B$ .

Index the vertices of  $A$ ,  $B$ , and each cluster using indices  $1, \dots, \ell$ . For the copy of  $F$  in  $G'$  replacing an edge of  $G$  having the  $i$ th color, the vertices in the cluster for each endpoint represent the corresponding partite set of  $F$  with a circular shift of  $i$  on the indices. Since  $k$  is a multiple of  $\ell$ , each vertex represents each vertex of the corresponding partite set  $k/\ell$  times over all  $k$  colors. Hence all vertices of  $G'$  have the same degree.  $\square$

The same conclusion holds when  $G$  is not bipartite but  $k$  is a multiple of  $2\ell$ , using a decomposition of  $G$  into 2-regular spanning subgraphs instead of into matchings. We omit the details, since we will not need that case.

To obtain monochromatic even cycles, it will suffice to obtain a monochromatic  $P_4$ -blowup of a long path. Let  $\langle v_1, \dots, v_k \rangle$  denote the path with vertices  $v_1, \dots, v_k$  in order.

**Lemma 2.2.** *If  $H$  is a  $P_4$ -blowup of  $P_{t+1}$ , then  $H$  contains  $C_{2t}$ .*

*Proof.* For the path  $\langle z_1, \dots, z_{t+1} \rangle$ , let the corresponding clusters be  $Z_1, \dots, Z_{t+1}$  in  $H$ ; each  $Z_i$  has size 2. Joining  $Z_j$  and  $Z_{j+1}$  is a copy of  $P_4$  and hence a matching of size 2. Linking these matchings from  $Z_2$  through  $Z_t$  yields two disjoint copies of  $P_{t-1}$ . Because there is a copy of  $P_4$  joining  $Z_1$  and  $Z_2$ , some vertex in  $Z_1$  is adjacent to both vertices in  $Z_2$ . Similarly, some vertex in  $Z_{t+1}$  is adjacent to both vertices in  $Z_t$ . These vertices extend the union of the two copies of  $P_{t-1}$  to a  $2t$ -cycle.  $\square$

We next show how to force monochromatic  $P_4$ -blowups of long paths. An  $(X, Y)$ -*bigraph* is a bipartite graph with partite sets  $X$  and  $Y$ .

**Lemma 2.3.** *Let  $F$  be a bipartite graph with parts of size  $\ell$  such that  $F \xrightarrow{s} P_4$ . Let  $H_0$  be a  $2s\binom{\ell}{2}$ -regular  $(X, Y)$ -bigraph with girth more than  $2t$ . If  $H$  is an  $F$ -blowup of  $H_0$ , then every  $s$ -edge-coloring of  $H$  contains a monochromatic  $P_4$ -blowup of a  $t$ -vertex path in  $H_0$ .*

*Proof.* We use an  $s$ -edge-coloring of  $H$  to construct an  $s\binom{\ell}{2}$ -edge-coloring of  $H_0$ . For each vertex  $u$  in  $H_0$ , fix an indexing  $\{u^1, \dots, u^\ell\}$  of the cluster in  $H$  corresponding to  $u$ . Consider an edge  $uv \in E(H_0)$  with  $u \in X$  and  $v \in Y$ ; we assign a color to  $uv$ . Let  $F_{uv}$  be the copy of  $F$  in  $H$  arising from the edge  $uv$  in  $H_0$ .

Since  $F \xrightarrow{s} P_4$ , in  $F_{uv}$  we find a monochromatic copy  $P$  of  $P_4$ . We color  $uv$  with a 3-tuple. The first coordinate is the color of  $E(P)$ . The second coordinate is the pair of indices  $\{i, j\}$  such that the vertices of  $P$  in the cluster for  $u$  are  $u^i$  and  $u^j$ . Similarly, the third coordinate records the indices of the vertices of  $P$  in the cluster for  $v$ . With  $s$  choices for the first coordinate and  $\binom{\ell}{2}$  choices for the second and third,  $s\binom{\ell}{2}$  colors are used on  $E(H_0)$ .

