Degree-associated reconstruction number of graphs

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Abstract

A *card* of a graph G is a subgraph formed by deleting one vertex. The Reconstruction Conjecture states that each graph with at least three vertices is determined by its multiset of cards. A *dacard* specifies the degree of the deleted vertex along with the card. The *degree-associated reconstruction number* $\mathrm{drn}(G)$ is the minimum number of dacards that determine G. We show that $\mathrm{drn}(G) = 2$ for almost all graphs and determine when $\mathrm{drn}(G) = 1$. For k-regular n-vertex graphs, $\mathrm{drn}(G) \leq \min\{k+2, n-k+1\}$. For vertex-transitive graphs (not complete or edgeless), we show that $\mathrm{drn}(G) \geq 3$, give a sufficient condition for equality, and construct examples with large drn. Finally, we prove that $\mathrm{drn}(G) = 2$ for all caterpillars except stars and one 6-vertex example. We conjecture that $\mathrm{drn}(G) \leq 2$ for all but finitely many trees.

1 Introduction

The well-known Graph Reconstruction Conjecture of Kelly [8] and Ulam [22] has been open for more than 50 years. It asserts that every graph with at least three vertices can be (uniquely) reconstructed from its "deck" of vertex-deleted subgraphs. Here the *deck* of a graph G is the multiset of unlabeled induced subgraphs formed by deleting one vertex from G, and these subgraphs are *cards* in the deck. The conjecture has been proved for many special classes, and many properties of G may be deduced from the deck. Nevertheless, the full conjecture remains open. Surveys of results on reconstruction include [3, 4, 10, 11].

Usually, a graph is determined by less than its full deck. Harary and Plantholt [7] defined the *reconstruction number* of a graph G, denoted $\mathrm{rn}(G)$, to be the minimum number of cards from the deck that suffice to determine G. The Reconstruction Conjecture is the statement that $\mathrm{rn}(G)$ is defined (at most $|V(G)|$) for each graph G with at least three vertices. Reconstruction numbers are known for various classes of graphs; see [1, 7, 13, 14, 15, 16].

Motivated by reconstruction questions for directed graphs, Ramachandran [18] proposed a variation. A *degree-associated card* (or *dacard*) of a graph (or digraph) is a pair (C, d)

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consisting of a card C in the deck and the degree (or in/out-degree pair) d of the deleted vertex. The multiset of dacards is the *dadeck* (the *degree-associated deck*). For graphs with at least three vertices, knowing the degree of the deleted vertex is equivalent to knowing the total number of edges. A simple counting argument computes $|E(G)|$ when the entire deck is known, so the dadeck gives the same information as the deck. However, the counting argument requires the entire deck, so an individual dacard gives more information than the corresponding card. Ramachandran [20] defined the *degree-associated reconstruction number* $\mathrm{drn}(G)$ of a graph G to be the minimum number of dacards that suffice to determine G. Clearly $\mathrm{drn}(G) \le \mathrm{rn}(G)$. Ramachandran studied this parameter for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs.

In this paper we continue this study. Bollobás [2] proved that $\mathrm{rn}(G) = 3$ for almost every graph. In Section 2 we conclude from this that $\mathrm{drn}(G) \le 2$ for almost every graph, and we characterize the graphs G for which $\mathrm{drn}(G) = 1$. We also prove that $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$ when G is a k-regular graph with n vertices.

In Section 3 we study vertex-transitive graphs. Let G be vertex-transitive. We prove that $\mathrm{drn}(G) \ge 3$ when G is not complete or edgeless and give a sufficient condition for equality; it holds for the Petersen graph, the k-dimensional hypercube, and the cartesian product $K_n \square K_2$. Also, if G has nonadjacent vertices with distinct neighborhoods, and $G^{(m)}$ arises from G by expanding each vertex into a set of m independent vertices, then $\mathrm{drn}(G^{(m)}) = tm + 2$, where t is the maximum number of vertices in G having the same neighborhood.

In Sections 4–6 we study trees. Section 4 gives sufficient conditions for $\mathrm{drn}(G) = 2$ when G is a tree. These aid subsequently in computing the value for all trees whose non-leaf vertices form a path; these trees are called *caterpillars*. If G is a caterpillar, then $\mathrm{drn}(G) = 2$ unless G is a star or the one 6-vertex tree with four leaves and maximum degree 3. We consider special families of caterpillars in Section 5 and complete the general proof in Section 6.

Let the term "graph" exclude loops and multiedges. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G. Define the *(open) neighborhood* $N_G(v)$ and *closed neighborhood* $N_G[v]$ of a vertex v in G by $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{u\}$. Let $d_G(v)$ or simply $d(v)$ denote the degree in G of vertex v, which equals $|N_G(v)|$. The maximum and minimum vertex degrees are $\Delta(G)$ and $\delta(G)$. A vertex v in G is *isolated* if $d_G(v) = 0$, *pendant* if $d_G(v) = 1$, and *dominating* if $N_G[v] = V(G)$. Given $S \subseteq V(G)$, the subgraph $G[S]$ *induced by* S is the graph with vertex set S in which two vertices are adjacent if and only if they are adjacent in G. For $v \in V(G)$, denote $G[V(G) - v]$ by $G - v$.

The disjoint union and the join of graphs G and H are written as $G + H$ and $G \vee H$, respectively; thus mG is the disjoint union of m copies of G. The *cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent precisely when $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. When $H = K_2$, the special case $G \square K_2$ of the cartesian product is formed from $2G$ by adding a perfect matching joining the two copies of each vertex of G; this is the *prism over* G. We write $\mathrm{diam}(G)$ for the *diameter* of G, which is the largest distance between vertices in G. We denote a path with vertices $v_1, \ldots, v_n$ in order as $\langle v_1, \ldots, v_n \rangle$.

2 Small reconstruction numbers and regular graphs

In this section we show that $\mathrm{drn}(G) \le 2$ for almost every graph G, and we determine when $\mathrm{drn}(G) = 1$. Our observation relies heavily on the result of Bollobás [2] about $\mathrm{rn}(G)$, which also implies that almost every graph is reconstructible.

Theorem 2.1 ([2])**.** *Almost every graph has reconstruction number 3. Furthermore, for almost every graph, any two cards in the deck determine everything about the graph except whether the two deleted vertices are adjacent.*

The reconstruction number of any graph is at least 3, since $G - u$ and $G - v$ are cards for both G and G', where G and G' differ only on whether the edge uv is present. Thus, the previous result is sharp. The degree information determines the last unknown bit of information without introducing another card.

Corollary 2.2. *For almost every graph G,* $\mathrm{drn}(G) \le 2$*.*

Proof. Let G be a graph with two cards that determine the graph except for whether the deleted vertices are adjacent. In the dadeck of G the cards $G - u$ and $G - v$ are paired with $d_G(u)$ and $d_G(v)$. The degree information determines whether uv is present, thereby reconstructing G; thus $\mathrm{drn}(G) \le 2$. ☐

It is natural to ask when $\mathrm{drn}(G) = 1$. Our next aim is to characterize such graphs. Let $\overline{G}$ denote the complement of a graph G.

Lemma 2.3. *For any graph G,* $\mathrm{drn}(G) = \mathrm{drn}(\overline{G})$*.*

Proof. Let v be a vertex in an n-vertex graph G. Since $d_{\overline{G}}(v) = n - 1 - d_G(v)$ and $\overline{G - v} = \overline{G} - v$, it follows that (C, d) is a dacard of G if and only if $(\overline{C}, n - 1 - d)$ is a dacard of $\overline{G}$.

Consider a multiset $\{(C_1, d_1), \ldots, (C_r, d_r)\}$ of dacards that determine G. Since these can be obtained from $\{(\overline{C}_1, n-1-d_1), \ldots, (\overline{C}_r, n-1-d_r)\}$ and $\overline{G}$ can be obtained from G, we conclude that $\mathrm{drn}(\overline{G}) \le \mathrm{drn}(G)$. Reversing the roles of G and $\overline{G}$ yields $\mathrm{drn}(G) = \mathrm{drn}(\overline{G})$. ☐

Note that $\mathrm{drn}(G) = 1$ if and only if G has a dacard that does not occur in the dadeck of any other graph. We next determine all dacards of this type.

Theorem 2.4. *The dacard (C, d) belongs to the dadeck of only one graph (up to isomorphism) if and only if one of the following holds:*
(1) $d = 0$ *or* $d = |V(C)|$*;*
(2) $d = 1$ *or* $d = |V(C)| - 1$*, and C is vertex-transitive;*
(3) *C is complete or edgeless.*

Proof. Let $n = |V(C)|$. In each case listed, there is exactly one way (up to isomorphism) to form a graph G with $n + 1$ vertices by adding to C a vertex with d neighbors in C.

Suppose now that (C, d) is a dacard for only one graph. That is, adding a vertex adjacent to d vertices in C produces a graph in the same isomorphism class no matter which d vertices of C are chosen. If (C,d) is not in the list above, then $d \notin \{0,n\}$ and $C \notin \{K_n, \overline{K}_n\}$. We must show that then $d \in \{1, n-1\}$ and C is vertex-transtive.

Because (C,d) is a dacard for only one graph, the same isomorphism class is produced no matter what set of d vertices is chosen for the neighborhood of the added vertex v. Since isomorphic graphs have the same number of triangles, and the number of triangles after adding v is the number of triangles in C plus the number of edges in C induced by neighbors of v, we conclude that every induced subgraph of C with d vertices has the same number of edges. It is a well-known exercise (see Exercise 1.3.35 on page 50 of [23]) that when $1 < d < n-1$, this property forces $C \in \{K_n, \overline{K}_n\}$.

