

Decomposition of Sparse Graphs

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Joint work with
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Nine Dragon Tree (NDT) Conjecture:

(Montassier, Ossona de Mendez, Raspaud, Zhu [2010])

$\text{Arb}(G) \leq k + \frac{d}{k+d+1} \Rightarrow G$ decomposes into $k+1$ forests, with the last being d -bounded.

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Motivation: Applies to “game coloring number”, a bound on game chromatic number χ_g (always $\chi_g(G) \leq \text{col}_g(G)$).

- Zhu [1999]: If G decomposes into G_1 and G_2 , then $\text{col}_g(G) \leq \text{col}_g(G_1) + \Delta(G_2)$.
- Faigle-Kern-Kierstead-Trotter [1993]: If G is a forest, then $\text{col}_g(G) \leq 4$.

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We seek sparseness conditions (bounds on $\text{Mad}(G)$ or $\text{Arb}(G)$) to guarantee decomposition results.

Prior result

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If $\text{Mad}(G) < 2 + \frac{2d}{d+4-6/(d+2)}$, then G decomposes into a forest and a d -bounded graph. When $d \leq 3$, both graphs can be required to be forests.

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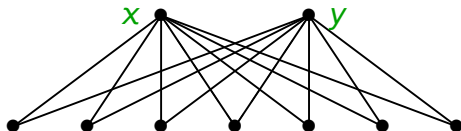
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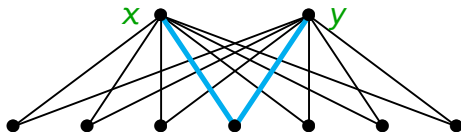


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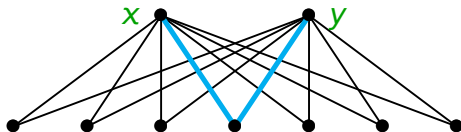
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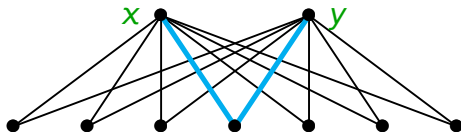
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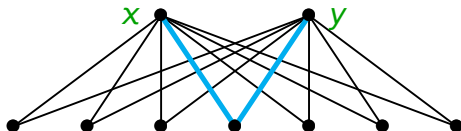
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• $\text{Mad}(K_{2,2d+2}) = \frac{8d+8}{2d+4} = 2 + \frac{2d}{d+2}$. (Same whenever we subdivide each edge of a $(2d+2)$ -regular multigraph.)

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5	$< 10/3$	4	$10/3$

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Thm. For $d = k + 1$, and for $d \leq 6$ when $k = 1$, the NDT Conjecture holds. That is, $\text{Arb}(G) \leq k + \frac{d}{k+d+1} \Rightarrow G$ decomposes into k forests and a d -bounded forest.

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Thm. (Lap Chi Lau and Tamás Király; weaker) The tighter bound on $\text{Arb}(G)$ that in the theorems above suffices for $(k, d - 1)$ -decomposition is shown to suffice for (k, d) -decomposition.

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We will use the induction hypoth. for contracted graphs, induced subgraphs, and edge-deleted subgraphs.

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Claim: $\beta_f(A') = \beta_{f'}(A) - (k + 1)k + \beta_f(B) + k^2$.

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Each edge in $G[A']$ is in $G_B[A]$ or $G[B]$ and contributes $(k + 1 + d)$ to both. Since $f'(z) = 0$, capacities are equal. Right gains $(k + 1)k$ from z and extra constant $-k^2$.

Lemma 1 - Contractions

Def. For $B \subseteq V(G)$, let G_B be the graph obtained by contracting B into a new vertex z .



Lem. 1 If f is feasible on G , and $B \subseteq V(G)$ with $|B| \geq 2$ and $\beta_f(B) \leq k$, then f' is feasible on G_B , where $f'(z) = 0$ and f' agrees with f on $V(G) - B$.

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Now $\beta_f(B) \leq k$ implies $\beta_{f'}(A) \geq \beta_f(A') \geq 0$. ■

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Pf. Pick a $(k, f|_B)$ -decomp. (F, D) of $G[B]$ and a (k, f') -decomp. (F', D') of G_B . Each edge of G is in $G[B]$ or G_B , with those of $[B, \bar{B}]$ incident to z in G_B .

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For forests $F_i \in F$ and $F'_i \in F'$ the union is a forest in G ; otherwise, contracting the part in F_i of a resulting cycle yields a cycle through z in F'_i . ■

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Having proved Claim 1, let $S = \{v \in V(G) : f(v) > 0\}$. We next show that S is an independent set.

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If **one** of $\{u, v\}$ is in A , then $A \subset V(G)$.