Since  $H_0$  has girth larger than  $2t$ , Theorem 1.2 implies  $H_0 \xrightarrow{s'} P_t$ , where  $s' = s\binom{\ell}{2}$ . A monochromatic copy of  $P_t$  in  $H_0$  yields a monochromatic  $P_4$ -blowup of  $P_t$  in  $H$ .  $\square$

**Theorem 2.4.** *If there is a bipartite  $s$ -color Ramsey host for  $P_4$  with parts of size  $\ell$  and average degree  $d$ , then  $R_\Delta(C_{2n}; s) \leq 2sd\binom{\ell}{2}$ .*

*Proof.* Let  $F$  be an  $s$ -color Ramsey host for  $P_n$  with parts of size  $\ell$  and average degree  $d$ . Let  $H_0$  be a  $2s\binom{\ell}{2}$ -regular bipartite graph with girth more than  $2(n+1)$ . By Lemma 2.1, there is a  $2sd\binom{\ell}{2}$ -regular  $F$ -blowup  $H$  of  $H_0$ . We claim that  $H \xrightarrow{s} C_{2n}$ . By Lemma 2.3, any  $s$ -edge-coloring of  $H$  contains a monochromatic  $P_4$ -blowup of  $P_{n+1}$ . By Lemma 2.2, such a coloring contains a monochromatic copy of  $C_{2n}$ .  $\square$

**Corollary 2.5.**  $R_\Delta(C_{2m}; s) \leq 16s^6$ .

*Proof.* A forest of stars in  $K_{2s, 2s}$  has at most  $4s - 2$  edges, but every  $s$ -edge-coloring puts at least  $4s$  edges in some color class. Hence  $K_{2s, 2s} \xrightarrow{s} P_4$ . Now apply Theorem 2.4 with  $\ell = d = 2s$ .  $\square$

The exponent in Corollary 2.5 seems weak in light of  $R_\Delta(C_{2m}; s) \leq R(C_{2m}; s) \leq c_m s^{m/(m-1)}$ , but the latter is for fixed  $m$  as  $s$  grows, while Corollary 2.5 is a statement for all  $m$  and  $s$ . In particular, for fixed  $s$  it yields a constant upper bound for all even cycles.

For  $s \leq 3$ , our main result of this section reduces that constant. For an edge-transitive graph  $G$ , let  $G^-$  be the graph obtained by deleting any one edge.

**Corollary 2.6.**  $R_\Delta(C_{2n}; 2) \leq 96$  and  $R_\Delta(C_{2n}; 3) \leq 864$ .

*Proof.* The bounds follow from Theorem 2.4 after showing  $K_{3,3}^- \rightarrow P_4$  and  $K_{4,4} \xrightarrow{3} P_4$ . (Unfortunately,  $K_{4,4}^-$  is not a 3-color Ramsey host for  $P_4$ . There is a 3-edge-coloring in which the graph in each color is  $2P_3 + P_2$ .)

A bipartite graph contains  $P_4$  unless it is a forest of stars. In  $K_{3,3}$  a forest of stars has at most four edges. Hence  $K_{3,3}^- \rightarrow P_4$  unless the graph in each color is  $2P_3$ . The centers of the two stars in one color have only each other as a possible remaining neighbor; hence they are nonadjacent in  $K_{3,3}^-$ . Now the remaining graph for the other color is  $C_4$ , not  $2P_3$ .

Since  $K_{4,4}$  has 16 edges, in any 3-edge-coloring of  $K_{4,4}$  some color class has at least six edges. The largest forest of stars has six edges, occurring only as  $2K_{1,3}$ . In this case, the graph for the two remaining colors contains  $K_{3,3}$ , forcing  $P_4$ .  $\square$

### 3 Lower Bound for Bipartite Graphs

Before we leave the realm of even cycles, we obtain lower bounds for degree Ramsey numbers of 4-cycles and of bipartite graphs more generally. We have observed that upper bounds for  $R(G; s)$  yield upper bounds for  $R_\Delta(G; s)$ . In particular, Chung and Graham [7] proved  $R(K_{p,q}; s) \leq (q-1)(s+s^{1/p})^p$ . For  $p=2$ , this was sharpened to  $R(K_{2,q}; s) \leq (q-1)s^2 + s + 2$ . For  $p=q=2$ , the upper bound is strict. We use the Local Lemma to provide lower bounds. The resulting lower bound is asymptotically stronger than the original lower bound by Chung and Graham on  $R(K_{p,q}; s)$ .

**Theorem 3.1.**  $\left(\frac{1}{epq}\right)^{1/(p+q-2)} s^{(pq-1)/(p+q-2)} \leq R_\Delta(K_{p,q}; s) < \lfloor (q-1)(s+s^{1/p})^p \rfloor$ .

With a stronger upper bound,  $\frac{1}{\sqrt{4e}} s^{3/2} \leq R_\Delta(C_4; s) \leq s^2 + s$ .