Hence we may assume that $d \in \{1, n-1\}$. Since (C,d) determines G if and only if $(\overline{C}, n-1-d)$ determines $\overline{G}$, we many assume that $d = 1$. Note that adding a vertex of degree 1 adds 1 to some vertex degree in C. In particular, (C,d) is a dacard for some graph with maximum degree $\Delta(C)+1$. If C is not regular, then also (C,d) is a dacard for some graph with maximum degree $\Delta(C)$. Hence C must be regular.

If C is regular of degree 0 or 1, then automatically C is vertex-transitive. For larger degree, every automorphism of the resulting graph G fixes v, since it is the only vertex of degree 1. Since attaching v to any vertex yields the same graph, C must have automorphisms taking each vertex to any other. Hence C is vertex-transitive. ☐

Corollary 2.5. *A graph G satisfies* $\mathrm{drn}(G) = 1$ *if and only if G or $\overline{G}$ has an isolated vertex or has a pendant vertex whose deletion leaves a vertex-transitive graph.*

Proof. We have $\mathrm{drn}(G) = 1$ if and only if the dadeck of G has a dacard (C,d) as described in Theorem 2.4. If C is complete or edgeless, or if $d \in \{0, |V(C)|\}$, then G or $\overline{G}$ has an isolated vertex. Case 2 of Theorem 2.4 yields the second possibility here. ☐

We close this section with a general bound for regular graphs. Regular graphs are well known to be reconstructible, since the degree list can be determined from the deck, and the deficient vertices in any card must be the neighbors of the missing vertex. One dacard gives the degree of the missing vertex, but it does not give the degree list and hence does not determine G. Nevertheless, we obtain an upper bound on $\mathrm{drn}(G)$.

Theorem 2.6. *If G is a k-regular graph on n vertices, then* $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$.

Proof. Since the complement of a k-regular graph is $(n-1-k)$-regular, by Lemma 2.3 it suffices to prove that $\mathrm{drn}(G) \le k+2$.

Since G is k-regular, each card has k vertices of degree $k-1$ and $n-1-k$ vertices of degree k. Let H be a graph that shares $k+2$ dacards with G. Let (C,k) be one shared dacard, so $C = H - u$ for some $u \in V(H)$.

If $H \not\cong G$, then u has a neighbor in H with degree k in C, so $\Delta(H) = k+1$. Each vertex of degree k in H whose deletion produces a card of G must neighbor all vertices of degree $k+1$ in H. If there are k such vertices, then vertices of degree $k+1$ in H have $k+2$ neighbors, a contradiction. ☐

Equality holds in the bound of Theorem 2.6 for graphs of the form $tK_{m,m}$ with $t > 1$, proved by Ramachandran [20]. Ramachandran [20] also proved for $k, t \geq 2$ that if G is a connected k-regular graph on n vertices, where $n \geq 3$, then $\mathrm{drn}(tG) \leq n - k + 2$.

We have observed that $\mathrm{drn}(G) = \mathrm{rn}(G) - 1$ almost always. Among regular graphs, they can differ by a lot: for $t, m > 1$, Ramachandran [20] showed that $\mathrm{drn}(tK_m) = 3$ even though $\mathrm{rn}(tK_m) = m+2$ (Myrvold [15]). However, this family contains all graphs we presently know where $\mathrm{drn}(G) \neq \mathrm{rn}(G)-1$. It would be worth finding others. We note that an argument like that of Theorem 2.6 proves that $\mathrm{rn}(G) \leq b + 1$ when G is k-regular, where b is the bound on drn in Theorem 2.6.

3 Vertex-transitive graphs

For regular graphs that are vertex-transitive, we obtain sharper results. A graph is vertex-transitive graphs if and only if its cards are pairwise isomorphic. Since vertex-transitive graphs are regular, Theorem 2.6 provides an upper bound. We will prove further lower and upper bounds and give sufficient conditions for equality in the bounds.

Since $\mathrm{drn}(G) = 2$ almost always, only special graphs need more dacards. When the dacards are identical, there is no clever choice of dacards, so one may expect vertex-transitive graphs to be hard to reconstruct. Ramachandran [20] showed that $\mathrm{drn}(tK_{m,m}) = m+2$ when $t > 1$. As noted above, $\mathrm{drn}(tK_m) = 3$. By setting $t = 2$ and applying $\mathrm{drn}(\overline{G}) = \mathrm{drn}(G)$, one also obtains $\mathrm{drn}(K_{m,m}) = 3$. We show first that 3 is a lower bound for drn on vertex-transtive graphs other than complete graphs and their complements.

Definition 3.1. A *clone* of a vertex x in a graph is a vertex having the same closed neighborhood as x. A *twin* of v is a vertex having the same open neighborhood as v. When G is edge-transitive, let G^- denote the graph formed by deleting any edge of G.

Theorem 3.2. *If G is vertex-transitive but is not complete or edgeless, then* $\mathrm{drn}(G) \geq 3$.

Proof. Let (C, d) denote the only dacard of G. To show that $\mathrm{drn}(G) > 2$, we construct a graph H different from G that has at least two copies of (C, d) in its dadeck.

Let v be a vertex of G, so $C = G - v$. If every neighbor of v in G is a clone of v, then G is a disjoint union of complete graphs, say mK_r with $m \geq 2$ and $r \geq 2$, where $r = d + 1$. In this case, $C = (m-1)K_r + K_{r-1}$. Let $H = (m-2)K_r + K_{r+1}^- + K_{r-1}$. Now H has two copies of (C, d) in its dadeck, and $H \ncong G$.

Otherwise, let u be a neighbor of v with $N_G[u] \neq N_G[v]$. Form H by adding to $G - v$ a clone u' of u. Now $H - u = H - u' = C$ and $d_H(u) = d_H(u') = d$. However, with $x \in N_G[u] - N_G[v]$ and $y \in N_G[v] - N_G[u]$, we have $d_H(x) > d_C(x) > d_c(y) = d_H(y)$; hence H is not regular and $H \ncong G$. □

We will give sufficient conditions for equality in this bound. First note that the bound can be arbitrarily bad; Ramachandran [20] showed that $\mathrm{drn}(tK_{m,m}) = m + 2$. We extend Ramachandran's result to a more general family of vertex-transitive graphs. The construction produces $tK_{m,m}$ when the base graph is tK_2.

Definition 3.3. An *expansion* of a graph G is a graph H obtained by replacing each vertex of G with an independent set such that copies in H of two vertices of G are adjacent in H if and only if the original vertices were adjacent in G. The *m-fold expansion* $G^{(m)}$ is the expansion of G in which each vertex expands into an independent set of size m. A *twin-set* in a graph is a maximal vertex subset consisting of vertices with identical open neighborhoods.

Theorem 3.4. *Let G be a vertex-transitive graph other than a complete graph, and suppose that G has no twins. If $m \geq 2$, then $\mathrm{drn}(G^{(m)}) = m+2$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. In $G^{(m)}$, each vertex v_i of G becomes an independent set V_i of size m. All vertices in V_i have the same neighborhood, while vertices in distinct such sets have different neighborhoods, since G has no twins. Hence $V_1, \ldots, V_n$ are the twin-sets in G. Note that $G^{(m)}$ is vertex-transitive and km-regular, where G is k-regular, and every vertex neighborhood in $G^{(m)}$ is a union of twin-sets. Let C be the unique card of $G^{(m)}$.

To show that $\mathrm{drn}(G^{(m)}) \geq m+2$, we build H sharing $m+1$ dacards with G. Since G is not complete and has no twins, it has nonadjacent vertices v_i and v_j with distinct neighborhoods. View C as $G - x$, where $x \in V_i$. Construct H by adding to C a vertex u with neighborhood $N(V_j)$ (this enlarges the twin-set V_j). Since $x \notin N(V_j)$, we have $d_H(u) = km$. In $G^{(m)}$ every $m+1$ vertices contain two with distinct neighborhoods, but in H the $m+1$ vertices in $V_j \cup \{u\}$ have the same neighborhood. Hence $H \ncong G^{(m)}$. Also, the dacards for these vertices of H are all (C, km). Thus $\mathrm{drn}(G^{(m)}) \geq m+2$.

For the upper bound, suppose that H is a graph having vertices $u_1, \ldots, u_{m+2}$ of degree km such that $H - u_i \cong C$ for $1 \leq i \leq m+2$. Since $m \geq 2$, there are n twin-sets in C, one of which has size $m-1$; call it U. Treating the deleted vertex of $G^{(m)}$ as a member of V_1, we let $V_1 - \{x\}$, $V_2, \ldots, V_n$ be the twin-sets of C. There are exactly n distinct vertex neighborhoods in C.

Now view C as $H - u_1$, and suppose that $N_H(u_1)$ is none of the vertex neighborhoods in C. Since $|U| = m-1$, among $u_2, \ldots, u_{m+2}$ there is a vertex u_j not in U. In $C - u_j$, there remain n distinct neighborhoods ($m \geq 2$ implies that all n twin-sets remain nonempty), and none of them is $N_H(u_1) - \{u_j\}$. Replacing u_1, we find that $H - u_j$ has $n+1$ distinct neighborhoods, contradicting $H - u_j \cong C$.

