

Decomposition of Sparse Graphs into Forests and a Graph with Bounded Degree

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Abstract

Say that a graph with maximum degree at most d is d -bounded. For $d > k$, we prove a sharp sparseness condition for decomposition into k forests and a d -bounded graph. The condition holds for every graph with fractional arboricity at most $k + \frac{d}{k+d+1}$. For $k = 1$, it also implies that every graph with maximum average degree less than $2 + \frac{2d}{d+2}$ decomposes into one forest and a d -bounded graph. When $d = k + 1$, and when $k = 1$ and $d \leq 6$, the d -bounded graph in the decomposition can also be required to be a forest, and for $(k, d) = (1, 2)$ it can also be required to have at most d edges in each component. For $d \leq k$, we prove that every graph with fractional arboricity at most $k + \frac{d}{2k+2}$ decomposes into $k + 1$ forests, of which one is d -bounded.

1 Introduction

A *decomposition* of a graph G consists of edge-disjoint subgraphs with union G . The *arboricity* of G , written $\Upsilon(G)$, is the minimum number of forests needed to decompose it. The famous Nash-Williams Arboricity Theorem states that a necessary and sufficient condition for $\Upsilon(G) \leq k$ is that no subgraph H has more than $k(|V(H)| - 1)$ edges. This is a sparseness condition. A slightly different sparseness condition places a bound on the average vertex degree in all subgraphs. The *maximum average degree* of a graph G , denoted $\text{Mad}(G)$, is

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$\max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$; it is the maximum over subgraphs H of the average vertex degree in H . (Our model of “graph” allows multiedges but no loops.)

Many papers have obtained various types of decompositions from bounds on $\text{Mad}(G)$. Our results extend some of these and the Nash-Williams Theorem, which states that $\Upsilon(G) = \left\lceil \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1} \right\rceil$. We consider the *fractional arboricity* $\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$, introduced by Payan [13]; for this we use the notation $\text{Arb}(G)$, by analogy with $\text{Mad}(G)$.

Three forests are needed to decompose a graph with fractional arboricity $2 + \epsilon$, but since this is just slightly above 2 one may hope that some restrictions can be placed on the third forest. Say that a graph is d -bounded if it has maximum degree at most d . Montassier et al. [11] posed the *Nine Dragon Tree (NDT) Conjecture* (honoring a famous tree in Kaohsiung, Taiwan that is far from acyclic): If $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$, then G decomposes into $k+1$ forests with one being d -bounded. They proved the cases $(k, d) = (1, 1)$ and $(k, d) = (1, 2)$, and they showed that no larger value of $\text{Arb}(G)$ is sufficient. In Section 3 we will prove the case $d = k + 1$. In Sections 4–6 we will prove the cases $(k, d) = (1, d)$ for $d \leq 6$.

Another line of research considers decomposing a planar graph into a forest and a d -bounded graph, following the seminal paper [8], which motivated the topic by its application to “game coloring number”. For a planar graph with girth g to decompose into a forest and a matching, $g \geq 8$ suffices [11, 14] (earlier sufficiency was proved for $g \geq 11$ in [8], for $g \geq 10$ in [2], and for $g \geq 9$ in [6]). Also, the graph left by deleting the edges of a forest can be guaranteed to be 2-bounded when $g \geq 7$ [8] (improved to $g \geq 6$ in [9]) and 4-bounded when $g \geq 5$ [8]. Borodin, Ivanova, and Stechkin [3] disproved the conjecture from [8] that every planar graph G decomposes into a forest and a $(\lceil \Delta(G)/2 \rceil + 1)$ -bounded graph. In [4], there are sufficient conditions for a planar graph with triangles to decompose into a forest and a matching, and [5] shows that a planar graph without 4-cycles (3-cycles are allowed) decomposes into a forest and a 5-bounded graph.

Many conclusions on planar graphs with large girth hold more generally when only the corresponding bound on $\text{Mad}(G)$ is assumed. If G is a planar graph with girth g , then G has at most $\frac{g}{g-2}(n-2)$ edges, by Euler’s Formula. This holds for all subgraphs, so girth g implies $\text{Mad}(G) < \frac{2g}{g-2}$. Montassier et al. [10] posed the question of finding the weakest bound on $\text{Mad}(G)$ to guarantee decomposition into one forest and a d -bounded graph. They proved that $\text{Mad}(G) < 4 - \frac{8d+12}{d^2+6d+6}$ is sufficient and that $\text{Mad}(G) = 4 - \frac{4}{d+2}$ is not (seen by subdividing every edge of a $(2d+2)$ -regular graph). The case $k = 1$ of our Theorem 1.1 completely solves this problem, implying that $\text{Mad}(G) < 4 - \frac{4}{d+2}$ suffices.

Our result also implies the previous girth results for decomposition of planar graphs into one forest and a d -bounded graph. Girth 8, 6, and 5 imply that $\text{Mad}(G)$ is less than $8/3$, 3, and $10/3$, respectively, which are precisely the bounds that by our result guarantee decomposition into one forest and a graph with maximum degree at most 1, 2, or 4, respectively.

Other work brought these problems closer together, requiring the leftover d -bounded graph to be a forest or considering the leftover after deleting more than one forest. For convenience in this paper, let a (k, d) -decomposition of a graph G be a decomposition of G into k forests and one d -bounded graph, and let a $(k, d)^*$ -decomposition be a (k, d) -decomposition in which the “leftover” d -bounded graph also is a forest. Graphs having such decompositions are (k, d) -decomposable or $(k, d)^*$ -decomposable, respectively.

Examples of planar graphs with girth 7 having no $(1, 1)$ -decomposition and examples with girth 5 having no $(1, 2)$ -decomposition appear in [11, 9]. Gonçalves [7] proved the conjecture of Balogh et al. [1] that every planar graph is $(2, 4)^*$ -decomposable. He also proved that planar graphs with girth at least 6 are $(1, 4)^*$ -decomposable and with girth at least 7 are $(1, 2)^*$ -decomposable.

The NDT Conjecture is that $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ guarantees a $(k, d)^*$ -decomposition. The fractional arboricity of a planar graph can be arbitrarily close to 3, which is not small enough for the NDT Conjecture to guarantee $(2, d)^*$ -decomposability for any constant d . However, requiring girth at least 6 or 7 yields fractional arboricity less than $6/4$ or $7/5$, respectively, in which case the NDT Conjecture would guarantee $(1, 4)^*$ - or $(1, 2)^*$ -decompositions, respectively. Hence the NDT Conjecture implies the results of Gonçalves for $(1, d)^*$ -decomposition of planar graphs with large girth, but not his result on $(2, 4)^*$ -decomposition.

Our Theorem 1.1 holds whenever $d > k$ but yields only a (k, d) -decomposition, weaker than the NDT Conjecture. To understand the relationship between the two problems and develop the statement of Theorem 1.1, we compare $\text{Mad}(G)$ and $\text{Arb}(G)$. Always $\text{Mad}(G) < 2\text{Arb}(G)$, but the conditions $\text{Mad}(G) < 2a$ and $\text{Arb}(G) \leq a$ are not equivalent.

To compute $\text{Arb}(G)$ or $\text{Mad}(G)$, it suffices to perform the maximization only over induced subgraphs. Letting $G[A]$ denote the subgraph of G induced by a vertex set A , we write $\|A\|$ for the number of edges in $G[A]$ (and $|A|$ for the number of vertices). We restate the conditions as integer inequalities. Since $k(k + d + 1) + d = (k + 1)(k + d)$, we have the following comparison, introducing an intermediate condition we call (k, d) -sparse:

Condition	Equivalent constraint (when imposed for all $A \subseteq V(G)$)
$\text{Arb}(G) \leq k + \frac{d}{k+d+1}$	$(k + 1)(k + d) A - (k + d + 1) \ A\ - (k + 1)(k + d) \geq 0$
$\text{Mad}(G) < 2k + \frac{2d}{k+d+1}$	$(k + 1)(k + d) A - (k + d + 1) \ A\ - 1 \geq 0$
(k, d) -sparse	$(k + 1)(k + d) A - (k + d + 1) \ A\ - k^2 \geq 0$

Since $(k + 1)(k + d) > k^2 \geq 1$, the condition on $\text{Arb}(G)$ implies (k, d) -sparseness, which in turn implies the condition on $\text{Mad}(G)$. By showing that (k, d) -sparseness suffices, Theorem 1.1 thus implies that $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ suffices for G to be (k, d) -decomposable, but $\text{Mad}(G) < 2k + \frac{2d}{k+d+1}$ might not. However, since $k^2 = 1$ when $k = 1$, the $(1, d)$ -sparseness condition is the same as the desired condition $\text{Mad}(G) < 4 - \frac{4}{d+2}$ for the problem in [10].

Theorem 1.1. *For $d > k$, every (k, d) -sparse graph is (k, d) -decomposable. Furthermore, the condition is sharp.*

Our proof of Theorem 1.1 in Section 2 is purely inductive, proving a technically stronger statement involving a more general restriction on the degrees of vertices in the leftover d -bounded graph. Further motivation for the constant in the (k, d) -sparseness condition comes from the sharpness example in Section 2.