*Proof.* The upper bounds follow from  $R_\Delta(G; s) < R(G; s)$ . For the lower bound, consider any graph  $H$  with maximum degree less than  $cs^{(pq-1)/(p+q-2)}$ , where  $c = \left(\frac{1}{epq}\right)^{1/(p+q-2)}$ . Color the edges of  $H$  independently at random from  $s$  colors. The probability that any copy of  $K_{p,q}$  is monochromatic is  $1/s^{pq-1}$ .

The number of copies of  $K_{p,q}$  containing a given edge of  $H$  is at most  $\binom{\Delta(H)-1}{p-1} \binom{\Delta(H)-1}{q-1}$ , which is less than  $\Delta(H)^{p+q-2}$ . Thus the event that a particular copy of  $K_{p,q}$  is monochromatic is mutually independent of a set of all but fewer than  $pq[\Delta(H)]^{p+q-2}$  other events. Since  $\Delta(H) \leq \left(\frac{1}{epq} s^{pq-1}\right)^{1/(p+q-2)}$ , the Local Lemma guarantees an  $s$ -coloring of  $E(H)$  with no monochromatic copy of  $K_{p,q}$ .  $\square$

With a more delicate argument, we can improve the lower bound on  $R_\Delta(C_4; s)$  for large  $s$ . For convenience, we will call an edge-coloring that contains no monochromatic  $C_4$  a  $C_4$ -free

coloring. Recall that if  $k - 1$  is a prime power, then  $R(C_4; k) \geq k^2 - k + 2$  ([7, 19]; see [22] for a slight improvement). This implies (with room to spare) that  $K_n$  has a  $C_4$ -free edge-coloring using at most  $\lfloor 4\sqrt{n} \rfloor$  colors.

**Lemma 3.2.** *Every bipartite graph with maximum degree at most  $d$  has a  $C_4$ -free edge-coloring using fewer than  $8.7d^{9/14}$  colors.*

*Proof.* Let  $H$  be such a graph, with partite sets  $X$  and  $Y$ . We decompose  $H$  into two subgraphs  $H'$  and  $H''$ , which will be colored using disjoint sets of colors. The claim clearly holds when  $d \leq 2$ , so we may assume  $d \geq 3$ .

We form  $H'$  by iteratively including certain subgraphs with “large” minimum degree. Let  $H_1 = H$ . As long as  $H_i$  has an edge that lies in at least  $2d^{27/14}$  4-cycles, we will extract the edges of a subgraph  $F_i$ , letting  $H_{i+1} = H_i - E(F_i)$ . Eventually the process must end with a subgraph having no such edge, say  $H_m$ . We then let  $H' = \bigcup_{i=1}^{m-1} F_i$  and  $H'' = H_m$ .

While  $i < m$ , determine  $F_i$  as follows. Let  $xy$  be an edge in at least  $2d^{27/14}$  4-cycles in  $H_i$ . These 4-cycles correspond to edges in the subgraph  $L_{xy}$  induced by  $N(x) \cup N(y) - \{x, y\}$ . Since  $\Delta(H) \leq d$ , there are fewer than  $2d$  vertices in  $L_{xy}$ , so its average degree is at least  $2d^{13/14}$ . Every nontrivial graph  $G$  has a subgraph whose minimum degree is at least half the average degree in  $G$ . Let  $F_i$  be a subgraph of  $L_{xy}$  having minimum degree at least  $d^{13/14}$ .

We color the edges of the residual graph  $H''$  as in the proof of Theorem 3.1. By construction, each edge of  $H''$  lies in fewer than  $2d^{27/14}$  4-cycles in  $H''$ . In a random  $s$ -coloring, each 4-cycle has probability  $1/s^3$  of being monochromatic. The event that a particular 4-cycle is monochromatic is mutually independent of a set of all but fewer than  $4(2d^{27/14})$  other such events. When  $e(8d^{27/14})/s^3 \leq 1$ , the Local Lemma ensures that some outcome has no monochromatic 4-cycle. The inequality holds for some integer  $s$  less than  $3d^{9/14}$  when  $d \geq 3$ .

Now consider  $H'$ , the union of  $F_1, \dots, F_{m-1}$ . We have noted that  $|V(F_i)| < 2d$ . As noted before the proof,  $F_i$  therefore has a  $C_4$ -free edge-coloring  $c_i$  using at most  $\lfloor 4\sqrt{2d} \rfloor$  colors. Since  $\delta(F_i) \geq d^{13/14}$ , each vertex of  $H'$  lies in at most  $d^{1/14}$  of these subgraphs. Consider  $uv \in E(H')$ , where  $u \in X$ ,  $v \in Y$ , and  $uv \in E(F_i)$ . Let  $\phi(uv) = (a, b, c_i(uv))$ , where  $a = |\{j: u \in V(F_j) \text{ and } j < i\}|$  and  $b = |\{j: v \in V(F_j) \text{ and } j < i\}|$ . The number of colors used by the edge-coloring  $\phi$  of  $H'$  is less than  $(d^{1/14})^2 \cdot 4\sqrt{2d}$ , which is less than  $5.7d^{9/14}$ .