Thus $N_H(u_1)$ is a vertex neighborhood in C. If it is the neighborhood of the deficient set, then $H \cong G^{(m)}$. Otherwise, H is an expansion of G in which one twin-set T has size $m+1$, one twin-set U has size $m-1$, and the others have size m. The only way to delete a vertex from H so that the twin-sets in the resulting graph have the same sizes as in C is to delete a vertex of T. Since $|T| = m+1$, the dacard (C, km) cannot occur $m+2$ times for H. □

In a vertex-transitive graph, the twin-sets all have the same size.

Corollary 3.5. *If G is a vertex-transitive graph other than a complete multipartite graph, then $\mathrm{drn}(G^{(m)}) = tm+2$ for every $m \geq 2$, where t is the size of the twin-sets in G.*

Proof. Collapsing the twin-sets of G into single vertices yields a vertex-transitive graph G_0 having no twins, and $G = G_0^{(t)}$. Since G is not a complete multipartite graph, G_0 is not a complete graph. Hence Theorem 3.4 applies to G_0, and $\mathrm{drn}(G^{(m)}) = \mathrm{drn}(G_0^{(tm)}) = tm+2$.□

In the remainder of this section we study sharpness in the lower bound of Theorem 3.2. We give a sufficient condition for $\mathrm{drn}(G) = 3$ in the family of vertex-transitive graphs and show that hypercubes and some other products satisfy it.

Definition 3.6. A vertex-transitive graph G is *coherent* if a card C of G formed by adding one vertex z to a two-vertex-deleted subgraph $G - \{x, y\}$ can only be formed by making z adjacent to $N_{G-y}(x)$ or $N_{G-x}(y)$.

Coherence prevents the deletion of two vertices from G in such a way that the card can be recreated by adding a vertex adjacent to some set of deficient vertices other than the full neighborhood of one of the deleted vertices.

Theorem 3.7. *Let G be a k-regular vertex-transitive graph. If G is coherent and has no clones or twins, then* $\mathrm{drn}(G) = 3$.

Proof. Let C be the unique card of G. We must show that if some graph H has vertices u, v, w of degree k such that deleting any one yields C, then $H \cong G$.

Let S be the set of vertices of degree $k-1$ in $H - u$. Since $H - u \cong C \cong G - x$, we may assume that $H - u = G - x$ (using the same vertex names), so $N_G(x) = S$ and $|S| = k$. Now $H - u - v$ is obtained by deleting x and v from G. The card $H - v$ is obtained by adding u and the appropriate edges to $H - u - v$; doing this adds u and appropriate edges to $G - x - v$ to produce a graph isomorphic to C. By coherence, $N_{H-v}(u)$ is $N_{G-v}(x)$ or $N_{G-x}(v)$.

If $N_{H-v}(u) = N_{G-v}(x)$, then $S - \{v\} \subseteq N_H(u)$. Also, $|S - \{v\}|$ is $k-1$ or k, depending on whether $v \in N_G(x)$. Since we are given $d_H(u) = k$, we obtain $N_H(u) = S$ and $H \cong G$.

If $N_{H-v}(u) = N_{G-x}(v)$, then $|N_H(u) \cap N_H(v)| \in \{k-1, k\}$, depending on whether $v \in N_G(x)$. This makes u and v clones or twins in H, respectively, since $d_H(u) = k$. Now we look at $H - w$. Whether w is adjacent to neither or both of $\{u, v\}$ in H, still u and v are clones or twins in $H - w$. Since G is regular, $H - w \cong C \cong G - x$, and $d_{H-w}(u) = d_{H-w}(v)$, forming G from $H - w$ makes w adjacent to neither or both of $\{u, v\}$. As a result, u and v are clones or twins in G, which contradicts the prohibition of such pairs. $\square$

It is easy to see that $tK_{m,m}$ and tK_m are coherent, but $tK_{m,m}$ has twins and tK_m has clones. We have noted that $\mathrm{drn}(tK_{m,m}) = m + 2$ and $\mathrm{drn}(tK_m) = 3$.

Proposition 3.8. *If G is a coherent 2-connected vertex-transitive graph, then tG is coherent.*

Proof. Vertices u and v to be deleted from tG may lie in the same component or not. If they don't, then a vertex added to turn $tG - u - v$ into the card C must restore one of the components of G. If u and v lie in the same component of tG, then the needed property follows from the coherence of G. $\square$

We close this section with several natural examples to illustrate the role of coherence.

Example 3.9. *If G is the Petersen graph, then* $\mathrm{drn}(G) = 3$. Any two nonadjacent vertices in G have exactly one common neighbor, and any two adjacent vertices have no common neighbors; hence G has no twins or clones. It therefore suffices to check coherence. Let C be the card. There are only two types of vertex pairs in G; adjacent or nonadjacent.

Deleting two adjacent vertices leaves four vertices with degree 2. Any two of them that did not have a common neighbor among the deleted vertices have a common neighbor among the remaining vertices. Adding a vertex adjacent to both of them creates a 4-cycle, which does not exist in C.

Deleting two nonadjacent vertices leaves one vertex with degree 1, and the vertices having degree 2 induce $2K_2$. A vertex added to form C must be adjacent to the leaf and to one vertex from each edge of this $2K_2$. To avoid creating a 4-cycle, only two of the four such choices are allowable, and these yield the vertex neighborhoods of the deleted vertices. $\square$

We next consider the k-dimensional hypercube Q_k, the graph with vertex set $\{0,1\}^k$ in which two vertices are adjacent if and only if they differ in exactly one coordinate. It is well known that vertices separated by distance 2 in Q_k have exactly two common neighbors.

Theorem 3.10. *If* $k \geq 2$, *then* $\mathrm{drn}(Q_k) = 3$.

Proof. The lower bound follows from Theorem 3.2. Ramachandran [20] showed that $\mathrm{drn}(C_4) = 3$. Since $Q_2 \cong C_4$, we may assume that $k \geq 3$. Since Q_k has no clones or twins, it suffices by Theorem 3.7 to show that Q_k is coherent. Let C be the unique card of Q_k. Given $u, v \in V(Q_k)$, let $F = Q_k - \{u, v\}$, and let $S = N_{Q_k - v}(u)$ and $S' = N_{Q_k - u}(v)$. Let z be a vertex added to F to obtain C; we must show that $N_C(z) \in \{S, S'\}$.

The vertex z cannot have have neighbors in both partite sets of F, since C is bipartite. Also it has no neighbor with degree k in F, since $\Delta(C) \leq k$. Hence $N_C(z) \in \{S, S'\}$ when u and v lie in opposite partite sets.

Now consider u and v in the same partite set. Since $\delta(C) = k-1$ and $\Delta(C) \leq k$, we have $S \cap S' \subseteq N_C(z) \subseteq S \cup S'$. If $N_C(z) \notin \{S, S'\}$, then z has neighbors in both $S - S'$ and $S' - S$. Since $d_C(z) = k = |S| = |S'|$, there also exist $w \in S - S'$ and $w' \in S' - S$ outside $N_C(z)$. Now $d_C(w) = d_C(w') = k - 1$. Hence w and w' have a common neighbor in Q_k deleted to obtain C. Since the distance between them in Q_k is 2, they have exactly two common neighbors in Q_k, and hence exactly one remains in C. However, since by choice neither lies in $S \cap S'$, neither u nor v is one of their common neighbors. Hence their common neighbors in Q_k both remain in F and hence in C. The contradiction implies that $N_C(z) \in \{S, S'\}$. $\square$

The hypercube Q_k is the cartesian product of k factors isomorphic to K_2. It would be nice to generalize Theorem 3.10 to all cartesian products of complete graphs. Our next result does this for one special case. As noted earlier, the cartesian product $G \square K_2$ is also called the *prism over* G. A *k-clique* in a graph is a set of k pairwise adjacent vertices.

Unfortunately, $K_3 \square K_2$ is not coherent, since it has C_4 as a double-vertex-deleted subgraph, and the card can be obtained by adding z adjacent to any two consecutive vertices on the cycle. Hence we cannot apply Theorem 3.7 to this graph.

Lemma 3.11. $\mathrm{drn}(K_3 \square K_2) \leq 3$.

Proof. It suffices to show that three dacards determine $K_3\square K_2$. Let C be the unique card of $K_3\square K_2$, and consider a graph H having three cards isomorphic to C, obtained by deleting any one of $\{u,v,w\}$, all having degree 3 in H.

If H has a vertex x of degree 4, then $\{u,v,w\}\subseteq N_H(x)$, since $\Delta(C)=3$. Let y be the unique nonneighbor of x. Since $K_{2,3}\not\subseteq C$, we have $d_H(y)\le 2$ or $N_H(y)=\{u,v,w\}$.

If $d_H(y)\le 2$, then since $xy\notin E(H)$, deleting one of $\{u,v,w\}$ leaves $d_C(y)\le 1$, contradicting $\delta(C)=2$. If $N_H(y)=\{u,v,w\}$, then $\{u,v,w\}$ is an independent set, since otherwise deleting one of them leaves two triangles sharing an edge, which does not occur in C. Having degree 3 now forces each vertex in $\{u,v,w\}$ to neighbor the remaining vertex z, and deleting any one of them again yields two triangles with a common edge.

Hence $\Delta(H)=3$, which yields $G\cong K_3\square K_2$. $\square$

Theorem 3.12. *If* $k\ge 2$, *then* $\mathrm{drn}(K_k\square K_2)=3$.

