

Longest Cycles in k -connected Graphs with Given Independence Number

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Joint work with
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slides and paper at
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Spanning Cycles

Sufficient cond. for spanning cycles in n -vertex graphs:

Thm. (Dirac [1952]) $\delta(G) \geq n/2$.

Thm. (Ore [1960]) $d(x)+d(y) \geq n$ whenever $xy \notin E(G)$.

Thm. (Chvátal–Erdős [1972]) $\kappa(G) \geq \alpha(G)$.

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Long-cycle versions for 2-connected graphs
($c(G)$ = circumference = length of longest cycle):

Thm. (Dirac [1952]) $c(G) \geq \min\{n, 2\delta(G)\}$.

Thm. (Bondy [1971]; also Bermond [1976], Linial [1976])
 $c(G) \geq \min\{n, s\}$ if $d(x)+d(y) \geq s$ whenever $xy \notin E(G)$.

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Long-cycle version of Chvátal–Erdős Theorem?

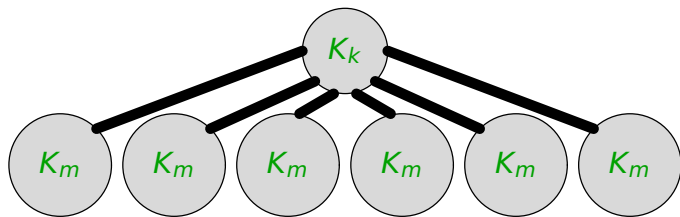
Conjecture

Conj. (Fouquet–Jolivet [1976]) If $\kappa(G) \leq \alpha(G)$, then $c(G) \geq \frac{k(n+a-k)}{a}$, where $k = \kappa(G)$ and $a = \alpha(G)$.

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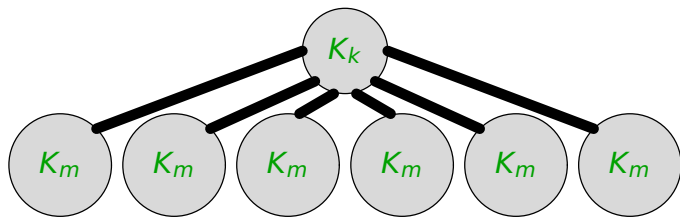
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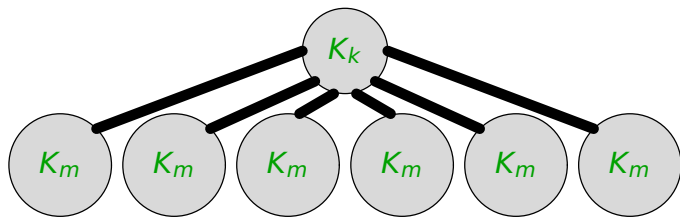


$$n = k + am, \quad \alpha(G) = a, \quad \kappa(G) = k, \quad c(G) = k(1+m) = \frac{k(n+a-k)}{a}.$$

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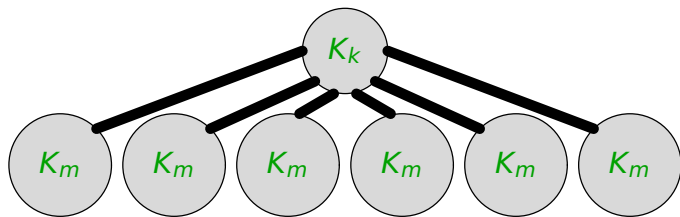
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- Known true for $a \in \{k+1, k+2\}$ (Fournier [1982]),
 $k = 2$ (Fournier [1984]), $k = 3$ (Manoussakis [2009]),
 $k = 4$ & $a < 2k - 1$ (Chen–Hu–Y.Wu [2010(a&b)+])

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Thm. (W.-O.-W.) The F–J Conjecture is true.

Stronger Conjecture

Conj. (Chen–Chen–Liu) If C and C' are distinct cycles in a k -connected graph, then there are cycles D and D' with $V(C) \cup V(C') \subseteq V(D) \cup V(D')$ and $|V(D) \cap V(D')| \geq k$.

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We prove the F–J Conjecture **without** proving the C–C–L Conjecture.

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Thm. (Kouider [1994]) If H is a subgraph of a k -connected graph G , then G has a cycle C such that $\alpha(H - V(C)) \leq \max\{0, \alpha(H) - k\}$.

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Lem. (Multi-Cycle Lemma) If G is a k -connected graph with independence number α , and $0 \leq \ell \leq \alpha - k$, then there exist cycles C_0, \dots, C_ℓ satisfying the following:

- (1) $\alpha(G - \bigcup_{i=0}^{\ell} V(C_i)) \leq \alpha - k - \ell$,
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Path Lem. \Rightarrow Cycle Lem. \Rightarrow Multi-Cycle Lem. \Rightarrow F-J Conj.