We claim that  $\phi$  is  $C_4$ -free. If the first two coordinates of  $\phi$  agree on two incident edges  $e$  and  $e'$ , then the common vertex of these edges has appeared in the same number of previous subgraphs, so  $e$  and  $e'$  belong to the same subgraph  $F_i$ . Since  $c_i$  is not constant on any 4-cycle in  $F_i$ , also  $\phi$  is not.

Thus  $\phi$  and the coloring obtained for  $H''$  together define a  $C_4$ -free edge-coloring of  $H$  using fewer than  $8.7d^{9/14}$  colors.  $\square$

**Lemma 3.3.** *Every graph with maximum degree at most  $d$  has a  $C_4$ -free edge-coloring using fewer than  $24.2d^{9/14}$  colors.*

*Proof.* We use induction on  $d$ . By Theorem 3.1,  $H$  has a  $C_4$ -free  $s$ -coloring when  $d < (1/\sqrt{4e})s^{3/2}$ , or equivalently when  $s > (4e)^{1/3}d^{2/3}$ . If  $d \leq (10.9)^{42}$ , then  $\lceil (4e)^{1/3}d^{2/3} \rceil < \lfloor 24.2d^{9/14} \rfloor$ , and the desired coloring exists.

For larger  $d$ , let  $H_1$  be a largest bipartite subgraph of  $H$ ; let  $X_1$  and  $Y_1$  be its partite sets. By the choice of  $H_1$ , each vertex of  $H$  has at least as many neighbors in the opposite partite set as in its own. Hence  $H[X_1]$  and  $H[Y_1]$  both have maximum degree at most  $d/2$ .

By the induction hypothesis, each of  $H[X_1]$  and  $H[Y_1]$  has a  $C_4$ -free edge-coloring using fewer than  $24.2(d/2)^{9/14}$  colors. Since they share no vertices, their union has such a coloring. Since  $H_1$  is bipartite with maximum degree at most  $d$ , Lemma 3.2 gives  $H_1$  a  $C_4$ -free edge-coloring using fewer than  $9d^{9/14}$  colors. The two colorings together form a  $C_4$ -free edge-coloring of  $H$  using fewer than  $9d^{9/14} + 24.2(d/2)^{9/14}$  colors. Since  $9 + 24.2/2^{9/14} < 24.2$ , the proof is complete.  $\square$

**Theorem 3.4.**  $R_\Delta(C_4; s) \geq .007s^{14/9}$ .

*Proof.* By Lemma 3.3, every graph with maximum degree at most  $d$  has a  $C_4$ -free edge-coloring using fewer than  $24.2d^{9/14}$  colors. When  $d = \lfloor .007s^{14/9} \rfloor$ , we have  $24.2d^{9/14} \leq s$ .  $\square$

We have seen that  $R(C_4; s)$  is approximately  $s^2$  for infinitely many  $s$  ([7]). Although this is an upper bound for  $R_\Delta(C_4; s)$ , as a lower bound we have only been able to prove  $\Omega(s^{14/9})$ . This prompts the natural question of determining the order of growth of  $R_\Delta(C_4; s)$  and also the following general question:

**Question 3.5.** For which graphs  $G$  is it true that  $\lim_{s \rightarrow \infty} \frac{R_\Delta(G; s)}{R(G; s)} = 0$ ?

## 4 Upper Bound for Odd Cycles

To prove  $R_\Delta(C_n; 2) \leq 3458$  for all  $n$ , we need another simple fact about  $K_{3,3}^-$  and  $P_4$ .

**Proposition 4.1.** *Deleting any one vertex from each part of a bipartition of  $K_{3,3}^-$  leaves a copy of  $P_4$ .*

*Proof.* Deleting one vertex from each part yields  $K_{2,2}$  or  $K_{2,2}^-$ , each of which contains  $P_4$ .  $\square$

To complete our proof for odd cycles, we need variations on Lemma 2.2. Let  $[n]$  denote  $\{1, \dots, n\}$ .