Meanwhile, Theorem 1.1 says nothing about the case $d \leq k$. In Section 3, we prove a result implying that a stronger condition on $\text{Arb}(G)$ than in the NDT Conjecture suffices to guarantee the stronger property of $(k, d)^*$ -decomposability when $d \leq k + 1$. This condition is $\text{Arb}(G) \leq k + \frac{d}{2k+2}$. When $d = k + 1$, this bound equals $k + \frac{d}{k+d+1}$, so this theorem implies the case $d = k + 1$ of the NDT Conjecture.

In Sections 4–6, we prove the NDT Conjecture for $(k, d) = (1, d)$ with $d \leq 6$, in a form that requires only (k, d) -sparseness as long as small graphs violating $\text{Arb}(G) \leq k + 1$ are forbidden. Meanwhile, the *Strong NDT Conjecture* asserts that $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ guarantees a $(k, d)^*$ -decomposition in which every component of the d -bounded forest has at most d edges. We prove this for $(k, d) = (1, 2)$ in Section 7 (the result of [11] implies it for $(k, d) = (1, 1)$). The results of Sections 4–7 use reducible configurations and discharging.

2 (k, d) -decomposition for $d > k$

We begin with a general example showing that Theorem 1.1 is sharp. This example also motivates the constant in the condition for (k, d) -sharpness. In studying (k, d) -decomposability of a graph G , define $\beta(A) = (k + 1)(k + d) |A| - (k + 1 + d) \|A\| - k^2$ for $A \subseteq V(G)$. The (k, d) -sparseness condition is that $\beta(A) \geq 0$ for all nonempty A .

Example 2.1. We construct a bipartite graph G with partite sets X and Y of sizes s and t , respectively. Let $s = t(k + d) - k + 1$, so $|V(G)| = t(k + d + 1) - k + 1$. With $X = \{x_1, \dots, x_s\}$ and $Y = \{y_1, \dots, y_t\}$, make x_i adjacent to y_i, \dots, y_{i+k} , where indices are taken modulo t . Every vertex in X has degree $k + 1$, so $|E(G)| = (k + 1)(k + d)t - k^2 + 1$.

A d -bounded subgraph of G has at most dt edges. Deleting a d -bounded subgraph thus leaves at least $k(k + d)t + kt - k^2 + 1$ edges. However, k forests in G cover at most $k[t(k + d + 1) - k]$ edges. Hence G is not (k, d) -decomposable.

On the other hand, G just barely fails to be (k, d) -sparse. If $|A| = 1$, then $\beta(A) = kd + k + d$. Now choose A to minimize $\beta(A)$ among subsets of $V(G)$ with size at least 2. If some vertex $v \in A$ has at most k neighbors in A , then $\beta(A - v) \leq \beta(A) - d$, contradicting the choice of A . Therefore, all $k + 1$ neighbors of each vertex in $A \cap X$ are also in A . Let $s' = |A \cap X|$ and $t' = |A \cap Y|$. Now

$$\begin{aligned}\beta(A) &= (k+1)(k+d)(s'+t') - (k+d+1)(k+1)s' - k^2 \\ &= (k+1)(k+d)t' - s'(k+1) - k^2 = (k+1)[(k+d)t' - s' - k + 1] - 1.\end{aligned}$$

We conclude that $\beta(A) \geq 0$ if and only if $s' \leq (k+d)t' - k$. When $t' = t$, this yields $\beta(A) < 0$ if and only if $A = V(G)$.

If $t' < t$, then each vertex of $Y - A$ forbids all its neighbors from A . For fixed t' , we maximize s' and minimize $\beta(A)$ for such A by letting $Y \cap A = \{y_1, \dots, y_{t'}\}$ (this makes the forbidden subsets of X overlap as much as possible). Writing $i = qt + r$ with $q \geq 0$ and $1 \leq r \leq t$, this allows $x_i \in A$ only when $1 \leq r \leq t' - k$. With $s = t(k+d) - k + 1$, we have $s' \leq (k+d)(t' - k) < (k+d)t' - k$.

We conclude that $\beta(A) \geq 0$ except when $A = V(G)$. The choice of the constant k^2 in the definition of β has enabled us to construct a graph that fails to be (k, d) -decomposable with the slightest possible failure of (k, d) -sparseness. \square

We prove Theorem 1.1 in a seemingly more general form to facilitate the inductive proof, but we will show at the end of this section that the more general form is equivalent to Theorem 1.1. Prior results in this area have been proved by the discharging method, which uses properties of a minimal counterexample G to contradict the hypothesized sparseness. Replacing the constant bound d on vertex degrees by an individual bound for each vertex permits a simple inductive proof without using discharging.

Definition 2.2. Fix positive integers d and k . A *capacity function* on a graph G is a function $f: V(G) \rightarrow \{0, \dots, d\}$. A (k, f) -*decomposition* of G decomposes it into k forests and a graph D such that each vertex v has degree at most $f(v)$ in D . For each vertex set A in G , let

$$\beta_f(A) = (k+1) \sum_{v \in A} (k + f(v)) - (k+d+1) \|A\| - k^2.$$

A capacity function f on G is *feasible* if $\beta_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$.

The idea is to reserve an edge uv for use in D by deleting it and reducing the capacity of its endpoints (when both have positive capacity). If the reduced function f' is feasible on $G - uv$, then the induction hypothesis will complete a (k, f) -decomposition. We will use this idea to reduce to the case where the vertices with positive capacity form an independent set.

To prove feasibility for f' , we must show $\beta_{f'}(A) \geq 0$ for $A \neq \emptyset$. The endpoints of the deleted edge may be both outside A (no problem), both in A (still easy), or just one in A . The latter case is problematic when $\beta_f(A) \leq k$, since $\beta_{f'}(A) = \beta_f(A) - (k+1)$. In this situation we will assemble a (k, f) -decomposition inductively by combining a decomposition

of $G[A]$ with a decomposition of the subgraph obtained by contracting A to one vertex. We begin with the definitions and lemmas needed to do that.

Definition 2.3. For $B \subseteq V(G)$, let G_B denote the graph obtained by contracting B into a new vertex z . The degree of z in G_B is the number of edges joining B to $V(G) - B$ in G ; edges of G with both endpoints in B disappear.

Lemma 2.4. *If f is a feasible capacity function on G , and B is a proper subset of $V(G)$ such that $|B| \geq 2$ and $\beta_f(B) \leq k$, then f^* is a feasible capacity function on G_B , where $f^*(z) = 0$ and f^* agrees with f on $V(G) - B$.*

Proof. For $A \subseteq V(G_B)$, we have $\beta_{f^*}(A) = \beta_f(A) \geq 0$ if $z \notin A$. When $z \in A$, we compute $\beta_{f^*}(A)$ by comparison with $\beta_f(A')$, where $A' = (A - \{z\}) \cup B$. Every edge in $G[A']$ appears in $G_B[A]$ or $G[B]$; hence the edges contribute the same to both sides of the equation below. Comparing the terms for constants and the terms for vertices (using $f^*(z) = 0$) yields

$$\beta_f(A') = \beta_{f^*}(A) - (k + 1)k + \beta_f(B) + k^2.$$

If $\beta_f(B) \leq k$, then $\beta_{f^*}(A) \geq \beta_f(A') \geq 0$. □

Lemma 2.5. *Let f be a capacity function on a graph G , and let B be a proper subset of $V(G)$. If $G[B]$ is $(k, f|_B)$ -decomposable and G_B is (k, f^*) -decomposable, where f^* is defined from f as in Lemma 2.4, then G is (k, f) -decomposable.*

Proof. Let (F, D) be a $(k, f|_B)$ -decomposition of $G[B]$, and let (F', D') be a (k, f^*) -decomposition of G_B , where F and F' are unions of k forests. Each edge of G is in $G[B]$ or G_B , becoming incident to z in G_B if it joins B to $V(G) - B$ in G . View $(F \cup F', D \cup D')$ as a decomposition of G by viewing the edges incident to z in F' as the corresponding edges in G .

The resulting decomposition is a (k, f) -decomposition of G . Since $f^*(z) = 0$, vertex z has degree 0 in D' , and all edges joining B to $V(G) - B$ lie in F' . Hence the restrictions from f are satisfied by $D \cup D'$. For each forest F_i among the k forests in F , its union with the corresponding forest F'_i in F' is still a forest, since otherwise contracting the portion in F_i of a resulting cycle would yield a cycle through z in F'_i when viewed as a forest in G' . □

Theorem 2.6. *If $d > k$ and G is a graph with a feasible capacity function f , then G is (k, f) -decomposable.*

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most k edges. For the induction step, suppose that G is larger.

If $\beta_f(B) \leq k$ for some proper subset B of $V(G)$ with $|B| \geq 2$, then the capacity function f^* on G_B that agrees with f except for $f^*(z) = 0$ is feasible, by Lemma 2.4. Since $G[B]$

is an induced subgraph of G , the restriction of f to B is feasible on $G[B]$. Since G_B and $G[B]$ are smaller than G , by the induction hypothesis G_B is $(k, f|_B)$ -decomposable and G_B is (k, f^*) -decomposable. By Lemma 2.5, G is $(k\mathcal{F}, \mathcal{D}_f)$ -decomposable.