Proof. Again the lower bound is from Theorem 3.2. Let $G=K_k\square K_2$. We have observed that $\mathrm{drn}(G)\le 3$ when $k\le 3$, so consider $k\ge 4$. Let C be the unique card of G. Since G has no clones or twins, by Theorem 3.7 it suffices to show that G is coherent. Given $u,v\in V(G)$, let $F=G-\{u,v\}$, and let $S=N_{G-v}(u)$ and $S'=N_{G-u}(v)$. Let z be a vertex added to F to obtain C; we must show that $N_C(z)\in\{S,S'\}$. Let A and B be the two k-cliques in G. By symmetry, we have two cases.

Case 1: $u,v\in A$. Vertices remaining in A have degree $k-2$ in F, and the neighbors of u and v in B have degree $k-1$ in F. Since $\delta(C)=k-1$ and $\Delta(C)=k$, we conclude that $N_C(z)$ contains all of $A-\{u,v\}$ and the neighbor of u or v in B. Hence $N_C(z)\in\{S,S'\}$.

Case 2: $u\in A$, $v\in B$. Here $F\subseteq K_{k-1}\square K_2$, with equality if $uv\in E(G)$ and one missing "cross-edge" if $uv\notin E(G)$. Since $k\ge 4$, the only $(k-1)$-cliques in F are $A-u$ and $B-v$. Since C has a k-clique, z must be adjacent to all of $A-u$ or $B-v$. Since C has exactly k vertices of degree $k-1$, z has no other neighbor if $uv\in E(G)$ and is adjacent to the remaining vertex of degree $k-2$ in F if $uv\notin E(G)$. In either case, $N_C(z)\in\{S,S'\}$. $\square$

Similar arguments can be made for other families of vertex-transitive graphs. For example, it follows also that $\mathrm{drn}(C_k\square K_2)=3$ for $k\ge 3$, where C_k is the k-cycle. We ask which vertex-transitive graphs are coherent, or at least which vertex-transitive graphs have coherent cartesian products with K_2.

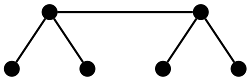
4 Trees

In one of the first papers on reconstruction, Kelly [8] proved that trees with at least three vertices are reconstructible. Several papers have studied reconstruction of trees given only some of the cards from the deck. Harary and Palmer [6] showed that every tree is uniquely determined by its leaf-deleted subgraphs, and Lauri [9] showed that every tree with at least three cutvertices is reconstructible from its cutvertex-deleted subgraphs.

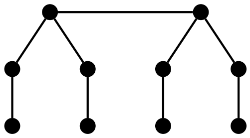
Myrvold [16] proved that every tree with at least 5 vertices has reconstruction number 3. Together with Corollary 2.5, this implies the following.

Corollary 4.1. *If T is a tree, then* $\mathrm{drn}(T) \le 3$, *and* $\mathrm{drn}(T) = 1$ *if and only if T is a star.*$\square$

By Corollary 2.2, almost every graph has degree-associated reconstruction number 2, and Prince [17] proved the "almost-always" statement also for the class of all trees. The trees H_1 and H_2 below do satisfy $\mathrm{drn}(H_1) = \mathrm{drn}(H_2) = 3$.



H_1



H_2

Example 4.2. $\mathrm{drn}(H_1) = 3$. The graph H_1 has only two distinct dacards. They are $(P_3 + 2K_1, 3)$ and $(S, 1)$, where S is the tree obtained by subdividing one edge of $K_{1,3}$. Hence there are three ways to take two dacards; two of the first, two of the second, and one of each. For these three cases, other graphs having the same two dacards are the graph obtained from $2K_1 + K_4$ by deleting one edge, the tree obtained from $K_{1,4}$ by subdividing one edge, and the tree obtained from $K_{1,3}$ by subdividing one edge twice, respectively.

Three dacards suffice, using one leaf and the two central vertices. For any reconstruction G, the leaf card forces G to be a tree, and the other two force G to have two vertices of degree 3. Hence G is obtained from S by appending a leaf to the one vertex of degree 2. $\square$

The argument for H_2 is similar but longer. We have particular interest in H_1 because it lies in the family we will study for the rest of this paper. First, the fact that we know of no tree T other than H_1 and H_2 such that $\mathrm{drn}(T) = 3$ suggests a conjecture.

Conjecture 4.3. *Only finitely many trees T satisfy* $\mathrm{drn}(T) = 3$.

A *caterpillar* is a tree whose non-leaf vertices induce a path called the *spine* of the caterpillar. In the remainder of this paper, we prove that the tree H_1 and the stars $K_{1,m}$ are the only caterpillars T such that $\mathrm{drn}(T) \ne 2$.

By Corollary 4.1, it suffices to prove that $\mathrm{drn}(T) \le 2$ for caterpillars other than H_1. In this section we give sufficient conditions for $\mathrm{drn}(T) \le 2$ when T is a tree. In the subsequent sections, we prove this inequality for various classes of caterpillars described by conditions on the list of degrees of the spine vertices, culminating in the full proof. The task is to select for each caterpillar T a pair of dacards that together determine T.

The *skeleton* of a tree T is the subtree T' obtained by deleting all leaves from T. Thus caterpillars are the trees whose skeletons are paths, and the spine of a caterpillar is its skeleton. We use $C(a_1, \dots, a_s)$ to denote a caterpillar with spine $\langle v_1, \dots, v_s \rangle$ by attaching a_i leaf neighbors to v_i for each $i \in \{1, \dots, s\}$. We call $(a_1, \dots, a_s)$ the *spine list.* Note that $C(a_1, \dots, a_s) \cong C(a_s, \dots, a_1)$ and that a_1 and a_s are both positive. Where convenient, we denote a repeated string in this notation by enclosing it in parentheses and writing its multiplicity as a superscript in parentheses. For example, $C(a, b, c, d, b, c, d, b, c, d, e, f) = C(a, (b, c, d)^{(3)}, e, f)$.

The *weight* $w(u)$ of a vertex u in a tree T is the maximum number of vertices in a component of $T-u$; note that all leaves in an n-vertex tree have weight $n-1$. The *centroid* of a tree is the set of vertices having minimum weight. Myrvold [16] extensively used centroids of trees in her analysis of reconstruction number of trees. To keep our presentation self-contained, we include short proofs of some elementary observations.

Lemma 4.4 ([16]). *The centroid of an n-vertex tree T consists of one vertex or two adjacent vertices. Also, $w(v) \le n/2$ if and only if v is in the centroid of T, and the centroid of T has size 1 if and only if T has a vertex with weight strictly less than $n/2$.*

Proof. For each vertex v, mark an incident edge from v toward a largest component of $T-v$. If uv is an edge not marked by v, then $w(u) > w(v)$, since $T-u$ has a component strictly containing a largest component of $T-v$. Therefore, if u is a vertex in the centroid, and u marks the edge uv, then v must also mark the edge uv.

On the other hand, since deleting any edge leaves two components, with at least one having at least half the vertices, every edge is marked by at least one of its endpoints. Since T has n vertices and $n-1$ edges, we conclude that exactly one edge is marked twice; the only candidates for the centroid are its endpoints.

If edge uv is marked twice, then $w(u) = w(v)$ if and only if each component of $T-uv$ has exactly half the vertices. Otherwise, the endpoint of uv in the larger component is the only vertex of the centroid and has weight less than $n/2$. □

A tree is *unicentroidal* or *bicentroidal* depending on whether its centroid has size 1 or 2, respectively. For simplicity, we refer to the centroid vertex of a unicentroidal tree as the centroid. A *centroidal vertex* is a vertex in the centroid.

Lemma 4.5 ([16]). *Let v be the centroid in a unicentroidal tree T. If ℓ is a leaf in T, then v is centroidal in $T-\ell$.*

Proof. Let T have n vertices. By Lemma 4.4, $w(v) < n/2$. The weight of v in T' is at most $(n-1)/2$, since deleting ℓ simply reduces one component of $T-v$. By Lemma 4.4, v is centroidal in T'. □

These facts about centroids can be useful in reconstructing a tree from its dacards. Note that if G has a card that is a tree obtained by deleting a vertex of degree 1, then G is a tree.

Proposition 4.6. *If T is a unicentroidal tree with a leaf ℓ adjacent to the centroid vertex, and $T-\ell$ is unicentroidal, then* $\mathrm{drn}(T) \le 2$.

Proof. Let $T' = T-\ell$, and let $\hat{T}$ be the card obtained by deleting the centroid from T. Thus $(T', 1)$ and $(\hat{T}, d)$ are the corresponding dacards, and ℓ is an isolated vertex in $\hat{T}$.

Let G be a graph having these dacards, obtained by deleting vertices u and v, respectively. From the first dacard, G is a tree. From the sizes of the components of $\hat{T}$, Lemma 4.4 tells us that G is unicentroidal with centroid v.

Since u is a leaf and $G-u$ is unicentroidal (being isomorphic to T'), Lemma 4.5 identifies v in $G-u$ as the centroid of $G-u$, Since $\hat{T}=G-v$, the d components of $\hat{T}$ agree with the components obtained by deleting the centroid from T', except that one may have u as an extra leaf. However, we know from T that instead $\hat{T}$ has one more component than $T'-v$, an isolated vertex. This forces u to be adjacent to v in G, yielding $G \cong T$. □

We have noted that having a dacard $(G-v,1)$ in which $G-v$ is a tree forces G to be a tree. Our next lemma gives another sufficient condition on dacards for G to be a tree.

Lemma 4.7. *Let G be a graph with dacards $(A,2)$ and $(B,2)$. If A and B are forests with two components, and the sizes of the components of A do not equal those of B, then G is a tree.*

Proof. Let the sizes of the components in A and B be $\{a_1,a_2\}$ and $\{b_1,b_2\}$, respectively. Let u and v be the vertices such that $G-u=A$ and $G-v=B$.