If $|A| \geq 2$, then $\beta_{\hat{f}}(A) = \beta_f(A) - (k + 1) \geq 0$.

If $|A| = 1$, then $\beta_{\hat{f}}(A) \geq (k + 1)k - 0 - k^2 \geq k$ (no loops).

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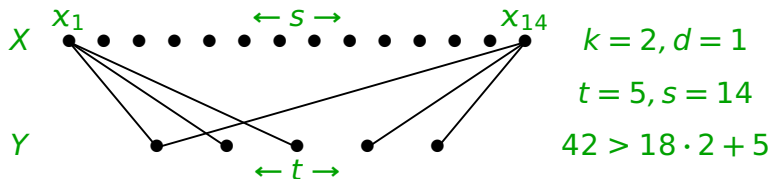
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Contradiction! Thus G is (k, f) -decomposable. ■

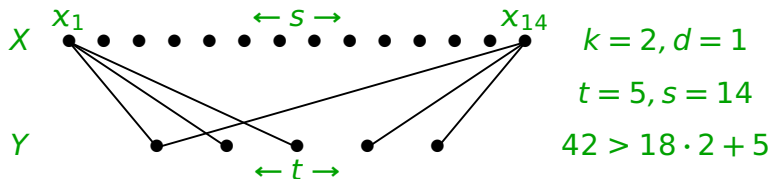
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Ex. Construct an X, Y -bigraph G with $|X| = s$ and $|Y| = t$ where $s = t(k + d) - k + 1$. Put $x_i \leftrightarrow \{y_i, \dots, y_{i+k}\}$ (indices modulo t), so $|E(G)| = (k + 1)(k + d)t - k^2 + 1$.



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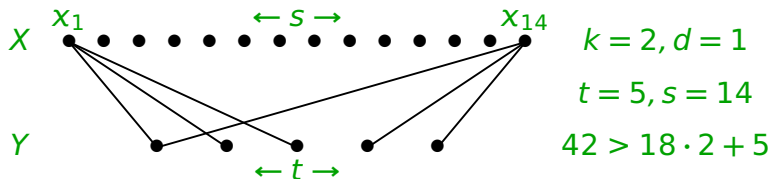
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Almost feasible: A set A minimizing $\beta_f(A)$ contains all nbrs of $X \cap A$ and also has $Y \cap A = \{y_0, \dots, y_{t'-1}\}$. Finally, $\beta_f(A) \geq 0$ for $A \subset V(G)$, but $\beta_f(V(G)) = -1$. ■

NDT Conjecture for $d = k + 1$

Require $(k, f)^*$ -decomposition: the leftover f -bounded graph is also a forest. Feasibility of f is not sufficient: an edge of multiplicity $k + 2$, with $f(u) = f(v) = d$, has $\beta_f(A) \geq 0$ for all A but no decomp. into $k + 1$ forests.

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Cor. The NDT Conjecture holds for $d = k + 1$.

Pf. $\text{Arb}(G) \leq k + \frac{d}{k+d+1} \Rightarrow f$ is feasible.

$S = V(G) \Rightarrow \alpha_f(A) \geq 1$ becomes $\|A\| \leq (k+1)(|A| - 1)$. ■

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$$\beta_{f'}(A') = \beta_f(G) + (k+1) \sum_{v \in A} (d - f(v)) - (k+1) \sum_{v \in A} (d - f(v)).$$

Thus the uniform function f' is feasible, and G' has a (k, d) -decomposition. Deleting the added ghost vertices yields a (k, f) -decomposition of G . ■

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Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define **sparseness** $\beta_G(A) = (k+1)(k+d)|A| - (k+d+1)\|A\|$.

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Def. Let $m_{k,d} = 2k + \frac{2d}{k+d+1}$. For $A \subseteq V(G)$, define **sparseness** $\beta_G(A) = (k+1)(k+d)|A| - (k+d+1)\|A\|$.

$$\text{Arb}(G) \leq m_{k,d}/2 \quad \Leftrightarrow \quad \beta_G(A) \geq (k+1)(k+d) \quad \forall A \subseteq V(G)$$

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Aim: Replace $\text{Arb} \leq m_{k,d}/2$ (**NDT hypothesis**) with (k, d) -sparse plus something else.

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$$(k+1)(k+d)|A| - (k+d+1)[(k+1)(|A|-1) + 1] \geq k^2,$$

which simplifies to $|A| \leq \frac{k}{k+1}(d+1)$.

A Revised Conjecture

Def. Graph G is **feasible** if $\beta_G(A) \geq k^2$ for $\emptyset \neq A \subseteq V(G)$.
A set $A \subseteq V(G)$ is **overfull** if $\|A\| > (k+1)(|A| - 1)$.