Hence we may assume that $\beta_f(B) \geq k+1$ for all such B . Let $S = \{v \in V(G) : f(v) > 0\}$. If S has adjacent vertices u and v , then let f' be the capacity function on $G - uv$ that agrees with f except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. If f' is feasible, then since $G - uv$ is smaller than G , it has a (k, f') -decomposition, and we add uv to the degree-bounded subgraph to obtain a (k, f) -decomposition of G .

To show that f' is feasible, consider $A \subseteq V(G') = V(G)$. If $u, v \notin A$, then $\beta_{f'}(A) = \beta_f(A)$. If $u, v \in A$, then the reduction in f and loss of one edge yield $\beta_{f'}(A) = \beta_f(A) - 2(k+1) + (k+d+1) \geq \beta_f(A)$, where the last inequality uses $d > k$. If exactly one of $\{u, v\}$ is in A , then A is a proper subset of $V(G)$. If $|A| \geq 2$, then $\beta_{f'}(A) = \beta_f(A) - (k+1) \geq 0$. If $|A| = 1$, then $\beta_{f'}(A) \geq k$, since G' has no loops.

Hence we may assume that S is independent. In this case, we show that G decomposes into k forests, yielding a (k, f) -decomposition of G in which the last graph has no edges. If $\Upsilon(G) > k$, then $V(G)$ has a minimal subset A such that $\|A\| \geq k(|A| - 1) + 1$ (note that $|A| \geq 2$). By this minimality, every vertex of A has at least $k+1$ neighbors in A . Let $A' = S \cap A$. Since S is independent, $\|A\| \geq (k+1)|A'|$. Taking $k+1$ times the first lower bound on $\|A\|$ plus d times the second yields

$$(k+1+d)\|A\| \geq (k+1)k(|A| - 1) + (k+1) + d(k+1)|A'|.$$

Now we compute

$$\begin{aligned} \beta_f(A) &= (k+1)k|A| + (k+1) \sum_{v \in A'} f(v) - (k+d+1)\|A\| - k^2 \\ &\leq (k+1)k|A| + (k+1)d|A'| - (k+1)k(|A| - 1) - (k+1) - d(k+1)|A'| - k^2 \\ &= (k+1)k - (k+1) - k^2 = -1. \end{aligned}$$

This contradicts the feasibility of f , and hence the desired decomposition of G exists. \square

The generality of the capacity function facilitates the inductive proof, and the desired statement about (k, d) -decomposition is a special case, but in fact the special case with capacity d for all v implies the general statement, making Theorem 1.1 and Theorem 2.6 equivalent. The equivalence uses the notion of “ghost” that will be helpful in Sections 4.

Definition 2.7. When considering (k, d) -decomposition, a *ghost* is a vertex of degree $k+1$ having only one neighbor (via all $k+1$ incident edges). A neighbor of v that is a ghost is a *ghost neighbor* of v .

Proposition 2.8. *Theorem 1.1 implies Theorem 2.6.*

Proof. Assume Theorem 1.1 and consider a feasible capacity function f on G . Form G' by giving $d - f(v)$ ghost neighbors to each vertex v .

We claim that G' is (k, d) -sharp. Adding a ghost neighbor of a vertex in a set adds 1 to the size of the set and $k + 1$ to the number of edges induced. Hence it changes the value of β by $(k + 1)(k + d) - (k + d + 1)(k + 1)$, which equals $-(k + 1)$. It therefore suffices to prove that $\beta(A') \geq 0$ for subsets A' of $V(G')$ that contain all the ghost neighbors of their vertices. Let $A = A' \cap V(G)$. Counting the increase in capacity from $f(v)$ to d and the cost of the ghost neighbors, we have

$$\beta(A') = \beta_f(A) + (k + 1) \sum_{v \in A} (d - f(v)) - \sum_{v \in A} (k + 1)(d - f(v)) = \beta_f(A) \geq 0,$$

where the last inequality holds because f is feasible. By Theorem 1.1, G' has a (k, d) -decomposition. Deleting the ghost vertices yields a (k, f) -decomposition of G . \square

In essence, we have shown that ghosts have the same effect as reduced capacity on the existence of decompositions.

3 $(k, d)^*$ -decomposition for $d \leq k + 1$

The capacity function f in Section 2 does the job of controlling vertex degrees to facilitate inductive construction of a (k, d) -decomposition. However, it cannot control the creation of cycles when we return a deleted edge to a decomposition satisfying reduced capacity. To do this, we impose another condition on the decomposition.

Definition 3.1. A *strong $(k, f)^*$ -decomposition* is a $(k, f)^*$ -decomposition in which each component of the degree-bounded forest contains at most one vertex v such that $f(v) < d$.

The strong decomposition condition will control the introduction of cycles. We will apply the induction hypothesis to $G - uv$ with reduced capacity function f' only when at least one endpoint of uv has capacity d . In $G - uv$, both endpoints have capacity less than d and will be the only such vertex in their components in D , so they will be in different components. We can thus add uv to D ; since one endpoint returns to capacity d , the strong condition continues to hold. This inductive approach will allow us to assume that no edge joins a vertex with capacity d to a vertex with positive capacity. For such graphs, the hypotheses will yield a decomposition into k forests, as in the final step of Theorem ??.

We will also need to strengthen the sparseness condition; feasibility of f is not sufficient. For example, if G consists of two vertices and an edge of multiplicity $k + 2$, and $f(u) =$

$f(v) = d$, then $\beta(A) \geq 0$ for all A , but G does not decompose into $k + 1$ forests. We will need another auxiliary function that excludes such examples. Also, in order to impose a stronger sparseness condition, we introduce a modified version of β_f .

Definition 3.2. Given a capacity function f on $V(G)$ using capacities at most d , let $S = \{v \in V(G) : f(v) = d\}$. For $A \subseteq V(G)$, let $f(A) = \sum_{v \in A} f(v)$ and $\hat{f}(A) = \min\{f(x) : x \in A\}$. Define α_f and β_f^* on subsets of G as follows:

$$\begin{aligned}\alpha_f(A) &= k|A| - k - \|A\| + |A \cap S|, \\ \beta_f^*(A) &= (2k + 2 - d)k|A| + (k + 1)[f(A) - 2\|A\|] - (k - 1)(2k + 2 - d).\end{aligned}$$

Say that f is *strongly feasible* when $\beta_f^*(A) > 0$ and $\alpha_f(A) \geq 0$ for all nonempty $A \subseteq V(G)$, with $\alpha_f(A) > 0$ whenever $A \subseteq S$.

With these definitions, we can state the main result of this section.

Theorem 3.3. *If $d \leq k + 1$ and f is a strongly feasible capacity function on a graph G , then G has a strong $(k, f)^*$ -decomposition.*

Once again the sparseness condition is motivated by and weaker than the desired fractional arboricity condition. The condition $\text{Arb}(G) \leq k + \frac{d}{2k+2}$ is equivalent to

$$(2k + 2 - d)k|A| + (k + 1)[d|A| - 2\|A\|] - k(2k + 2) - d \geq 0 \quad \text{for } A \subseteq V(G).$$

When $f(v) = d$ for all v , this is the same as $\beta_f^*(A) \geq 0$, except that we are subtracting $k(2k + 2) + d$ instead of $(k - 1)(2k + 2 - d)$.

Corollary 3.4. *$\text{Arb}(G) \leq k + \frac{d}{2k+2}$ guarantees $(k, d)^*$ -decomposability. In particular, the NDT Conjecture holds when $d = k + 1$.*

Proof. Since the constant subtracted in the inequality for $\text{Arb}(G)$ is larger, $\text{Arb}(G) \leq k + \frac{d}{2k+2}$ implies $\beta_f^*(A) > 0$ for all A when $f(v) = d$ for all v . With this capacity function, $|A \cap S| = |A|$ for all $A \subseteq V(G)$, and the condition $\alpha_f(A) \geq 1$ (since $A \subseteq S$) becomes $\|A\| \leq (k + 1)(|A| - 1)$, true for all A when $\text{Arb}(G) < k + 1$. Hence Theorem 3.3 applies.

When $d = k + 1$, we have $d + k + 1 = 2k + 2$, and $\text{Arb}(G) \leq k + \frac{d}{k+d+1}$ is sufficient. \square

The condition on α_f is necessary for a strong $(k, f)^*$ -decomposition. Nonnegativity of $\alpha_f(A)$ states that A has at most $|A \cap S|$ edges plus the number that k forests can absorb. Each vertex of A in S permits one more edge in a degree-bounded forest D , by allowing an edge joining two components. If $A \subseteq S$, then we reach the allowable spanning tree in $G[A]$ before the last vertex, so the requirement must increase to $\alpha(A) \geq 1$ when $A \subseteq S$.

We prove a useful bound on β_f^* in terms of α_f .

Lemma 3.5. For a capacity function f on a graph G and a set $A \subseteq V(G)$ with $|A| \geq 2$,

$$\beta_f^*(A) \leq (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S| + 1].$$

In particular, if $\alpha_f(A) \leq 0$ and $\beta_f^*(A) > 0$ with $A \not\subseteq S$, then $f(x) \geq |A \cap S|$ for all $x \in A$.