**Lemma 4.2.** *Let  $H$  contain a  $P_4$ -blowup of a path  $\langle z_1, \dots, z_t \rangle$ , where each vertex  $z_j$  expands into a set  $Z_j$  of size 2. If the vertices of  $Z_1$  are adjacent in  $H$ , then  $H$  contains cycles of every odd length from 3 to  $2t - 1$ .*

*Proof.* For  $1 \leq j \leq t - 1$ , a copy of  $P_4$  in  $H$  with vertex set  $Z_j \cup Z_{j+1}$  contains a matching of  $Z_j$  into  $Z_{j+1}$ . Choose  $k \in [t - 1]$ . Linking the matchings from  $Z_1$  through  $Z_k$  yields two disjoint copies of  $P_k$ . In the copy of  $P_4$  joining  $Z_k$  and  $Z_{k+1}$ , some vertex in  $Z_{k+1}$  is adjacent to both vertices in  $Z_k$ , linking the two copies of  $P_k$  to form a path of length  $2k$  with both endpoints in  $Z_1$ . The edge within  $Z_1$  completes a cycle of length  $2k + 1$ .  $\square$

**Lemma 4.3.** *Fix  $t \in \mathbb{N}$  with  $t \geq 5$ . Let  $H$  contain a  $P_4$ -blowup of a path  $\langle z_1, \dots, z_t \rangle$ , where each vertex  $z_j$  expands to a set  $Z_j$  of two vertices in  $H$ . If  $H$  contains an edge  $e$  joining  $Z_1$  and  $Z_3$ , then  $H$  contains cycles of every odd length from 7 to  $2t - 3$ .*

*Proof.* As in the proof of Lemma 4.2, the matchings from  $Z_1$  through  $Z_t$  yield disjoint copies of  $P_t$ . Let  $Z_j = \{z_j^1, z_j^2\}$ , indexed so that  $\langle z_1^1, \dots, z_t^1 \rangle$  and  $\langle z_1^2, \dots, z_t^2 \rangle$  are paths in  $H$ .

For  $4 \leq k \leq t$ , some vertex in  $Z_k$  is adjacent to both  $z_{k-1}^1$  and  $z_{k-1}^2$  and completes a path of length  $2k - 6$  with  $\langle z_3^1, \dots, z_{k-1}^1 \rangle$  and  $\langle z_3^2, \dots, z_{k-1}^2 \rangle$ . It therefore suffices to prove that there is a path of length 3 or length 5 joining  $z_3^1$  and  $z_3^2$  in the subgraph of  $H$  induced by  $Z_1 \cup Z_2 \cup Z_3$ . (In the first case, we obtain paths of lengths 5 through  $2t - 3$ ; in the second case, we obtain lengths 7 through  $2t - 1$ .)

If  $e$  has endpoints with different superscripts, then we may assume  $e = z_1^1 z_3^2$ , and the desired path is  $\langle z_3^1, z_2^1, z_1^1, z_3^2 \rangle$ . Otherwise, we may assume  $e = z_1^1 z_3^1$ . The copy of  $P_4$  joining  $Z_1$  and  $Z_2$  uses either  $z_1^1 z_2^2$  or  $z_2^1 z_1^2$ . In the first case, the desired path is  $\langle z_3^1, z_1^1, z_2^2, z_3^2 \rangle$ . Otherwise, it is  $\langle z_3^1, z_1^1, z_2^1, z_1^2, z_2^2, z_3^2 \rangle$ .  $\square$

The *square* of a graph  $G$ , denoted  $G^2$ , is the graph with vertex set  $V(G)$  in which two vertices are adjacent if the distance between them in  $G$  is at most 2. Note that  $G \subseteq G^2$ .

**Theorem 4.4.**  $R_\Delta(C_n; 2) \leq 3458$ .

*Proof.* When  $n$  is even, the result follows from Theorem 2.5. If  $n \in \{3, 5\}$ , then  $n$  is small enough so that a complete graph with sufficiently small maximum degree is a Ramsey host for  $C_n$ . We thus assume that  $n$  is odd and at least 7.

Let  $H_0$  be a 36-regular  $(X_0, Y_0)$ -bigraph with girth at least  $2t$ , where  $t$  is sufficiently large in terms of  $n$ . Let  $H_1 = H_0^2$ . Since  $H_0$  is bipartite, the extra edges in  $H_1$  have both endpoints in the same partite set of  $H_0$ . Each vertex in  $H_1$  is incident to 36 edges that cross the bipartition of  $H_0$  and to  $36 \cdot 35$  edges inside its partite set, since  $H_0$  has no triangles. Thus  $H_1$  is 1296-regular. By Lemma 2.1, there is a 3456-regular  $K_{3,3}^-$ -blowup  $H_2$  of  $H_1$ . Let  $H$  be the 3458-regular graph obtained from  $H_2$  by adding a triangle on each cluster in  $H_2$ .