If G is disconnected, then the neighbors of u in G belong to the same component of A, which we may call A_1. Now G has two components with orders a_1+1 and a_2, and the component of G containing A_1 is not a tree. To make B a forest, v must lie on all cycles in G and hence must lie in A_1. Since G and B both have two components, v is not a cutvertex of A_1. Now $\{a_1,a_2\}=\{b_1,b_2\}$, a contradiction.

Hence G is connected. Since $d_G(u)=2$, it follows that G is a tree. □

By the characterization in Corollary 2.5 of when $\mathrm{drn}(G)=1$, the only such graphs that are trees are stars. We have also observed that $\mathrm{drn}(H_1)=3$. To complete our analysis of caterpillars, in the remainder of the paper we only need to prove results showing that caterpillars other than H_1 have degree-associated reconstruction number at most 2. General arguments for reconstruction of trees often must exclude the special case of paths; we treat them separately here.

Proposition 4.8. *If $n \ge 4$, then* $\mathrm{drn}(P_n)=2$.

Proof. For $n=4$, use the two dacards $(P_3,1)$ and $(P_1+P_2,2)$. The first forces every reconstruction to be a tree, and hence in the second the missing vertex has a neighbor in each component, yielding P_4.

For $n \ge 5$, let $a=\left\lfloor \frac{n-1}{2} \right\rfloor$ and $b=\left\lceil \frac{n-1}{2} \right\rceil$. Let G be a graph having the two dacards $(P_a+P_b,2)$ and $(P_{a-1}+P_{b+1},2)$, associated with u and v, respectively. By Lemma 4.7, G is a tree. (Here $a-1 \ge 1$ requires $n \ge 5$.)

Let w be a neighbor of u in G. If w is not a leaf in $G-u$, then $d_G(w)=3$. Since $\Delta(G-v)=2$, we have $v \in N_G(w)$. Now the component of $G-v$ containing u has at least $a+3$ vertices, since it contains all of one component of P_a+P_b plus u, w, and another neighbor of w. Since the components of $G-v$ have at most $a+2$ vertices, we conclude that u has no neighbor with degree 3 in G, and hence $G=P_n$. □

Our general arguments fail also for several other classes of caterpillars where we will need alternative choices of dacards. It is worth noting that P_n is forced by two dacards only when they correspond to a centroidal vertex and a noncentroidal neighbor of the centroid.

5 Caterpillars of the form $C(1,0,a_3,\ldots,a_{s-2},0,1)$

We begin with a technical lemma that will restrict the form of caterpillars with special symmetry properties. A *palindrome* is a list unchanged under reversal.

Lemma 5.1. *Let $B=(b_1,\ldots,b_s)$. If $(b_1,\ldots,b_s)$ and $(b_3,\ldots,b_s)$ are palindromes, then either B is constant, or s is odd and B alternates two values. If $(b_1,\ldots,b_{s-1})$ and $(b_2,\ldots,b_s)$ are palindromes, then either B is constant, or s is even and B alternates two values.*

Proof. Define a graph R with vertex set $\{v_1,\ldots,v_s\}$ such that $v_iv_j\in E(R)$ if and only if the palindrome requirements force $b_i=b_j$. If R consists of one component, then B is constant. If R consists of two components, one containing the odd-indexed and the other the even-indexed vertices, then B is constant or alternates between two values.

If $(b_1,\ldots,b_s)$ and $(b_3,\ldots,b_s)$ are palindromes, then $v_iv_j\in E(R)$ if and only if $i+j=s+1$ or $i+j=s+3$. If s is even, then R is the path $\langle v_1,v_s,v_3,v_{s-2},\ldots,v_{s-1},v_2\rangle$. If s is odd, then R consists of two paths, $\langle v_1,v_s,v_3,v_{s-2},\ldots\rangle$ containing the odd-indexed vertices, and $\langle v_2,v_{s-1},v_4,v_{s-3},\ldots\rangle$ containing the even-indexed vertices.

If $(b_1,\ldots,b_{s-1})$ and $(b_2,\ldots,b_s)$ are palindromes, then $v_iv_j\in E(R)$ if and only if $i+j=s$ or $i+j=s+2$. If s is odd, then R is the path $\langle v_1,v_{s-1},v_3,v_{s-3},\ldots,v_{s-2},v_2,v_s\rangle$. If s is even, then R consists of two paths, $\langle v_1,v_{s-1},v_3,v_{s-3},\ldots\rangle$ containing the odd-indexed vertices, and $\langle v_s,v_2,v_{s-2},v_4\ldots\rangle$ containing the even-indexed vertices. □

In the remainder of the paper, $T=C(a_1,\ldots,a_s)$, with spine $\langle v_1,\ldots,v_s\rangle$, where v_i has degree a_i in T. In the rest of this section, $T=C(1,0,a_3,\ldots,a_{s-2},0,1)$. By Proposition 4.8, $\mathrm{drn}(P_{s+2})=2$. Since P_{s+2} is the case $a_3=\cdots=a_{s-2}=0$, we may let $r=\min\{i\colon\ a_i>0$ and $3\le i\le s-2\}$. To show $\mathrm{drn}(T)\le 2$, we present two dacards that determine T. Consider the dacards for leaves adjacent to v_1 and v_r, writing

$$C_1=C(1,0^{(r-3)},a_r,\ldots,a_{s-2},0,1), \qquad D_1=(C_1,1),$$
$$C_2=C(1,0^{(r-2)},a_r-1,a_{r+1},\ldots,a_{s-2},0,1), \qquad D_2=(C_2,1).$$

Let G be a graph reconstructed from dacards D_1 and D_2, with vertices u and v being the corresponding deleted vertices. Since $d_G(u)=d_G(v)=1$, either card forces G to be a tree. We show that $G\cong T$, with some exceptions where we will later use other dacards.

Lemma 5.2. *If $T=C(1,0,a_3,\ldots,a_{s-2},0,1)$ and T is not a path, then the dacards D_1 and D_2 determine T unless T satisfies one of the following conditions:*
(1) $T=C(1,0^{(p)},1,0^{(q)},1)$, where $p,q\ge 1$;
(2) $T=C(1,0^{(p+1)},k,(\alpha),k-1,0^{(p)},1)$, where $k\ge 1$, $p\ge 0$, and (α) is a palindrome.

Proof. From D_2 it follows that G is a tree with diameter at least $s+1$. Since $\mathrm{diam}(G-u)=s$ and $s\ge 5$, it follows that u is adjacent in G to an endpoint of a longest path in $G-u$. Hence G is T or is $C(1,0^{(r-3)},a_r,\ldots,a_{s-2},0,0,1)$. Suppose the latter.

Since $G-v\cong C_2$, and both G and C_2 have spines with s vertices, decreasing one term of the spine list L for G yields the spine list L' for C_2 or its reverse, L''.

Let L_i, L_i', L_i'' denote the ith entry in L, L', L'', respectively. Since $L_{r-1} = a_r > 0 = L'_{r-1}$, changing L into L' by decreasing one L_i requires $i = r-1$ and $a_r = 1$. Since no other change is allowed, we have

$$\begin{array}{ccccccccl} a_r - 1 & = & L_{r-1} - 1 & = & L'_{r-1} & = & 0, & & \\ a_{r+1} & = & L_r & = & L'_r & = & a_r - 1, & & \\ a_{i+1} & = & L_i & = & L'_i & = & a_i & & \text{for } r+1 \le i \le s-3, \\ 0 & = & L_{s-2} & = & L'_{s-2} & = & a_{s-2}, & & \end{array}$$

and thus $a_r = 1$ and $a_{r+1} = \cdots = a_{s-2} = 0$. Hence $T = C(1, 0^{(r-2)}, 1, 0^{(s-r-1)}, 1)$, as in (1).

Suppose instead that decreasing some L_j by 1 changes L into L''; we first restrict the choices for j. By construction, $3 \le r \le s-2$ and $s \ge 5$. We compare the expressions below.

$$\begin{array}{rl} T = & C(1,\ 0^{(r-2)},\quad a_r,\quad \ldots,\quad a_{s-2},\ 0,\ 1) \\ G = & C(1,\ 0^{(r-3)}, a_r,\quad \ldots,\quad a_{s-2},\ 0,\ 0,\ 1) = C(L) \\ C_2 = & C(1, 0, a_{s-2}, \ldots,\ a_{r+1},\quad a_r - 1,\quad 0^{(r-2)}\ ,\ 1) = C(L'') \\ \text{positions} = & \quad 1, 2,\ 3,\ \ldots,\ s-r,\ s-r+1,\ \ldots,\ s-2, s-1, s \end{array}$$

Since $L_i = a_{i+1}$ for $2 \le i \le s-2$, we have $L_{r-1} + L_{s-r+1} = a_r + a_{s-r+2}$. Since $L_i'' = a_{s+1-i}$ for $i \ne s-r+1$ (and $L''_{s-r+1} = a_r - 1$), setting $i = r-1$ yields $L''_{r-1} + L''_{s-r+1} = a_{s-r+2} + a_r - 1$, except that $L''_{r-1} + L''_{s-r+1} = a_{s-r+2} + a_r - 2$ when $r - 1 = s - r + 1$. In either case, $L''_{r-1} + L''_{s-r+1} < L_{r-1} + L_{s-r+1}$, and hence $j \in \{r-1, s-r+1\}$.