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$\text{Arb}(G) \leq m_{k,d}/2$ implies feasibility and forbids overfull sets. Feasibility prohibits overfull sets with more than $\frac{(d+1)k}{k+1}$ vertices. Hence the **NDT Conjecture** follows from:

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Ghosts help control $(k, d)^*$ -decompositions, since the edges at a ghost must lie in $k + 1$ distinct forests.

Minimal Counterexamples

Def. For fixed k, d , a **minimal counterexample** to the conjecture is one having the fewest ghosts among those with the fewest non-ghosts. A set $A \subseteq V(G)$ is **nontrivial** if it has at least two but not all non-ghosts.

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Lem. Properties of a minimal counterexample G :

- 1) G is $(k + 1)$ -edge-connected.
- 2) If $d_G(v) \leq 2k + 1$, then v has no ghost neighbor.
- 3) If $\beta_G(A) \leq k(k + 1)$, then G_A is feasible.
- 4) If G_A is $(k, d)^*$ -decomposable, then so is G .
- 5) If A is nontrivial, then $\beta_G(A) > k(k + 1)$.
- 6) If $\beta_G(A) \leq k(k + 1)$ and A has just one non-ghost v , then $d_G(v) \geq (k + 1)(d + 1)$.
- 7) If $d_G(v) < (k + 1)(d + 1)$, then v has no non-ghost neighbor with degree $k + 1$.
- 8) If v has $q(\geq 1)$ ghost nbrs, then $d_G(v) > kq + k + d$.
- 9) Edges with multiplicity $k + 1$ have a ghost endpoint.

Discharging Argument

Aim: Show that if G has the properties derived for a minimal counterexample, then $\text{Mad}(G) \geq m_{k,d}$. We need a bit more but first suggest the discharging argument.

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Thus a ghost takes $m_{k,d} - (k+1)$ in total from its nbr. By force, $\mu(v) = m_{k,d}$ when $d_G(v) = k+1$, since $(k+1)$ -vertices are not adjacent.

Other Vertex Degrees

If all neighbors of v have degree $k + 1$, then each edge takes $\frac{k+d-1}{k+d+1}$, so $\mu(v) = d_G(v) \frac{2}{k+d+1}$.

Hence $\mu(v) \geq m_{k,d}$ if and only if $d_G(v) \geq (k + 1)(k + d)$.

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How to handle degrees between $k+1$ and $(k+1)(k+d)$?

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A vertex v with degree less than $(k + 1)(k + d)$ and more than $k + 1$ has no non-ghost $(k + 1)$ -neighbor, but it has some neighbors with higher degrees and hence won't give as much as $(k + 1) \frac{k+d+1}{k+d-1}$. More information is needed about the degrees of neighboring vertices.

The Case $k = 1$

When $(k, d) = (1, 1)$, only 2-vertices need charge. Their nbrs have high enough degree that **Rule 1** suffices. This proves the **NDT Conjecture** for $(k, d) = (1, 1)$ (Montassier et al.).

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Thm. For $d > k = 1$, if each 3-vertex in a minimal counterexample G has a neighbor with degree at least $d + 2$, then $\text{Mad}(G) \geq m_{1,d} = 2 + \frac{2d}{d+2}$.

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Rule 2: If $d_G(v) = 3$, and v has neighbor u with $d_G(u) \geq d + 2$, then v receives $\frac{d-2}{d+2}$ from u .

Everyone is Happy

Rules 1 and 2 yield $\mu(v) \geq m_{1,d}$ when $d_G(v) \leq 3$
(from $3 + \frac{d-2}{d+2} = 2 + \frac{2d}{d+2}$). Since $\frac{d-2}{d+2} < \frac{d}{d+2}$, the general
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If $4 \leq d_G(v) \leq d+1$, then v has no non-ghost 2-nbr, and Rule 2 takes no charge from v . If v has q ghost 2-neighbors with $q \geq 1$, then $d_G(v) \geq q + d + 2$. Since $q \leq d/2$, we compute $\mu(v) \geq 2 + \frac{2d}{d+2} + \frac{d}{2} > m_{1,d}$.

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If $d+2 \leq d_G(v) \leq 2d+1$, again no non-ghost 2-nbr; v may give charge to q ghost neighbors and to $d_G(v) - 2q$ neighbors of degree 3.

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$$\begin{aligned}\mu(v) &\geq d_G(v) - \frac{d}{d+2}2q - [d_G(v) - 2q]\frac{d-2}{d+2} \\ &= \frac{4(d_G(v)-q)}{d+2} \geq \frac{4(d+2)}{d+2} = 4 > m_{1,d}.\end{aligned}$$

Hence the final charge at each vertex is at least $m_{1,d}$. ■

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Thm. The **Strong NDT Conj.** is true when $(k, d) = (1, 2)$.