Let  $X$  be the union of the clusters in  $H_2$  arising from  $X_0$ , and let  $Y$  be the union of the clusters in  $H_2$  arising from  $Y_0$ . Note that  $X$  and  $Y$  partition  $V(H)$ , and the subgraph of  $H$  given by the cut  $[X, Y]$  is a  $K_{3,3}^-$ -blowup of  $H_0$ .

We claim that  $H \rightarrow C_n$ . Consider a 2-edge-coloring of  $H$ . Apply Lemma 2.3 to the cut  $[X, Y]$  to obtain a monochromatic  $P_4$ -blowup of a path  $\langle z_1, \dots, z_t \rangle$  in  $H_0$  that alternates between  $X_0$  and  $Y_0$ . Let red be the color of this  $P_4$ -blowup, and let blue be the other color. For  $1 \leq j \leq t$ , let  $Z_j$  be the 3-vertex cluster in  $H$  corresponding to  $z_j$ . Let  $z_j^1$  and  $z_j^2$  be the vertices of  $Z_j$  used in the  $P_4$ -blowup, and let  $Z'_j = \{z_j^1, z_j^2\}$ .

By construction,  $z_j^1 z_j^2 \in E(H)$ ; call these edges *internal*. Since the distance in  $H_0$  between  $z_j$  and  $z_{j+2}$  is 2, in  $H_1$  they are adjacent, and hence in  $H$  the clusters  $Z_j$  and  $Z_{j+2}$  are joined by a copy of  $K_{3,3}^-$ . By Proposition 4.1, in  $H$  there is a (not necessarily monochromatic) copy of  $P_4$  joining  $Z'_j$  and  $Z'_{j+2}$ . Call the edges of these copies of  $P_4$  *skip edges*.

If the internal edge in  $Z'_j$  is red, then Lemma 4.2 applies to the longer of  $\langle z_j, \dots, z_t \rangle$  and  $\langle z_j, \dots, z_1 \rangle$  to yield a red  $n$ -cycle. If some skip edge joining  $Z'_j$  and  $Z'_{j+2}$  is red, then Lemma 4.3 applies to the longer of  $\langle z_j, \dots, z_t \rangle$  and  $\langle z_{j+2}, \dots, z_1 \rangle$  to yield a red  $n$ -cycle.

Hence we may assume that every internal edge and every skip edge is blue. Now  $H$  contains a blue  $P_4$ -blowup of the path  $\langle z_1, z_3, z_5, \dots, z_n \rangle$ . Because the internal edges are blue, Lemma 4.2 yields a blue  $n$ -cycle.  $\square$

## 5 $R_\Delta$ -bounded Families

In this section we prove that the family of all (open) blowups of trees is  $R_\Delta$ -bounded. Since the family of trees is  $R_\Delta$ -bounded, by Corollary 1.4, this is an immediate consequence of our main result in this section, which is a bound on the  $s$ -color degree Ramsey number of a  $d$ -blowup of  $G$  in terms of  $d$ ,  $s$ , and  $R_\Delta(G; s)$ .

Our proof is similar to the proof of Burr, Erdős, and Lovász [6] that the chromatic Ramsey number of a graph equals the ordinary Ramsey number of its family of homomorphic images. The main tool is the bipartite version of Ramsey's Theorem, perhaps observed first by Erdős and Rado [12]: for each  $d$  and  $s$ , there is an integer  $m$  such that  $K_{m,m} \xrightarrow{s} K_{d,d}$ . Let  $B_s(d)$  be the least  $m$  such that  $K_{m,m} \xrightarrow{s} K_{d,d}$ . Currently, the best known bound for  $s = 2$  is  $B(d) \leq (1 + o(1))2^{d+1} \log_2 d$  (Conlon [8]).

Let  $B_s^k$  be the iterated composition of  $B_s$  with itself  $k$  times. For example,  $B_s^2(d) = B_s(B_s(d))$  and  $B_s^0(d) = d$ .

**Theorem 5.1.** *If  $k = R_\Delta(G; s)$  and  $G'$  is the  $d$ -blowup of  $G$ , then  $R_\Delta(G'; s) \leq kB_s^{k+1}(d)$ . In particular, the family of blowups of graphs in a  $R_\Delta$ -bounded family is  $R_\Delta$ -bounded.*

*Proof.* Let  $H$  be a graph with maximum degree  $k$  such that  $H \xrightarrow{s} G$ , let  $m_j = B_s^j(d)$  for  $j \geq 0$ , and let  $H'$  be the  $m_{k+1}$ -blowup of  $H$ . We show that  $H' \xrightarrow{s} G'$ .