Since $L_i = 0$ for $2 \le i \le r-2$, we have $j \ge r-1$. Since only position j changes, the first $r-2$ positions agree in L and L''. Hence $a_i = 0$ for $s-r+3 \le i \le s-1$ (when $r = 3$ this conclusion is empty). If $r - 1 \ge s - r + 2$, then this statement includes $a_r - 1 = 0$, since $L''_{s-r+1} = a_r - 1$. In this case $T = C(1, 0^{r-2}, 1, 0^{(s-1-r)}, 1)$, which satisfies description (1). If $r - 1 = s - r + 1$, then $s - r + 3 = r + 1$; we obtain $T = C(1, 0^{(r-2)}, a_r, 0^{(r-3)}, 1)$ and $G = C(1, 0^{(r-3)}, a_r, 0^{(r-2)}, 1)$, and hence $G \cong T$.

Hence we may assume that $r-1 < s-r+1$. Now $a_{i+1} = L_i = L_i'' = a_{s+1-i}$ for $r \le i \le s-r$. Hence $(a_{r+1}, \ldots, a_{s-r+1})$ is a palindrome, and a_{s-r+2} equals $a_r - 1$ (if $j = r-1$) or a_r (if $j = s-r+1$). Letting $\alpha = (a_{r+1}, \ldots, a_{s-r+1})$, we have $T = C(1, 0^{(r-2)}, k, (\alpha), k', 0^{(r-3)}, 1)$ and $G = C(1, 0^{(r-3)}, k, (\alpha), k', 0^{(r-2)}, 1)$, where $k = a_r \ge 1$ and $k' \in \{k, k-1\}$. If $k' = k$, then $G \cong T$; otherwise, T satisfies description (2). $\square$

Since $C(a_1, \ldots, a_s) \cong C(a_s, \ldots, a_1)$ for every caterpillar by reversing the spine, we have shown that a caterpillar of the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$ is determined by the stated choice of dacards taken from one end or the other unless under both directions the caterpillar has one of the exceptional forms in described in Lemma 5.2.

Our argument to handle these exceptional forms has exceptions itself. The difficulty is that in the exceptional cases the two dacards D_1 and D_2 chosen for Lemma 5.2 do not determine T. Nevertheless, in all exceptional cases, we find two dacards that work. We show first that the type (1) exceptional form in Lemma 5.2 causes no difficulty.

Proposition 5.3. *If* $T = C(1, 0^{(p)}, 1, 0^{(q)}, 1)$*, where* $p, q \geq 0$*, then* $\operatorname{drn}(T) \leq 2$*.*

Proof. The caterpillar T contains one vertex of degree 3, which has exactly one leaf neighbor. Use the dacards for these two vertices: $D_1 = (P_{p+q+5}, 1)$ and $D_2 = (P_{p+2} + K_1 + P_{q+2}, 3)$. Let G be a reconstruction from these dacards, with u and v being the respective deleted vertices. As a leaf deletion, D_1 forces G to be a tree. Since $G - u$ is a path, v is the only vertex of degree 3 in G. Hence v must have a neighbor in each component of $P_{p+2} + K_1 + P_{q+2}$, and that neighbor cannot have degree 2 in its component. We obtain $G \cong T$. $\square$

Among the type (2) exceptions in Lemma 5.2, we consider several special forms.

Proposition 5.4. *If* $T = C(1, 0^{(p+1)}, (2, 0)^{(q)}, 1, 0^{(p)}, 1)$*, where* $p, q \geq 1$*, then* $\operatorname{drn}(T) \leq 2$*.*

Proof. Let $j = p + 3 + 2\lfloor q/2 \rfloor$. The spine vertex v_j has degree 4. Consider the dacards obtained by deleting v_j or a leaf ℓ adjacent to v_j. Deleting ℓ leaves a tree with $2p + 4q + 6$ vertices, and hence any reconstruction G is a tree with $2p + 4q + 7$ vertices. The card when we delete v_j consists of two isolated vertices and two caterpillars, which have $p + 3 + 4\lfloor q/2 \rfloor$ and $p + 1 + 4\lceil q/2 \rceil$ vertices. For either parity of q, the maximum of these is $p + 3 + 2q$.

Let u and v be the leaf and the non-leaf vertices deleted from G to obtain these dacards. Since $p + 3 + 2q < (2p + 4q + 7)/2$, Lemma 4.4 implies that v is the centroid of G. The tree $G - u$ has $2p + 4q + 6$ vertices and is bicentroidal, with centroid vertices v_j and $v_{j\pm1}$ ($+1$ when q is odd, -1 when q is even); each of these vertices has weight $p + 2q + 3$. By Lemma 4.5, v is one of these two vertices. Since $d_G(v) = 4$ and the spine neighbors of v_j have no leaf neighbors, $v = v_j$. Since $d_{G-u}(v_j) = 3$, we obtain G from the leaf card $G - u$ by adding u adjacent to v_j. Thus $G \cong T$. $\square$

Proposition 5.5. *If* $T = C(1, 0^{(p)}, 1^{(q)}, 0^{(p)}, 1)$*, where* $p \geq 1$ *and* $q \geq 0$*, then* $\operatorname{drn}(T) \leq 2$*.*

Proof. If $q = 0$, then T is a path, and Proposition 4.8 applies. If $q = 1$, then Proposition 5.3 applies. Now consider $q \geq 2$. Note that $s = 2p + q + 2$, so $\operatorname{diam}(T) = 2p + q + 3$.

Let x be the leaf adjacent to v_{p+2}. Consider the dacards obtained by deleting v_p (with degree 2) and x. Note that $T - x = C(1, 0^{(p+1)}, 1^{(q-1)}, 0^{(p)}, 1)$ and $T - v_p = P_p + C(2, 1^{(q-1)}, 0^{(p)}, 1)$. Let G be a reconstruction from these two dacards, with $G - u \cong T - x$ and $G - v \cong T - v_p$. As usual, the leaf dacard forces G to be a tree. Since $\operatorname{diam}(G - u) = 2p + q + 3 = \operatorname{diam} T$, the neighbors of v in G must be endpoints of longest paths in the two components of $G - v$. Hence $G \cong T$ or $G = C(2, 1^{(q-1)}, 0^{(2p+1)}, 1)$, depending on which end of the longest path in the non-path component in $G - v$ is adjacent to v.

In the latter case, since the spine endpoints in $G - u$ each have only one leaf neighbor, u must be adjacent in G to the spine vertex having two leaf neighbors. Now $G - u \cong C(1^{(q)}, 0^{(2p+1)}, 1)$. Since $p \geq 1$ and $q \geq 2$, this graph is not isomorphic to $T - x$, a contradiction. Hence this case does not arise, and $G \cong T$. $\square$

We now have the tools to prove the main result of this section.

Theorem 5.6. *If* $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$*, then* $\mathrm{drn}(T) = 2$.

Proof. By Proposition 4.8, we may assume that T is not a path. In Lemma 5.2, we proved that the dacards for the leaves adjacent to v_1 and the next spine vertex having a leaf neighbor determine T unless both T and its reverse description $C(a_s, \ldots, a_1)$ have the forms specified in Lemma 5.2. If the description is as in (1) of Lemma 5.2, then T is a path plus one pendant edge, and Proposition 5.3 yields $\mathrm{drn}(T) \le 2$.

Hence we may assume that both T and the reverse description T' are as in (2) of Lemma 5.2. If $L = (a_1, \ldots, a_s)$, then

$$L = (1, 0^{(p+1)}, k, (\alpha), k-1, 0^{(p)}, 1) = (1, 0^{(q)}, \ell - 1, (\beta), \ell, 0^{(q+1)}, 1)$$

for some palindromes (α) and (β) and integers p, q, k, ℓ such that $p, q \ge 0$ and $k, \ell \ge 1$.

Suppose that $k \ge 2$. The last nonzero entry of L before a_s is both a_{s-p-1} and a_{s-q-2}, so $q = p - 1$ and $\ell = k - 1$. Hence

$$L = (1, 0^{(p+1)}, k, (\alpha), k-1, 0^{(p)}, 1) = (1, 0^{(p-1)}, k-2, (\beta), k-1, 0^{(p)}, 1),$$

which implies that $k = 2$ and that both $(a_{p+4}, \ldots, a_{s-p-2})$ and $(a_{p+2}, \ldots, a_{s-p-2})$ are palindromes. Since $a_{p+2} = 0 \ne k = a_{p+3}$, Lemma 5.1 yields $T = C(1, 0^{(p+1)}, (2, 0)^{(s/2-p-2)}, 1, 0^{(p)}, 1)$, where s is even and $p \ge 1$. Since L contains at least one 2, Proposition 5.4 yields $\mathrm{drn}(T) \le 2$.

By reversing L, the same argument holds when $\ell \ge 2$. Finally, when $k = \ell = 1$,

$$L = (1, 0^{(p+1)}, 1, (\alpha), 0^{(p+1)}, 1) = (1, 0^{(q+1)}, (\beta), 1, 0^{(q+1)}, 1).$$

Since $a_{p+3} = 1$ and $a_2 = \cdots = a_{q+2} = 0$, we have $p \ge q$. Since $a_{s-q-2} = 1$ and $a_{s-p-1} = \cdots = a_{s-1} = 0$, we have $q \ge p$. Thus $p = q$, and $(a_{p+4}, \ldots, a_{s-p-2})$ and $(a_{p+3}, \ldots, a_{s-p-3})$ are palindromes. Since $a_{p+3} = a_{s-p-2} = 1$, Lemma 5.1 implies that $a_{p+3} = \cdots = a_{s-p-2} = 1$, so $T = C(1, 0^{(p+1)}, 1^{(s-2p-4)}, 0^{(p+1)}, 1)$. By Proposition 5.5, again $\mathrm{drn}(T) \le 2$. $\square$

6 General caterpillars

Having shown that $\mathrm{drn}(T) \le 2$ whenever T has the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, we may exclude such caterpillars (and stars) from our study of general caterpillars. In the general case, we will use the dacards obtained by deleting the first spine vertex v_1 and one of its leaf neighbors. These determine T except in some cases. Again we handle the exceptional cases separately, using other dacards. The next several propositions handle these cases. Note that setting $k = 0$ in the first yields a path.