Proof. Substituting $\|A\| = k|A| - k - \alpha_f(A) + |A \cap S|$ into the formula for $\beta_f^*(A)$ yields

$$\beta_f^*(A) = -dk|A| + (k+1)[2\alpha_f(A) + f(A) - 2|A \cap S|] + (2k+2) + d(k-1).$$

Summing capacities over $x \in A$ yields $f(A) \leq (d-1)|A| + |A \cap S| + \hat{f}(A) - (d-1)$ (the inequality is strict when $A \subseteq S$). Substituting this into the formula above yields

$$\begin{aligned} \beta_f^*(A) &\leq -dk|A| + (k+1)(d-1)|A| + (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + 3k + 3 - 2d \\ &= (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + (d-k-1)|A| + 3k + 3 - 2d \\ &\leq (k+1)[2\alpha_f(A) + \hat{f}(A) - |A \cap S|] + k + 1, \end{aligned}$$

where the last inequality uses $|A| \geq 2$. □

We need an analogue of Lemma 2.4, with G_B as defined there.

Lemma 3.6. For $d \leq k+1$, let f be a strongly feasible capacity function on G , and let B be a proper subset of $V(G)$ with $|B| \geq 2$. Define f^* and \bar{f} on G_B by $f^*(z) = \hat{f}(B) - |B \cap S|$ and $\bar{f}(z) = 0$, letting both functions agree with f on $V(G) - B$. If $\alpha_f(B) = 0$, then f^* is strongly feasible. If $\beta_f^*(B) \leq k+1$, then \bar{f} is strongly feasible.

Proof. First consider the case $\alpha_f(B) = 0$. As observed in Lemma 3.5, $\hat{f}(B) \geq |B \cap S|$ when $\alpha_f(B) = 0$. Hence $f^*(z) \geq 0$, so f^* is a capacity function. Since f is strongly feasible and $\alpha_f(B) = 0$, we have $\beta_f^*(B) > 0$ and $B \not\subseteq S$. Since $\hat{f}(B) = d$ only if $B \subseteq S$, we must have $f^*(z) < d$, so the set S is the same for f^* and f .

If $z \notin A \subseteq V(G_B)$, then $\beta_{f^*}^*(A) = \beta_f^*(A)$ and $\alpha_{f^*}(A) = \alpha_f(A)$. When $z \in A$, we compute $\alpha_{f^*}(A)$ and $\beta_{f^*}^*(A)$ from $\alpha_f(A')$ and $\beta_f^*(A')$, where $A' = (A - \{z\}) \cup B$. As in Lemma 2.4, $|A'| = |A| - 1 + |B|$ and $\|A'\| = \|A\| + \|B\|$, where $\|A\|$ counts edges in G_B . Hence

$$\begin{aligned} \alpha_f(A') &= \alpha_{f^*}(A) + \alpha_f(B); \\ \beta_f^*(A') &= \beta_{f^*}^*(A) + \beta_f^*(B) - (k+1)f^*(z) - (2k+2-d). \end{aligned}$$

Since $\alpha_f(B) = 0$, we obtain $\alpha_{f^*}(A) = \alpha_f(A') \geq 0$, as desired since $f^*(z) < d$. By Lemma 3.5, $\alpha_f(B) = 0$ implies $\beta_f^*(B) \leq (k+1)[\hat{f}(B) - |B \cap S| + 1] = (k+1)f^*(z) + (k+1)$. Now $\beta_{f^*}^*(A) \geq \beta_f^*(A') + k + 1 - d \geq \beta_f^*(A') > 0$.

For \bar{f} , again it suffices to check A with $z \in A \subseteq V(G_B)$ and let $A' = (A - \{z\}) \cup B$. Now

$$\beta_{\bar{f}}^*(A') = \beta_{\bar{f}}^*(A) + \beta_{\bar{f}}^*(B) - (2k + 2 - d) \leq \beta_{\bar{f}}^*(A),$$

where we have used $\beta_{\bar{f}}^*(B) \leq k + 1$, $\bar{f}(z) = 0$, and $k + 1 - d \geq 0$. We also need $\alpha_{\bar{f}}(A) \geq 0$. With $\beta_{\bar{f}}^*(A) \geq \beta_{\bar{f}}^*(A') > 0$ and $\bar{f}(z) = 0$, this follows from Lemma 3.5. \square

Lemma 3.7. *Let f be a capacity function on G , and let B be a proper subset of $V(G)$ with $|B| \geq 2$. If $G[B]$ is strongly $(k, f|_B)^*$ -decomposable and G_B is strongly $(k, f^*)^*$ -decomposable, with f^* defined from f as in Lemma 3.6, then G is strongly $(k, f)^*$ -decomposable.*

Proof. Let (F, D) be a strong $(k, f|_B)^*$ -decomposition of $G[B]$, and let (F', D') be a strong $(k, f^*)^*$ -decomposition of G_B , where F and F' are unions of k forests. Each edge of G is in $G[B]$ or G_B , becoming incident to z in G_B if it joins B to $V(G) - B$ in G . Viewing F' and D' as subgraphs of G , we show that $(F \cup F', D \cup D')$ is a strong $(k, f)^*$ -decomposition of G .

As in Lemma 2.5, the union of any forest F_i in F with the corresponding forest F'_i in F' is still a forest, since otherwise contracting the portion in F_i of a resulting cycle would yield a cycle through z in F'_i when viewed as a forest in G' . This argument applies also to $D \cup D'$.

Recall that $S = \{v \in V(G) : f(v) = d\}$. If $\hat{f}(B) = d$, then $B \subseteq S$; we conclude that $f^*(z) < d$. Since (F', D') is a strong $(k, f^*)^*$ -decomposition, $f^*(z) < d$ implies that vertices other than z in its component in D' lie in S . Therefore, each component of $D \cup D'$ in G has at most one vertex outside S .

Since $D \subseteq G[B]$ and each component of D has at most one vertex outside S , each vertex v of B has at most $|B \cap S|$ neighbors in D . By the definition of $f^*(z)$, vertex v gains at most $\hat{f}(B) - |B \cap S|$ neighbors in D' ; together it has at most $f(v)$ neighbors in $D \cup D'$. \square

Proof of Theorem 3.3: *If $d \leq k + 1$ and f is a strongly feasible capacity function on a graph G , then G has a strong $(k, f)^*$ -decomposition.*

Proof. We use induction on the number of vertices plus the number of edges; the statement is trivial when there are at most k edges. For the induction step, suppose that G is larger.

Recall that $S = \{v \in V(G) : f(v) = d\}$. Let $R = \{v \in V(G) : f(v) = 0\}$, and let $T = V(G) - S - R$. We prove the structural claim that if G has no strong $(k, f)^*$ -decomposition, then S is independent and no edge joins S and T .

Suppose that G has an edge uv such that $u \in S$ and $v \in S \cup T$. We choose such an edge with $v \in T$ if one exists; otherwise, $v \in S$. Let $G' = G - uv$, and let f' be the capacity function on G' that agrees with f except for $f'(u) = f(u) - 1$ and $f'(v) = f(v) - 1$. Note that $u \notin \{x : f'(x) = d\}$. If f' is strongly feasible, then since $G - uv$ is smaller than G , it has a strong $(k, f')^*$ -decomposition (F, D) . Since $f'(u) < d$ and $f(u) = d$, adding the edge uv to D yields a strong $(k, f)^*$ -decomposition of G .

To prove the structural claim, it thus suffices to show that f' is strongly feasible. We consider $\alpha_{f'}(A)$ and $\beta_{f'}^*(A)$. If $|A| = 1$, then $\alpha_{f'}(A) = |A \cap S|$ (positive if $A \subseteq S$). Also, $\beta_{f'}^*(A) = (2k + 2 - d) + (k + 1)f(A) \geq 2k + 2 - d > 0$, since $d \leq k + 1$.

Next consider $A = V(G)$. Since $u, v \in A$, we have $\beta_{f'}^*(A) = \beta_f^*(A) > 0$. Also, $\alpha_{f'}(A) < \alpha_f(A)$ requires $u, v \in S$. Not all vertices satisfy $f'(x) = d$, since $f'(u) < d$. Therefore, having $\alpha_f(A) \geq 1$ and $\alpha_{f'}(A) \geq 0$ suffices, so we may assume that $\alpha_f(A) = 0$. With $A = V(G)$ and $u, v \in S$, the choice of uv in defining f' implies that no edges join S and T . Since $\alpha_f(A) = 0$ implies $A \not\subseteq S$, we have $R \cup T \neq \emptyset$. If $R \neq \emptyset$, then $\hat{f}(A) = 0$, contradicting Lemma 3.5. Hence $R = \emptyset$. Since no edges join S and T , now G is disconnected, and we can combine strong decompositions of the components obtained from the induction hypothesis.

Finally, suppose $2 \leq |A| < |V(G)|$. If $\alpha_f(A) = 0$, then the capacity function f^* on G_A that agrees with f except for $f^*(z) = \hat{f}(A) - |A \cap S|$ is strongly feasible, by Lemma 3.6. Also, the restriction of f to A is strongly feasible on $G[A]$. Since G_A and $G[A]$ are smaller than G , by the induction hypothesis $G[A]$ is strongly $(k, f|_A)^*$ -decomposable and G_A is strongly $(k, f^*)^*$ -decomposable. By Lemma 3.7, G is strongly $(k, f)^*$ -decomposable.