Fix an  $s$ -edge-coloring of  $H'$ . By Vizing's Theorem [28], the edges of  $H$  can be partitioned into  $k+1$  matchings  $M_1, \dots, M_{k+1}$ . Each edge  $uv$  in  $H$  corresponds to a copy of  $K_{m_{k+1}, m_{k+1}}$  in  $H'$ . Because  $m_{k+1} = B_s(m_k)$ , each  $s$ -edge-coloring of  $K_{m_{k+1}, m_{k+1}}$  contains a monochromatic

copy of  $K_{m_k, m_k}$ . For each edge  $uv$  in  $M_{k+1}$ , find a monochromatic copy of  $K_{m_k, m_k}$  in its blowup in  $H'$ . In the clusters for  $u$  and  $v$ , retain only the vertices in this subgraph, and for each vertex not incident to an edge of  $M_{k+1}$ , retain an arbitrary set of  $m_k$  vertices in its cluster. Since  $M_{k+1}$  is a matching, from each cluster we discard vertices only once; the resulting induced subgraph  $H'_k$  of  $H'$  has  $m_k$  vertices from each cluster.

Iterate this process for the remaining matchings  $M_k, \dots, M_1$ , obtaining successive induced subgraphs  $H'_{k-1}, \dots, H'_0$ . In the final subgraph  $H'_0$  of  $H'$ , the clusters correspond to the vertices of  $H$ , each cluster consists of  $d$  vertices, and each copy of  $K_{d,d}$  joining two clusters is monochromatic in the given  $s$ -edge-coloring.

The edge-colored graph  $H'_0$  corresponds to an  $s$ -edge coloring of  $H$  by assigning  $uv$  the color of the edges in the copy of  $K_{d,d}$  joining the  $u$ -cluster and  $v$ -cluster in  $H'_0$ . Because  $H \xrightarrow{s} G$ , a monochromatic copy of  $G$  occurs, and this copy lifts to a monochromatic copy of  $G'$  in  $H'_0$ . It follows that  $H' \xrightarrow{s} G'$ .  $\square$

Theorem 5.1 yields an upper bound on  $R_\Delta(G'; s)$  that is a tower function of height  $k$ . Our next aim is to reduce this upper bound when  $G$  is bipartite. If  $G$  has a bipartite  $s$ -color Ramsey host  $H$  with maximum degree  $k$ , then the  $d$ -blowup  $G'$  of  $G$  satisfies  $R_\Delta(G'; s) \leq kd \cdot 4^{ds^{k+1}}$ . This substantial improvement in the bound from Theorem 5.1 is obtained by using a highly unbalanced complete bipartite graph to force  $K_{d,d}$  instead of a symmetric one. We need a simple and standard lemma; we include its proof for completeness.

**Lemma 5.2.**  $K_{sm, \binom{sm}{m}N} \xrightarrow{s} K_{m, N}$ .

*Proof.* Let  $H = K_{sm, \binom{sm}{m}N}$ . Let the partite sets of  $H$  be  $X$  and  $Y$  with  $|X| = sm$  and  $|Y| = \binom{sm}{m}N$ . Consider an  $s$ -coloring of  $H$ , and let  $L$  be a largest color class;  $L$  has at least  $mN \binom{sm}{m}$  edges. Let  $\lambda$  be the number of copies of  $K_{m,1}$  in  $L$  with center in  $Y$ . Using the convexity of  $\binom{x}{m}$  and  $\sum_{y \in Y} d_L(y) \geq mN \binom{sm}{m}$ , we have

$$\lambda = \sum_{y \in Y} \binom{d_L(y)}{m} \geq \binom{sm}{m} N \cdot \binom{m}{m} = \binom{sm}{m} N.$$

Since  $X$  has  $\binom{sm}{m}$  subsets of size  $m$ , some  $m$ -subset of  $X$  is the leaf set of at least  $N$  of these copies of  $K_{m,1}$ . Thus  $K_{m, N} \subseteq L$ .  $\square$

**Theorem 5.3.** *If a bipartite graph  $H$  is an  $s$ -color Ramsey host for  $G$  with  $\Delta(H) = k$ , and  $G'$  is a  $d$ -blowup of  $G$ , then  $R_\Delta(G'; s) \leq kd \cdot 4^{ds^{k+1}}$ .*

*Proof.* Let  $m_0 = n_0 = d$ . For  $i \geq 0$ , let  $m_{i+1} = sm_i$  and  $n_{i+1} = \binom{sm_i}{m_i} n_i$ . Note that  $n_i \geq m_i = ds^i$ . By Lemma 5.2,  $K_{m_{i+1}, n_{i+1}} \xrightarrow{s} K_{m_i, n_i}$ . Since  $n_{i+1} \leq 2^{sm_i} n_i$  and  $n_0 = d$  (and  $s \geq 2$ ), we have by induction on  $i$  that  $n_{i+1} \leq d \cdot 4^{ds^{i+1}}$ .