Proposition 6.1. *If* $T = C(k+1, k^{(m)}, k+1)$*, where* $k, m \ge 1$*, then* $\mathrm{drn}(T) = 2$.

Proof. The cards obtained by deleting leaf neighbors of v_1 and v_2 are $C(k^{(m+1)}, k+1)$ and $C(k+1, k-1, k^{(m-1)}, k+1)$. Let G be a reconstruction from these dacards, with u and v respectively being the added vertices of degree 1; G must be a tree. Since the endpoints of the spine in $G-v$ both have $k+1$ leaf neighbors, G has two vertices at distance $m+1$ that each have at least $k+1$ leaf neighbors. Since $G-u$ has only one vertex with $k+1$ leaf neighbors, the neighbor of u in $G-u$ must have distance $m+1$ from the spine endpoint having $k+1$ leaf neighbors. There is only one such vertex, so $G \cong C(k+1, k^{(m)}, k+1)$. $\square$

A *branch vertex* is a vertex with degree at least 3. Let B_k denote the caterpillar formed by giving two leaf neighbors to one end of P_k. Let z_k denote the third leaf in B_k.

Proposition 6.2. *If* $T = C(2, 0^{(s-2)}, 2)$*, where* $s \geq 3$*, then* $\mathrm{drn}(T) = 2$.

Proof. Let $p = \lceil s/2 \rceil$. Note that v_p is centroidal in T and v_{p-1} is not. The cards C_1 and C_2 obtained by deleting v_p and v_{p-1} are $B_{p-1} + B_{s-p}$ and $B_{p-2} + B_{s-p+1}$, respectively. Let $D_1 = (C_1, 2)$ and $D_2 = (C_2, 2)$; these are the dacards for v_p and v_{p-1} when $s \geq 5$. We postpone the special cases $s = 4$ and $s = 3$ (when $s = 2$, the caterpillar reduces to H_1).

Let G be a reconstruction from $\{D_1, D_2\}$, where $C_1 = G-u$ and $C_2 = G-v$. By Lemma 4.7, G is a tree, and each of u and v has one neighbor in each component of its dacard.

Case 1: $uv \in E(G)$. Since $d_G(u) = d_G(v) = 2$, vertex v is a leaf in $G-u$, and u is a leaf in $G-v$. Thus $G-v$ can be obtained from $G-u$ by deleting the leaf v in $G-u$ and attaching u to one vertex in the other component of $G-u$. Since $p-2 < p-1 \leq s-p < s-p+1$, with the components of $G-u$ being isomorphic when $p-1 = s-p$, obtaining a component of $G-v$ by deleting a leaf of a component of $G-u$ happens only by deleting z_{p-1} from B_{p-1} to obtain B_{p-2}. Hence B_{s-p+1} is the component of $G-v$ containing u, and it arises from B_{s-p} only by attaching u to z_{s-p}. Now $G \cong T$.

Case 2: $uv \notin E(G)$. Let Q and Q' be the components of $G-u$, with $v \in V(Q)$. Since $uv \notin E(G)$, we have $d_{G-u}(v) = 2$. Now v is a cut-vertex of Q. Let q be the order of the component of $Q-v$ not containing the neighbor of u in $V(Q)$. It follows that $G-v$ has components of orders q and $s+3-q$; we also know that these values are p and $s-p+3$. Since the orders of Q and Q' differ by at most one, we have $q < s+3-q$. We conclude that $q = p$. To accommodate the inclusion of vertex v and another vertex, Q needs at least $p+2$ vertices, so $Q = B_{s-p} \cong B_p$ (with s even), v is the vertex of B_{s-p} adjacent to z_{s-p}, and u is adjacent to z_{s-p}. Now examination of $G-v$ shows that the neighbor of u in B_{p-1} is z_{p-1}, and again $G \cong T$.

In either case, when $s \geq 5$, we conclude that $G \cong T$. For $s \in \{3, 4\}$, we again use dacards for v_p and v_{p-1}, but now $p = 2$, and we obtain $C_1 = P_3 + B_{s-p}$ and $C_2 = 2K_1 + B_{s-p+1}$, with $D_1 = (C_1, 2)$ and $D_2 = (C_2, 3)$. Although Lemma 4.7 does not apply, still every reconstruction G (with $C_1 = G-u$ and $C_2 = G-v$) is a tree. This holds because D_1 implies that G has no isolated vertex, and then D_2 gives v a neighbor in each component of $G-v$.

If $s = 3$, then $C_1 = 2P_3$, which yields $\Delta(G) \leq 3$. Hence we cannot make v adjacent to the center of B_{s-p+1} (which equals $K_{1,3}$), and making it adjacent to a leaf of B_{s-p+1} yields $G \cong T$.

If $s = 4$, then $T = C(2, 0, 0, 2)$, with $C_1 = P_3 + K_{1,3}$ and $C_2 = 2K_1 + B_3$. If v is adjacent to z_3 in the component B_3 of $G-v$, then $G \cong T$, so we exclude the other three possibilities. If G has a vertex x of degree 4, then $\Delta(G-u) = \Delta(G-v) = 3$ requires $u, v \in N_G(x)$. Now x has a neighbor v of degree 3, but restoring u to $G-u$ gives x no neighbor with degree more than 3. Hence $\Delta(G) = 3$. This requires u to be adjacent to the central vertex of P_3 and a leaf of $K_{1,3}$ in the two components of C_1, yielding $G \cong T$. $\square$

Proposition 6.3. *If $T = C(k+2, (0, k)^{(m)}, 0, k+2)$, with $k \geq 0$ and $m \geq 1$, then $\mathrm{drn}(T) \leq 2$.*

Proof. The case where $k = 0$ is a special case of Proposition 6.2, so we may assume that $k \geq 1$. In that case T is unicentroidal and has a leaf adjacent to the centroid whose deletion leaves a unicentroidal subtree. By Proposition 4.6, $\mathrm{drn}(T) = 2$. $\square$

For a general caterpillar T, with $T = C(a_1, \ldots, a_s)$, we want to make a uniform choice of two dacards. The main lemma shows that this choice determines T unless T belongs to one of several exceptional classes of caterpillars. The proof of the theorem then uses the classes we have already discussed to handle the exceptional classes.

Lemma 6.4. *If $T = C(a_1, \ldots, a_s)$, then the dacards for an endpoint of the spine and one of its leaf neighbors determine T unless T is* Type t *for* t $\in \{1, 2, 3, 4\}$*, defined as follows:*
(1) $T = C(1, 0, a_3, \ldots, a_s)$ *with* $s \geq 3$;
(2) $T = C(2, (0, 0)^{(m)}, (1, 0)^{(n)}, 2)$ *with* $m, n \geq 0$;
(3) $T = C(k+1, k^{(m)}, (k+1)^{(n)})$ *with* $k, m, n \geq 1$;
(4) $T = C(k+2, (0, k)^{(m)}, (0, k+1)^{(n)}, 0, k+2)$ *with* $k, n \geq 0$ *and* $m \geq 1$.

Proof. Since $\mathrm{drn}(K_{1,t}) = 1$, we may assume that $s \geq 2$. Let $\langle v_1, \ldots, v_s \rangle$ be the spine of T. Recall that $a_1, a_s \geq 1$. Specify the dacards by deleting v_1 and by deleting a leaf neighbor ℓ of v_1. Let $T_1 = T - \ell$, and let T_2 be the nontrivial component of $T - v_1$. Thus the dacards are $(a_1 K_1 + T_2, a_1 + 1)$ and $(T_1, 1)$. From the dacard $(T_1, 1)$, any reconstruction G is a tree. Define u and v by $G - u = T_1$ and $G - v = a_1 K_1 + T_2$. Let x the neighbor of u in G, and let y be the non-leaf neighbor of v in G (since G is a tree, $d_G(v) = a_1 + 1$ forces v to have one neighbor in T_2).

Define r and s by letting the spine of T_2 be $\langle v_r, \ldots, v_s \rangle$ and the spine of T_1 be $\langle v_q, \ldots, v_s \rangle$. We list four events; always (U1 or U2) and (V1 or V2) occurs. Note that if U1 and V1 occur, then T has Type 1, so we may assume that this case does not occur (and also that $G \ncong T$).

$$\begin{aligned} &\text{U1:} \quad a_1 = 1, \quad q = 2, \quad \operatorname{diam} T_1 = s. \\ &\text{U2:} \quad a_1 > 1, \quad q = 1, \quad \operatorname{diam} T_1 = s + 1. \\ &\text{V1:} \quad a_2 = 0, \quad r = 3, \quad \operatorname{diam} T_2 = s - 1. \\ &\text{V2:} \quad a_2 > 0, \quad r = 2, \quad \operatorname{diam} T_2 = s. \end{aligned}$$

We call the descriptions of G obtained from $G - u$ and $G - v$ the *u-description* and the *v-description* of G. The cases depend on the location of y in T_2. Most importantly, this determines whether G is a caterpillar.