Hence we may assume that $\alpha_f(A) > 0$. For $\alpha_{f'}(A) < \alpha_f(A)$, we must have u or v in $A \cap S$, and the decline can only be by 1. Hence $\alpha_{f'}(A) \geq 0$, which is good enough since $f'(u), f'(v) < d$. If $\beta_{f'}^*(A) > 0$, then A causes no problem.

Otherwise, $\beta_f^*(A) \leq k + 1$, since reduction of β^* requires $|A \cap \{u, v\}| = 1$, and the reduction is then by $k + 1$. Now Lemma 3.6 implies that \bar{f} is strongly feasible on G_A , where $\bar{f}(z) = 0$ and otherwise \bar{f} agrees with f . By the induction hypothesis, G_A has a strong $(k, \bar{f})^*$ -decomposition (F, D) , and $G[A]$ has a strong $(k, f|_A)^*$ -decomposition (F', D') . As in Lemma 3.7, $(F \cup F', D \cup D')$ is a strong $(k, f)^*$ -decomposition of G ; since z is isolated in D , the components of D' do not extend.

Hence we may assume that S is independent and that no edge joins S and T . As in Theorem 2.6, we claim that G decomposes into k forests, completing the desired decomposition. Otherwise, we find a set A such that $\beta_f^*(A) \leq 0$, contradicting strong feasibility. Note that $\beta_f^*(A) = (2k + 2 - d)g(A) + h(A)$, where $g(A) = k(|A| - 1) - \|A\| + 1$ and $h(A) = (k + 1)f(A) - d\|A\|$. It suffices to find A such that $g(A) \leq 0$ and $h(A) \leq 0$.

If $\Upsilon(G) > k$, then $V(G)$ has a minimal subset A such that $\|A\| \geq k(|A| - 1) + 1$; that is, $g(A) \leq 0$. Minimality implies that every vertex of A has at least $k + 1$ neighbors in A .

If $A \cap T = \emptyset$, then $\|A\| \geq (k + 1)|A \cap S| = (k + 1)f(A)/d$, which simplifies to $h(A) \leq 0$. If $A \subseteq T$, then $|A \cap S| = 0$, so $\alpha_f(A) = g(A) - 1 < 0$, contradicting strong feasibility of f .

Hence we may assume that $A \cap T$ is a nonempty proper subset of A . The minimality of A implies that $\|A - T\| \leq k(|A - T| - 1)$, and hence more than $k|A \cap T|$ edges of $G[A]$ are incident to T . From the independence of S and the absence of edges joining S and T , we now have $\|A\| > (k + 1)|A \cap S| + k|A \cap T|$. Since $f(v) = d$ for $v \in S$ and $f(v) \leq d - 1$ for

$v \in T$, this yields $\|A\| \geq (k+1)\frac{f(A \cap S)}{d} + k\frac{f(A \cap T)}{d-1}$. Multiplying by d , we obtain

$$d\|A\| \geq (k+1)f(A \cap S) + kf(A \cap T)\frac{d}{d-1} \geq (k+1)f(A),$$

using $d/(d-1) \geq (k+1)/k$ and $f(R) = 0$. Thus $h(A) \leq 0$, which as we noted suffices to complete the proof. \square

4 Approach to $(k, d)^*$ -decomposition

For our remaining stronger conclusions in which the “leftover” subgraph D must also be a forest, the highly local approach of Section 2 that reserves one edge for D by reducing the degree capacity of its endpoints is not adequate. When $d > k+1$, it becomes harder to avoid creating a cycle when replacing a reserved edge.

We use the inductive approach of obtaining *reducible configurations* (structures that are forbidden from minimal counterexamples) and then the *discharging method*, showing that the average degree in any graph avoiding the reducible configurations is too high. This method can also be used to prove Theorem 1.1, but such a proof would be lengthier than that in the previous section. On the other hand, it may settle the case $k = d$ for (k, d) -decomposition.

For this discussion, we modify β by removing the term independent of A , and we drop the notation for the capacity function because each vertex will have capacity d .

Definition 4.1. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For a set A of vertices in a graph G , the *sparseness* $\beta_G(A)$ is defined by $\beta_G(A) = (k+1)(k+d)|A| - (k+d+1)\|A\|$.

The term “sparseness” here is natural, because if $\beta_G(A)$ is sufficiently large for all A , then G is sufficiently sparse to satisfy the relevant bound on $\text{Mad}(G)$ or $\text{Arb}(G)$. Sparseness also distinguishes between the conditions on $\text{Mad}(G)$ and $\text{Arb}(G)$. As mentioned previously, $\text{Arb}(G) \leq m_{k,d}/2$ may fail when $\text{Mad}(G) < m_{k,d}$ holds. The former requires a set A such that $\beta_G(A) < (k+1)(k+d)$, while the latter requires only that $\beta_G(A) \geq 1$ for all A .

Example 4.2. Let H be the (multi)graph consisting of $q+1$ vertices in which one vertex has degree $(k+1)q$ and the others have degree $k+1$ and form an independent set. We have $\text{Arb}(H) = k+1$, but $\text{Mad}(H) = 2q(k+1)/(q+1)$. If $d < q < k+d$, then $\text{Mad}(H) < m_{k,d}$, but H has no $(k, d)^*$ -decomposition.

This graph H can be excluded by requiring (k, d) -sparseness (note that $d < q < k+d$ requires $k \geq 2$, which is where (k, d) -sparse and $\text{Mad}(G) < m_{k,d}$ differ). For H , we have $(k+1)(k+d)|V(H)| - (k+d+1)\|V(H)\| = (k+1)(k+d-q)$, which violates (k, d) -sparseness if and only if $q > d$. Furthermore, $q > d$ if and only if H has no $(k, d)^*$ -decomposition.

Even $\beta_G(A) \geq k^2$ ((k, d) -sparseness) allows $\Upsilon(G) \leq k + 1$ to fail, but only on a small subgraph. Violating $\Upsilon(G) \leq k + 1$ requires an r -vertex subgraph with at least $(k + 1)(r - 1) + 1$ edges. If such a graph is also (k, d) -sparse, then

$$(k + 1)(k + d)r - (k + d + 1)[(k + 1)(r - 1) + 1] \geq k^2,$$

which simplifies to $r \leq \frac{k}{k+1}(d + 1)$. □

In the cases where we can guarantee a $(k, d)^*$ -decomposition, we obtain a stronger statement than the case (k, d) of the NDT Conjecture by weakening the hypothesis to require only (k, d) -sparseness, while excluding multigraphs with at most $(d + 1)k/(k + 1)$ vertices that satisfy this bound but fail to decompose into $k + 1$ forests.

Definition 4.3. Fix $k, d \in \mathbb{N}$. A graph G is *feasible* if $\beta_G(A) \geq k^2$ for all nonempty $A \subseteq V(G)$. A set $A \subseteq V(G)$ is *overfull* if $\|A\| > (k + 1)(|A| - 1)$.

Now that we are fixing (k, d) , “feasible” is a convenient abbreviation for “ (k, d) -sparse”. Theorem 1.1 showed that feasible graphs are (k, d) -decomposable (when $d > k$), and by Example 2.1 this condition on β_G is sharp. Graphs with overfull sets are not $(k, d)^*$ -decomposable. We have noted that the bound $\text{Arb}(G) \leq m_{k,d}/2$ both implies feasibility and prohibits overfull sets. Furthermore, feasibility prohibits overfull sets with more than $(d + 1)k/(k + 1)$ vertices. Hence the conjecture below is equivalent to the NDT Conjecture.

Conjecture 4.4. Fix $k, d \in \mathbb{N}$. If G is feasible and has no overfull set with at most $(d + 1)k/(k + 1)$ vertices, then G is $(k, d)^*$ -decomposable.

We will prove Conjecture 4.4 when $k = 1$ and $d \leq 6$. The advantage we gain when $k = 1$ is that $k^2 = 1$, so the feasibility condition reduces to $\beta_G(A) > 0$ for all A . We can then bring a variety of techniques to bear, including properties of submodular functions.

The basic framework of the proof holds for general k , so we maintain the general language throughout this section before specializing to $k = 1$. We do this to suggest the generalization to larger k and because the proofs of these lemmas are as short for general k as for $k = 1$.

We typically use (F, D) to denote a $(k, d)^*$ -decomposition of G , where F is a disjoint union of k forests and D is a forest with maximum degree at most d . Note that the hypotheses of Conjecture 4.4 remain satisfied under discarding edges or vertices.

Definition 4.5. A j -vertex is a vertex of degree j . Among the non- $(1, d)^*$ -decomposable graphs satisfying the hypotheses of Conjecture 4.4, a *minimal counterexample* is one that has the fewest ghosts among those with the fewest non-ghosts.

Ghosts help control $(k, d)^*$ -decompositions, because such a decomposition must put one edge at a ghost into D . Without loss of generality, the other k edges at the ghost may be placed arbitrarily into the forests in F .