Let  $(X, Y)$  be a bipartition of  $H$ . Obtain  $H'$  by blowing up vertices in  $X$  into  $m_{k+1}$  copies and vertices in  $Y$  into  $n_{k+1}$  copies. Since  $\Delta(H) \leq k$ , we have  $\Delta(H') \leq kd \cdot 4^{ds^{k+1}}$ . The argument that  $H' \xrightarrow{s} G'$  is essentially the same as in the proof of Theorem 5.1, except that in stage  $i$  we extract a monochromatic  $K_{m_i, n_i}$  from each  $s$ -colored copy of  $K_{m_{i+1}, n_{i+1}}$ .  $\square$

The use of bipartite graphs in Theorem 5.3 can be generalized to the case where the Ramsey host has bounded chromatic number. Theorem 5.3 greatly improves Theorem 5.1 for a family of bipartite graphs that is  $R_\Delta$ -bounded using bipartite Ramsey hosts. For example, since there are bipartite graphs satisfying given lower bounds on girth and average degree using constructions like those in [13], the holds for the family of trees, we can use Theorem 1.2 to apply Theorem 5.3 to the family of trees.

For the family of blowups of trees, we can get an even better bound by adapting the strategy used in Corollary 1.4.

**Theorem 5.4.** *Let  $T$  be a tree, and let  $G$  be the  $d$ -blowup of  $T$ . If  $m = B_s(d)$ , then  $R_\Delta(G; s) \leq 2sm \binom{m}{d}^2 (\Delta(T) - 1)$ .*

*Proof.* Let  $H_0$  be an  $(X, Y)$ -bigraph with girth larger than  $2|V(T)|$  that is regular of degree  $2s \binom{m}{d}^2 (\Delta(T) - 1)$ . Let  $H$  be the  $m$ -blowup of  $H_0$ . For each cluster  $U$  in  $H$  corresponding to a vertex  $u$  in  $H_0$ , fix an arbitrary indexing  $\{u_1, \dots, u_m\}$  of  $U$ . We claim that  $H \xrightarrow{s} G$ .

Given an  $s$ -edge-coloring of  $H$ , we construct a  $s \binom{m}{d}^2$ -edge-coloring of  $H_0$ . An edge  $uv$  in  $H_0$  with  $u \in X$  and  $v \in Y$  becomes a copy of  $K_{m,m}$  in  $H$  whose partite sets are the clusters  $U$  and  $V$  for  $u$  and  $v$ . Because  $m = B_s(d)$ , some subsets  $U_0 \subseteq U$  and  $V_0 \subseteq V$  of size  $d$  induce a monochromatic copy  $F$  of  $K_{d,d}$ . We color  $uv$  in  $H_0$  with a 3-tuple. The first coordinate records the color on  $E(F)$ ; the second and third coordinates record the subsets  $U_0$  and  $V_0$  (respectively) as subsets of the indices  $1, \dots, m$ . There are  $s \binom{m}{d}^2$  such colors.

Since  $H_0$  is regular of degree  $2s \binom{m}{d}^2 (\Delta(T) - 1)$  with girth larger than  $2|V(T)|$ , Theorem 1.2 implies that every  $s \binom{m}{d}^2$ -edge-coloring of  $H_0$  contains a monochromatic copy of  $T$ . Using the monochromatic copies of  $K_{d,d}$  corresponding to edges of  $H_0$  in this copy of  $T$ , we obtain a monochromatic copy of  $G$  in  $H$ . (The second and third coordinates in the coloring of  $H_0$  ensure that the copies of  $K_{d,d}$  come together at common vertices in the clusters.)  $\square$

A *grid* is the Cartesian product of two paths. Grids have maximum degree 4 but do not all arise as subgraphs of  $d$ -blowups of trees for any fixed  $d$ . Consequently, Theorem 5.4 does not give a constant upper bound on the degree Ramsey number of grids. Beyond trees and cycles, grids seem like a natural next family to consider.

**Question 5.5.** Is the family of grids  $R_\Delta$ -bounded? More simply, is there a constant upper bound on  $R_\Delta(G; 2)$  when  $G$  is a grid?

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