Case 1: y is in $\{v_{r+1}, \ldots, v_{s-1}\}$ or is a leaf neighbor of such a vertex. Since we make v (with its a_1 leaf neighbors) adjacent to y, in this case G is not a caterpillar. The u-description also produces G and hence is not a caterpillar. Thus x is a leaf neighbor of a vertex in $\{v_{q+1}, \ldots, v_{s-1}\}$. The skeleton G' has three leaves. In the v-description, the leaves are v_r, v_s, and v; also, G' has $s-r+1$ edges if y is in the spine of T_2, otherwise $s-r+2$. In the u-description, the leaves of G' are v_q, v_s and x, and G' has $s-q+1$ edges.

Let S_u and S_v denote the multiset of degrees in G of the leaves of G' under the u-description and v-description of G, respectively. Equating the numbers of edges of G' in the two descriptions yields several possibilities.

(i) If $q=1$, then $r=2$ and y is not in the spine of T_2. Now $S_u = \{a_1, a_s+1, 2\}$ and $S_v = \{a_2+1, a_s+1, a_1+1\}$. Equality requires $a_1 = 1$, which contradicts $q=1$.

(ii) If $q=2$, then $r=2$, since otherwise T has Type 1. Now $S_u = \{a_2+2, a_s+1, 2\}$ and $S_v = \{a_2+1, a_s+1, 2\}$, and equality cannot hold.

Case 2: y is a leaf neighbor of v_r or v_s. For such y, if $a_2 = 0$ and hence $r=3$, then $G \cong T$ or $G = C(a_3+1, a_4, \ldots, a_{s-1}, a_s-1, 0, a_1)$. If $a_2 > 0$ and hence $r=2$, then $G = C(a_1, 0, a_2-1, a_3, \ldots, a_s)$ or $G = C(a_2, \ldots, a_{s-1}, a_s-1, 0, a_1)$.

Subcase 2a: $a_2 = 0$. Here $G = C(a_3+1, a_4, \ldots, a_{s-1}, a_s-1, 0, a_1)$. Avoiding Type 1 requires $a_1 > 1$ and $q=1$, so $\operatorname{diam} T_1 = s+1 = \operatorname{diam} G$. Since G is a caterpillar with diameter $\operatorname{diam} T_1$, vertex x is on the spine of T_1, say $x = v_j$. With $G \not\cong T$, we have $j > 1$, and the u-description is $G = C(a_1-1, a_2, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_s)$, with $a_2 = 0$.

In obtaining the multiset of leaf degrees for G from that of T, in both the v-description and the u-description one term increases and one term decreases. The values that change must be the same in each instance; hence $a_1 = a_s$ and $a_3 = a_j$. Since $a_1 \neq a_1 - 1$, the descriptions match without reversal. Since $a_s = a_1 = a_3 + 2 = a_j + 2 > a_j + 1 > 0$, we conclude that $j \leq s-2$. Since $0 = a_{s-1} = a_{s-3} = \cdots$, we conclude that $s-j$ is even (otherwise $a_j + 1 = 0$).

If j and s are even, then $0 = a_2 = a_4 = \cdots = a_j = a_3$, so $a_1 = a_s = 2$. Also $0 = a_3 = \cdots = a_{s-1}$. Since $a_j+1 = a_{j+2} = \cdots = a_{s-2} = a_s - 1 = 1$, we find that T is Type 2.

If j and s are odd, then $0 = a_2 = \cdots a_{s-1}$ and $a_1 - 2 = a_3 = \cdots = a_j$ and $a_j + 1 = a_{j+2} = \cdots = a_{s-2} = a_s - 1$, with $j \geq 3$. Letting $k = a_3$, we find that T is Type 4.

Subcase 2b: $a_2 > 0$. Here $\operatorname{diam} T_2 = s$, and hence $\operatorname{diam} G = s+2$. Since adding u to T_1 can only add 1 to the diameter, $\operatorname{diam} T_1 = s+1$, and hence $q = 1$, $a_1 > 1$, and x is a leaf neighbor of v_1 or v_s. Now the u-description is $G = C(1, a_1-2, a_2, \ldots, a_s)$ or $G = C(a_1-1, a_2, \ldots, a_{s-1}, a_s-1, 1)$.

In both possibilities for the v-description with $a_2 > 0$, one end of the spine of G has a_1 leaf neighbors. Since $a_1 \notin \{1, a_1-1\}$, the second possibility for the u-description is forbidden. Furthermore, since $a_1 > 1$, the first possibility must be oriented so that a_s in the u-description matches up with a_1 in the v-description. We have two choices.

(i) $(a_s, \ldots, a_3, a_2-1, 0, a_1) = (1, a_1-2, a_2, \ldots, a_s)$. This is forbidden, since it requires $1 = a_s = a_1$, but $a_1 > 1$.

(ii) $(a_2, \ldots, a_{s-1}, a_s-1, 0, a_1) = (1, a_1-2, a_2, \ldots, a_s)$. Since $0 = a_{s-1} = a_{s-3} = \cdots$ and $1 = a_2 = a_4 = \cdots$, we conclude that s is even. Now $a_1 - 2 = 0$ and $a_s - 1 = 1$, and T is Type 2 with $m = 0$.

Case 3: $y \in \{v_r, v_s\}$. If $y = v_r$, then $G \cong T$, so we may assume $y = v_s$ and G is a caterpillar with diameter $s-r+3$. Since G is a caterpillar, x is a spine vertex of T_1 or a leaf neighbor of v_q or v_s.

If x is a leaf neighbor of v_q or v_s, then adding u to T_1 enlarges the diameter, so $\operatorname{diam} G = s-q+3$. Hence $q=r$, which requires $a_1=1$ and $a_2>0$, and $q=r=2$. Since $a_1=1$, setting x to a leaf neighbor of v_2 yields $G\cong T$. Hence the v-description is $G=C(a_2,\dots,a_s,a_1)$ and the u-description is $G=C(a_2+1,a_3,\dots,a_{s-1},a_s-1,1)$. Since $a_2>0$, the descriptions must match up without reversal, which fails because $a_2\neq a_2+1$.

Finally, we may assume that x is a spine vertex v_j in T_1. Now $\operatorname{diam} G = s-q+2$, so $q=r-1$. Avoiding Type 1 leaves only $q=r-1=1$, so $a_1>1$ and $a_2>0$. If $j=1$, then $G\cong T$, so $j>1$. Now the v-description is $G=C(a_2,\dots,a_s,a_1)$ and the u-description is $G=C(a_1-1,a_2,\dots,a_{j-1},a_j+1,a_{j+1},\dots,a_s)$. Since $a_1-1\neq a_1$, the descriptions must match up without reversal. Two posibilities remain.

(i) If $j=s$, then matching positions yields $a_1-1=a_2=\cdots=a_s$. Now the v-description of G is the reverse of the original description of T, and hence $G\cong T$.

(ii) If $1<j<s$, then matching positions yields $a_1-1=a_2=\cdots=a_j=a_{j+1}-1=\cdots=a_s-1$. Letting $a_2=k$, we have $T=C(k+1,k^{(j-1)},(k+1)^{(s-j)})$. We may assume that $k\geq 1$, since otherwise T is Type 1. Now T is Type 3. ☐

Theorem 6.5. *If T is a caterpillar that is neither H_1 nor a star, then* $\operatorname{drn}(T)=2$.

Proof. Let $T=C(a_1,\dots,a_s)$. As in Section 5, reversing the order of the spine vertices does not change the isomorphism class of a caterpillar; $T\cong T'$, where $T'=C(a_s,\dots,a_1)$. In Lemma 6.4 we used dacards corresponding to the first spine endpoint and a leaf adjacent to it, but similar results hold by taking dacards corresponding to the *last* spine vertex and a leaf adjacent to it. Thus our choice of two dacards, from one end of T or the other, uniquely determines T unless both T and T' have a Type listed in Lemma 6.4.

Suppose first that T is Type 1. If T' also is Type 1, then $T=C(1,0,a_3,\dots,a_{s-2},0,1)$, and $\operatorname{drn}(T)\leq 2$ by Proposition 5.6. Since all other Types end with $a_s>1$, but $a_1=1$, the reversal of a Type 1 caterpillar cannot be Type 2, 3, or 4. This completes the proof when T (or T') is Type 1.

Suppose next that T is Type 2. Since the length of the spine has different parity in Type 2 and Type 4, T' is not of Type 4. If T' is Type 2 or Type 3, then either $T=C(2,2)$ and $T\cong H_1$, or $T=C(2,(0,0)^{(m)},2)$ with $m\geq 1$, in which case $\operatorname{drn}(T)\leq 2$ by Proposition 6.2. This completes the proof when T (or T') is Type 2.

If T and T' are both Type 3, then $T=C(k+1,k^{(m)},k+1)$ with $k,m\geq 1$, and $\operatorname{drn}(T)\leq 2$ by Proposition 6.1. Since the entries in specifying a Type 3 caterpillar are all positive, and for Type 4 they are not, T and T' cannot be Type 3 and Type 4.

Finally, if T and T' are both Type 4, then $n=0$. Now $\operatorname{drn}(T)\leq 2$ by Proposition 6.3.

Having exhausted all cases, the proof is complete. ☐

There is hope to complete a proof that $\operatorname{drn}(T)\leq 2$ for all but finitely many trees. Building upon our result, one can try to make a choice of two dacards that determines T when T

is not a caterpillar, with finitely many exceptions. There may be several special classes in addition to caterpillars where the dacards needs to be chosen in other ways.

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