Lemma 4.6. *A minimal counterexample G is $(k + 1)$ -edge-connected (and hence has minimum degree at least $k + 1$).*

Proof. If G has an edge cut Q with size at most k , then $(k, d)^*$ -decompositions of the components of $G - Q$ combine to form an $(k, d)^*$ -decomposition of G by allowing each forest to acquire at most one edge of the cut. \square

Corollary 4.7. *In a minimal counterexample G , a vertex with degree at most $2k + 1$ cannot be a neighbor of a ghost.*

Proof. If such a vertex v is also a ghost, then G has two vertices and is $(k, d)^*$ -decomposable. Otherwise, the edges incident to v and not incident to the neighboring ghost form an edge cut of size at most k , contradicting Lemma 4.6. \square

Definition 4.8. A j -neighbor of a vertex is a neighbor that is a j -vertex. A ghost neighbor of a vertex is a neighbor that is a ghost. Adding a ghost neighbor at a vertex v means adding to the graph a vertex of degree $k + 1$ whose only neighbor is v . For a vertex set A in a graph G , contracting A to a vertex v^* means deleting all edges within A and replacing A with a single vertex v^* incident to all edges that joined A to $V(G) - A$. Let G_A denote the graph obtained from G by contracting A to v^* and adding d ghost neighbors at v^* .

Lemma 4.9. *If G is feasible and $\beta_G(A) \leq k(k + 1)$, then G_A is feasible.*

Proof. For $X \subseteq V(G_A)$, we show that $\beta_{G_A}(X) \geq k^2$. Let S be the set of d ghost neighbors added at v^* . If $v^* \notin X$, then the inequality is hardest when $S \cap X = \emptyset$, since each vertex of S adds $(k + 1)(k + d)$ to the sparseness of $X - S$. With $S \cap X = \emptyset$, we have $\beta_{G'}(X) = \beta_G(X) \geq k^2$.

If $v^* \in X$, then the inequality is hardest when $S \subseteq X$, since each addition of a ghost to a set containing its neighbor reduces the sparseness by $k + 1$. Before adding S , contracting A to v^* loses $|A| - 1$ vertices and $\|A\|$ edges. Let $X' = A \cup (X - S - v^*)$; note that $X' \subseteq V(G)$. We compute

$$\begin{aligned} \beta_{G_A}(X) &= \beta_G(X') - (k + 1)(k + d)(|A| - 1) + (k + d + 1)\|A\| - d(k + 1) \\ &= \beta_G(X') + k(k + 1) - \beta_G(A) \geq \beta_G(X') \geq k^2. \end{aligned} \quad \square$$

Lemma 4.10. *For $A \subseteq V(G)$, if $G[A]$ and G_A are $(k, d)^*$ -decomposable, then G is $(k, d)^*$ -decomposable.*

Proof. Let (F, D) be a $(k, d)^*$ -decomposition of G_A . Since v^* has d ghost neighbors in G_A , its neighbors in D are only those ghosts; no edges of D join v^* to vertices of G . Let (F', D') be a $(k, d)^*$ -decomposition of $G[A]$.

Combining (F', D') and (F, D) (after deleting the ghost neighbors of v^*) forms a $(k, d)^*$ -decomposition of G . All edges joining v^* to $V(G) - A$ lie in F and are incident to various vertices of A . Since v^* lies on no cycle in F , adding the edges of F' does not complete a cycle. That is, each forest in a $k\mathcal{F}$ -decomposition of F can be combined with any one of the forests in a $k\mathcal{F}$ -decomposition of F' . \square

Definition 4.11. Let $d_G(v)$ denote the degree of a vertex v in a graph G . A set $A \subseteq G$ is *nontrivial* if A contains at least two non-ghosts but not all non-ghosts in G .

We avoid confusion between the overall parameter d and the degree function by always using the relevant graph as a subscript when discussing individual vertex degrees.

Lemma 4.12. *Let A be a vertex set in a minimal counterexample G . If A is nontrivial, then $\beta_G(A) > k(k+1)$. If A is trivial with exactly one non-ghost vertex v , and $\beta_G(A) \leq k(k+1)$, then $d_G(v) \geq (k+1)(d+1)$.*

Proof. Suppose that $\beta_G(A) \leq k(k+1)$. By Lemma 4.9, G_A is feasible. If A is nontrivial, then G_A has fewer non-ghosts than G . The minimality of G then implies that both G_A and $G[A]$ are $(k, d)^*$ -decomposable. By Lemma 4.10, also G would be $(k, d)^*$ -decomposable.

Hence we may assume that A is trivial with non-ghost vertex v , so A consists of v and some number h of ghost neighbors of v . Now $\beta_G(A) = (k+1)(k+d-h)$, so $\beta_G(A) \leq k(k+1)$ requires $h \geq d$. If $h > d$, then already $d_G(v) \geq (k+1)(d+1)$. If $h = d$ and $A = V(G)$, then G is explicitly $(k, d)^*$ -decomposable. In the remaining case, G has vertices outside A , and the only vertex of A with outside neighbors is v . Since G is $(k+1)$ -edge-connected (by Lemma 4.6), we again have $d_G(v) \geq (k+1)(d+1)$. \square

Lemma 4.13. *If v is a vertex in a minimal counterexample G , and $d_G(v) < (k+1)(k+d)$, then v has no non-ghost $(k+1)$ -neighbor.*

Proof. Let u be a non-ghost $(k+1)$ -neighbor of v , and let W be the set of other neighbors of u . Since $d_G(u) = k+1$, no vertex in $W \cup \{v\}$ is a ghost. Form G' from G by deleting the edges incident to u and then adding $k+1$ edges joining u to v ; this makes u a ghost neighbor of v in G' . Note that G' and G have the same numbers of edges and vertices, but G' has fewer non-ghost vertices than G , since u and its neighbors are non-ghosts in G and at least u becomes a ghost in G' .

If G' is feasible, then the choice of G implies that G' has an $(k, d)^*$ -decomposition (F, D) . Now modify (F, D) : delete the copies of uv in F (keeping the copy in D), and add the k other edges at u in G to the k forests in F . This yields a $(k, d)^*$ -decomposition of G .

It thus suffices to show that G' is feasible. We need only consider A such that $u, v \in A$ and $W \not\subseteq A$; otherwise, $\beta_{G'}(A) \geq \beta_G(A) \geq k^2$, since G is feasible. With $u \in A$, we have $\beta_{G'}(A) = \beta_G(A - u) - (k + 1)$, since adding a ghost neighbor costs $k + 1$. We worry only if $\beta_G(A - u) \leq k(k + 1)$. Since $W \not\subseteq A$, the set A does not contain all non-ghosts in G . If v is the only non-ghost in $A - u$, then $d_G(v) \geq (k + 1)(k + d)$, by Lemma 4.12. Since our hypothesis is $d_G(v) < (k + 1)(k + d)$, we conclude that $A - u$ is nontrivial, and now Lemma 4.12 yields $\beta_G(A - u) > k(k + 1)$. \square

Lemma 4.14. *If a minimal counterexample G has a vertex v with q ghost neighbors, where $q \geq 1$, then $d_G(v) > kq + k + d$.*

Proof. Form G' from G by deleting the ghost neighbors of v . Since G' is an induced subgraph of G , it is feasible. Forming G' does not increase the number of non-ghost vertices, but it decreases the numbers of vertices and edges, so G' has an $(k, d)^*$ -decomposition (F', D') .

By Lemma 4.6, $d_{G'}(v) \geq k + 1$. We may assume that $d_{D'}(v) \leq d_{G'}(v) - k$, since edges of D' at v can be moved arbitrarily to F' until F' has at least k edges at v . Now restore each ghost vertex by adding one incident edge to each forest in F' and the remaining incident edge to D' , yielding (F, D) .

Since $F \in k\mathcal{F}$ and $D \in \mathcal{F}$, it suffices to check $d_D(v)$. We have $d_D(v) = d_{D'}(v) + q \leq d_{G'}(v) - k + q = d_G(v) - kq - k$. Thus $d_D(v) \leq d$ unless $d_G(v) > kq + k + d$. \square

If v has q ghost neighbors, then $d_G(v) \geq (k + 1)q$. Hence the lower bound in Lemma 4.14 strengthens the trivial lower bound when $q \leq k + d$.

Lemma 4.15. *If G is a minimal counterexample, then two vertices in G are joined by $k + 1$ edges only when one of them is a ghost.*

Proof. Since G has no overfull set, edge-multiplicity is at most $k + 1$. If two ghosts are adjacent, then G has two vertices and is $(k, d)^*$ -decomposable.

Suppose that non-ghosts u and v are joined by $k + 1$ edges. Obtain G' from G by contracting these edges into a single vertex v^* and adding a ghost neighbor w to v^* .

We claim that G' is feasible and has no overfull set. If $A \subseteq V(G') - \{v^*\}$, then $\beta_{G'}(A) \geq \beta_G(A - \{w\}) \geq k^2$. If $v^* \in A \subseteq V(G')$, then $\beta_{G'}(A) \geq \beta_{G'}(A \cup \{w\}) = \beta_G(A') \geq k^2$, where $A' = (A - \{v^*, w\}) \cup \{u, v\}$. Hence G' is feasible.

Since G has no overfull set, an overfull set in G' must contain v^* , and a smallest such set A does not contain w . Let $A' = (A - \{v^*\}) \cup \{u, v\}$. Now A' has one more vertex than A and induces $k + 1$ more edges in G than A induces in G' . Hence A' is overfull if and only if A is overfull. We conclude that G' has no overfull set.

Since G' has the same numbers of vertices and edges as G , but G' has fewer non-ghosts than G , minimality of G now implies that G' has a $(k, d)^*$ -decomposition (F, D) . At w there

is one edge in each forest in F and one edge in D . Replacing these with the edges joining u and v (one in each forest) yields a $(k, d)^*$ -decomposition of G , since the new degree of u or v in D is at most $d_D(v^*)$, and an edge joining u and v completes a cycle in its forest only if contracting that edge yields a cycle in the corresponding forest in (F, D) . \square

5 Discharging Argument and Submodularity

The lemmas of Section 4 provide a framework for a discharging argument. We would like to show that if G has the structural properties of a minimal counterexample, then $\text{Mad}(G) \geq m_{k,d}$; this would prove the conjecture. We have not yet proved sufficient structural properties to complete the argument. By outlining a discharging argument, we will suggest what else is needed. Section 6 will complete the proof for $k = 1$ and $d \leq 6$.

Let G be a minimal counterexample. Since G is feasible, $\text{Mad}(G) < m_{k,d} = 2k + \frac{2d}{k+d+1}$. Give each vertex an initial charge equal to its degree in G (by Lemma 4.6, each vertex has degree at least $k + 1$). We aim to redistribute charge to obtain a final charge $\mu(v)$ for each vertex v such that $\mu(v) \geq m_{k,d}$. This motivates our first discharging rule.

Rule 1: A vertex of degree $k + 1$ takes charge $m_{k,d}/(k + 1) - 1$ along each incident edge from the other endpoint of that edge. This amount equals $\frac{k+d-1}{k+d+1}$.

In particular, a ghost takes total charge $m_{k,d} - (k + 1)$ from its neighbor. By force, Rule 1 increases the charge of each $(k + 1)$ -vertex to $m_{k,d}$, since Lemma 4.13 implies that $(k + 1)$ -vertices are not adjacent unless G has just two vertices.

If all neighbors of v have degree $k + 1$, then $\mu(v) = d_G(v) \frac{2}{k+d+1}$, since each edge takes $\frac{k+d-1}{k+d+1}$. In this case, $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k + 1)(k + d)$.

The problem is how to handle vertices with degree between $k + 1$ and $(k + 1)(k + d)$. Vertices with degree at most $2k$ need additional charge (as do vertices with degree $2k + 1$ when $d > k + 1$), though they do not need as much as $(k + 1)$ -vertices need. Vertices with degree less than $(k + 1)(k + d)$ cannot afford to give away too much. The principle we need to quantify is that lower-degree vertices must have higher-degree neighbors.

A vertex v with degree less than $(k + 1)(k + d)$ cannot be adjacent only to $(k + 1)$ -vertices. By Lemma 4.13, v has no non-ghost $(k + 1)$ -neighbor. If v has only ghost neighbors, then G consists of one vertex plus ghost neighbors, but such a graph has the desired decomposition or is infeasible (see Example 4.2). Hence v has some neighbors with higher degrees and will not need to give away as much. More information is needed about the degrees of neighboring vertices to complete a proof.

When $(k, d) = (1, 1)$, only 2-vertices need charge. By Lemma 4.13, their neighbors have high enough degree that Rule 1 suffices to complete the discharging argument. Since a forest

with maximum degree 1 is a matching, this proves the result of [11] that the Strong NDT Conjecture holds when $(k, d) = (1, 1)$.

When $k = 1$ and $d > 1$, only 2-vertices and 3-vertices need charge. This leads to a sufficient condition for completing the discharging argument.

Theorem 5.1. *For $d > k = 1$, let G be a minimal counterexample in the sense of Section 4. If each 3-vertex in G has a neighbor with degree at least $d+2$, then $\text{Mad}(G) \geq m_{1,d} = 2 + \frac{d}{d+2}$.*

Proof. In addition to the special case for $k = 1$ of Rule 1 stated above, in which each 2-vertex receives $\frac{d}{d+2}$ along each edge, we add a rule to satisfy 3-vertices.

Rule 2: If $d_G(v) = 3$, and v has neighbor u with $d_G(u) \geq d + 2$, then v receives $\frac{d-2}{d+2}$ from u .

We show that the final charge of each vertex is at least $m_{1,d}$. Rules 1 and 2 ensure that $\mu(v) \geq m_{1,d}$ when $d_G(v) \in \{2, 3\}$ (since $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general argument for vertices with degree at least $2d + 2$ also remains valid.

If $4 \leq d_G(v) \leq 2d+1$, then v has no non-ghost 2-neighbor, by Lemma 4.13. If v has q ghost 2-neighbors with $q \geq 1$, then $d_G(v) \geq q + d + 2$, by Lemma 4.14. Hence $\mu(v) = d_G(v) > m_{1,d}$ if $4 \leq d_G(v) \leq d + 1$, since Rule 2 takes no charge from v .

If $d + 2 \leq d_G(v) \leq 2d + 1$, then v may give charge to q ghost neighbors (to each along two edges) and to $d_G(v) - 2q$ neighbors of degree 3. Using Lemma 4.14,

$$\mu(v) \geq d_G(v) - \frac{d}{d+2}2q - [d_G(v) - 2q]\frac{d-2}{d+2} = \frac{4(d_G(v) - q)}{d+2} \geq \frac{4(d+2)}{d+2} = 4 > m_{1,d}.$$

The final charge at each vertex is at least $m_{1,d}$, so no minimal counterexample is feasible. \square

This reduces Conjecture 4.4 for the case $k = 1$ to proving that in a minimal counterexample G , each 3-vertex has a neighbor with degree at least $d + 2$. Our proofs of this fact depend on d . In each case, we will use submodularity properties of the function β_G .

Definition 5.2. A function β on the subsets of a set is *submodular* if $\beta(X \cap Y) + \beta(X \cup Y) \leq \beta(X) + \beta(Y)$ for all subsets X and Y . When G' is an induced subgraph of G , define the *potential function* $\rho_{G'}$ by $\rho_{G'}(X) = \min\{\beta_G(W) : X \subseteq W \subseteq V(G')\}$.

Lemma 5.3. *For any graph G and any induced subgraph G' of G , the sparseness function β_G on the subsets of $V(G)$ is submodular.*

Proof. To compare $\beta_G(X \cap Y) + \beta_G(X \cup Y)$ with $\beta_G(X) + \beta_G(Y)$, note first that $|X \cup Y| + |X \cap Y| = |X| + |Y|$. Hence it suffices to show that $\|X \cup Y\| + \|X \cap Y\| \geq \|X\| + \|Y\|$. All edges contribute equally to both sides except edges joining $X - Y$ and $Y - X$, which contribute 1 to the left side but 0 to the right. \square

6 Neighbors of 3-vertices when $k = 1$

Throughout this section, $k = 1$. For $k = 1$, feasibility reduces to the statement that $\beta_G(A) = (2d + 2)|A| - (d + 2)\|A\| \geq 1$ for $A \subseteq V(G)$. When G is a minimal counterexample, Lemma 4.12 implies that $\beta_G(A) \geq 3$ when A is nontrivial (contains at least two non-ghosts but not all non-ghosts). Furthermore, if d is even, then always $\beta_G(A)$ is even, so in that case we may assume $\beta_G(A) \geq 4$ when A is nontrivial. By Theorem 5.1, to prove the NDT Conjecture when $k = 1$ it suffices to prove that every 3-vertex in a minimal counterexample has a neighbor with degree at least $d + 2$.

Lemma 6.1. *Fix d with $2 \leq d \leq 6$, and let G be a minimal counterexample. If v is a 3-vertex in G and has no neighbor with degree at least $d + 2$, then v has two neighbors u and u' such that $\rho_{G'}(\{u, u'\}) \geq d + 3$, where $G' = G - v$.*

Proof. Together, Corollary 4.7 and Lemma 4.15 imply that every 3-vertex has three distinct neighbors. Let U be the neighborhood of v , with $U = \{u_1, u_2, u_3\}$. Let $Z_i = U - \{u_i\}$. Suppose that $\rho_{G'}(U_i) \leq d + 2$ for all i .

For each i , let X_i be a subset of $V(G')$ such that $\rho_{G'}(Z_i) = \beta_G(X_i)$. For any permutation i, j, k of $\{1, 2, 3\}$,

$$2d + 4 \geq \beta_G(X_i) + \beta_G(X_j) \geq \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j).$$

For $X' \subseteq V(G')$, let $X = X' \cup \{v\}$. If $U \subseteq X' \subseteq V(G')$, then $\beta_G(X') = \beta_G(X) + d + 4$. If $X' \neq V(G')$, then $X \neq V(G)$, and X is nontrivial if it has at least two non-ghosts, which by Lemma 4.12 would yield $\beta_G(X') \geq d + 7 + \epsilon$, where $\epsilon = 1$ if d is even and $\epsilon = 0$ if d is odd. However, if $X' = V(G')$, then we only have $\beta_G(X') \geq d + 5 + \epsilon$.

Since each edge vu_i has multiplicity 1, no vertex in U is a ghost, and neither is v . Since $u_k \in X_i \cap X_j$ and $d_G(u_k) < d + 2$, Lemma 4.12 implies $\beta_G(X_i \cap X_j) \geq 3 + \epsilon$. Since $U \subseteq X_i \cup X_j$, we also conclude $\beta_G(X_i) + \beta_G(X_j) \geq d + 8 + 2\epsilon$ for all d , and the lower bound increases by 2 if $X_i \cup X_j \neq V(G')$.

Thus $\rho_{G'}(X_i) + \rho_{G'}(X_j) \geq d + 8 + 2\epsilon$. If $d \leq 4$, then $d + 8 + 2\epsilon > 2d + 4$, and the desired conclusion follows. Hence we may assume $d \in \{5, 6\}$; furthermore, $X_i \cup X_j = V(G')$ for all i, j , since otherwise the lower bound on $\beta_G(X_i) + \beta_G(X_j)$ again exceeds $2d + 4$.

In more detail, the computation of Lemma 5.3 is

$$\beta_G(X_i) + \beta_G(X_j) = \beta_G(X_i \cup X_j) + \beta_G(X_i \cap X_j) + (k + d + 1)m,$$

where m is the number of edges joining $X_i - X_j$ and $X_j - X_i$. If $m \geq 1$, then we obtain $\beta_G(X_i) + \beta_G(X_j) \geq 2d + 10 > 2d + 4$, which yields the desired conclusion. Hence $m = 0$ in

each case. That is, each $X_i \cap X_j$ is a separating set in G' . (If G' is disconnected, then some edge incident to v is a cut-edge, which contradicts Lemma 4.6.) Furthermore,

$$\beta_G(X_i \cap X_j) = \beta_G(X_i) + \beta_G(X_j) - \beta_G(X_i \cup X_j) \leq 2d + 4 - (d + 5 + \epsilon) = d - 1 - \epsilon.$$

Now let $Z = X_1 \cap X_2 \cap X_3$. Since $X_i \cup X_j = V(G')$, any vertex of $V(G') - Z$ misses exactly one of the three sets, so $\{Z, \overline{X}_1, \overline{X}_2, \overline{X}_3\}$ is a partition of $V(G')$. Since $\beta_G(X_i) \leq d + 2$ and $\beta_G(V(G')) \geq d + 5$, each \overline{X}_i is nonempty, so $Z \neq V(G')$. If Z contains only one non-ghost, then feasibility requires it to have at most d ghost neighbors, and $\beta_G(Z) \geq 2$. Otherwise, since $v \notin Z$, we conclude that Z is nontrivial, and hence $\beta_G(Z) \geq 3$.

Now, since $\overline{X}_i \subseteq X_j \cap X_k$, submodularity yields

$$2d + 1 - \epsilon \geq \beta_G(X_i) + \beta_G(X_j \cap X_k) \geq \beta_G(V(G')) + \beta_G(Z) \geq d + 7.$$

We conclude that $d \geq 6 + \epsilon$, which completes the proof for $d \leq 6$. □

Lemma 6.2. *If $3 \leq d \leq 6$ and G is a minimal counterexample, then every 3-vertex has a neighbor with degree at least $d + 2$.*

Proof. Let u_1, u_2, u_3 be the neighbors of a 3-vertex v , and let $U = \{u_1, u_2, u_3\}$. Suppose that $d_G(u) \leq d + 1$ for $u \in U$. Since each edge vu_i has multiplicity 1, no vertex in U is a ghost vertex, and any edge induced by U has multiplicity 1 (Lemma 4.15).

Let $G' = G - v$. By Lemma 6.1, we may assume by symmetry that $\rho_{G'}(\{u_1, u_2\}) \geq d + 3$. Form H from G' by adding an extra edge joining u_1 and u_2 . For $A \subseteq V(H) = V(G')$, we have $\beta_H(A) = \beta_G(A)$ unless $u_1, u_2 \in A$, but in the remaining case $\rho_{G'}(\{u_1, u_2\}) \geq d + 3$ yields $\beta_H(A) \geq 1$.

Hence H is feasible, and it has fewer non-ghosts than G . To have an $(1, d)^*$ -decomposition of H , we need only exclude overfull sets of size at most $(d + 1)/2$, which is at most 3. There are no triple-edges in H , since G has no double-edges within U . An overfull triple must include u_1 and u_2 , since G has no overfull triple. The third vertex w must be adjacent to u_1 or u_2 by two edges in G . Since those vertices are also adjacent to v , we have contradicted $d_G(u_1) = d_G(u_2) = 3$.

Let (F, D) be an $(1, d)^*$ -decomposition of H . Obtain a decomposition of G by (1) replacing the added edge u_1u_2 with vu_1 and vu_2 in whichever of F and D contains it, and (2) placing vu_3 in the other subgraph. The degree in D of u_1 and u_2 is the same as a subgraph of H or G , and cycles through v would correspond to cycles in the decomposition of H . The only worry is $d_D(u_3)$, since we have increased this by 1 if the added edge in H belonged to F . If $d_D(u_3)$ has increased to $d + 1$, then we have the desired conclusion unless

$d_G(u_3) = d + 1$, but now we can move any one edge incident to u_3 from D to F to complete a $(1, d)^*$ -decomposition of G . \square

7 The Strong NDT Conjecture for $(k, d) = (1, 2)$

In this section we prove our strongest conclusion for our most restrictive hypothesis. Many of the steps are quite similar to our previous arguments, so we put them all together in a single proof.

Theorem 7.1. *The Strong NDT Conjecture holds when $(k, d) = (1, 2)$. That is, if G is feasible, then G has an $(\mathcal{F}, \mathcal{F}_2)$ -decomposition (F, D) in which every component of D has at most two edges (a strong decomposition).*

Proof. Since $m_{1,2} = 3$, feasibility is equivalent to $\text{Mad}(G) < 3$. Let G be a counterexample with the fewest non-ghosts. By the argument of Lemma 4.6, G is 2-edge-connected.

If G has adjacent 2-vertices u and v , then at least one is not a ghost. Letting $G' = G - \{u, v\}$, the minimality of G yields a strong decomposition (F, D) of G' . Adding the edge uv to D and the other edges incident to u and v to F yields a strong decomposition of G .

If G has a vertex with three ghost neighbors, then G is infeasible, so every vertex has at most two ghost neighbors. If G has only one non-ghost, then G explicitly has a strong decomposition. Hence we may assume that G has at least two non-ghosts.

Since d is even, always β_G is even, so feasibility can be stated as $\beta_G(A) \geq 2$ for $A \subseteq V(G)$ (here $\beta_G(A) = 6|A| - 4\|A\|$). A set A is *tight* if $\beta_G(A) = 2$. A set consisting of a vertex with two ghost neighbors is a trivial tight set.

By Lemma 4.9, if A is a tight set, then G_A is feasible. The same argument as in Lemma 4.10 shows that if G is a minimal counterexample, $A \subseteq V(G)$, and G_A has a strong decomposition, then G has a strong decomposition. Hence we may assume, as in the earlier proofs, that $\beta_G(A) \geq 4$ for every nontrivial set A .

Suppose that G has a non-ghost 2-vertex v . Each neighbor of v has degree at least 3. If a neighbor u of v has at most one ghost neighbor, then form G' from $G - v$ by giving u one additional ghost neighbor w . Now G and G' have the same numbers of vertices and edges, but G' has fewer non-ghost vertices.

We claim also that G' is feasible. If $u \notin A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \notin A$, and then $\beta_{G'}(A) = \beta_G(A) \geq 2$. If $u \in A \subseteq V(G')$, then $\beta_{G'}(A)$ is minimized when $w \in A$, and then $\beta_{G'}(A) \geq \beta_G(A - \{w\} \cup \{v\}) - 2 \geq 2$, since $A - \{w\} \cup \{v\}$ is nontrivial.

We conclude that G' has a strong decomposition (F, D) , by the minimality of G . Each of F and D must have one edge incident to w . We obtain a strong decomposition of G by deleting w , adding vu to D , and adding the other edge at v to F .

We may therefore assume that every neighbor of a non-ghost 2-vertex has at least two ghost neighbors. Since G is 2-edge-connected, a q -vertex cannot have $(q - 1)/2$ ghost neighbors. In particular, a vertex with at least two ghost neighbors must have degree at least 6, so every neighbor of a non-ghost 2-vertex has degree at least 6.

Once again we have derived many properties of a minimal counterexample. We complete the proof by using discharging to show that if G has these properties, then $\text{Mad}(G) \geq 3$. This contradicts feasibility, which is equivalent to $\text{Mad}(G) < 3$; hence there is no minimal counterexample.

The initial charge of each vertex is its degree; we manipulate charge so that the final charge $\mu(v)$ of each vertex v is at least 3. The only discharging rule is that a 2-vertex takes charge $1/2$ along each incident edge from the other endpoint of that edge. Hence the final charge of a 2-vertex is 3.

Since each neighbor of a non-ghost 2-vertex has degree at least 6, vertices of degree 3, 4, or 5 give charge only to ghosts. If $d_G(v) = 3$, then v has no ghost neighbors, and $\mu(v) = 3$. If $d_G(v) \in \{4, 5\}$, then v has at most one ghost neighbor, and $\mu(v) \geq d_G(v) - 1 \geq 3$. If $d_G(v) \geq 6$, then v gives at most $1/2$ along each edge, so $\mu(v) \geq d_G(v) - d_G(v)/2 \geq 3$. \square

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