Multi-Cycle \Rightarrow Fouquet–Jolivet

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$$\begin{aligned} n &= |V(C_0)| + \sum_{i=1}^{\ell} \left| V(C_i) - \bigcup_{j=0}^{i-1} V(C_j) \right| \\ &\leq |V(C_0)| + (a - k) \left(\frac{|V(C_0)|}{k} - 1 \right). \end{aligned}$$

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Lem. (Multi-Cycle) For $0 \leq l \leq a - k$, $\exists C_0, \dots, C_l$ with

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The inequality simplifies to $|V(C_0)| \geq \frac{k(n+a-k)}{a}$. ■

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Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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Including C' in the list yields (1), but we also need (2).

Cycle Lemma \Rightarrow Multi-Cycle Lemma (cont.)

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(1) $\alpha(G - \bigcup_{i=0}^l V(C_i)) \leq a - k - l$

(2) $|V(C_i) - \bigcup_{j=0}^{i-1} V(C_j)| \leq \frac{|V(C_0)|}{k} - 1$ for $1 \leq i \leq l$.

Pf. Apply Cycle Lem. with C_0 as C to get C' :

$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Cycle Lemma \Rightarrow Multi-Cycle Lemma (cont.)

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$$\alpha(H - V(C')) \leq a - k - l \text{ and } |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1.$$

Case 1: $|V(C')| \leq |V(C_0)|$.

Complete the desired list by setting $C_l = C'$; okay since

$$\left| V(C') - \bigcup_{j=0}^{l-1} V(C_j) \right| \leq |V(C') - V(C_0)| \leq |V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1.$$

Cycle Lemma \Rightarrow Multi-Cycle Lemma (cont.)

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$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

Case 2: $|V(C')| > |V(C_0)|$.

New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

Cycle Lemma \Rightarrow Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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Case 2: $|V(C')| > |V(C_0)|$.

New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

$$i = 1: V(C'_1) - V(C'_0) = V(C_0) - V(C').$$

$$i > 1: V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j).$$

Cycle Lemma \Rightarrow Multi-Cycle Lemma (cont.)

Lem. (Cycle) Given disjoint H and cycle C (length $\geq k$), $\exists C'$ with $|V(C) - V(C')| \leq \frac{|V(C)|}{k} - 1$ and $\alpha(H - V(C')) \leq \alpha(H) - 1$.

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Pf. Apply Cycle Lem. with C_0 as C to get C' :

$\alpha(H - V(C')) \leq a - k - l$ and $|V(C_0) - V(C')| \leq \frac{|V(C_0)|}{k} - 1$.

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New list: Set $C'_0 = C'$, with $C'_i = C_{i-1}$ for $1 \leq i \leq l$.

$i = 1$: $V(C'_1) - V(C'_0) = V(C_0) - V(C')$.

$i > 1$: $V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j) \subseteq V(C_{i-1}) - \bigcup_{j=0}^{i-2} V(C_j)$.

So, $|V(C'_i) - \bigcup_{j=0}^{i-1} V(C'_j)| \leq \frac{|V(C_0)|}{k} - 1 \leq \frac{|V(C'_0)|}{k} - 1$. ■

Path Lemma \Rightarrow Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$,
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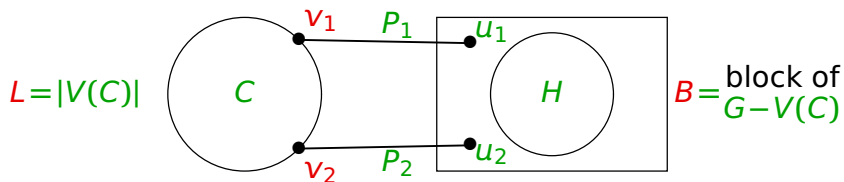
Pf. If H is disconnected or has a cut-vertex, then find C'
using an induced subgraph of H . Otherwise, . . .

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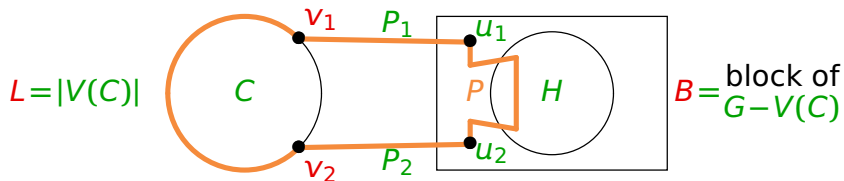


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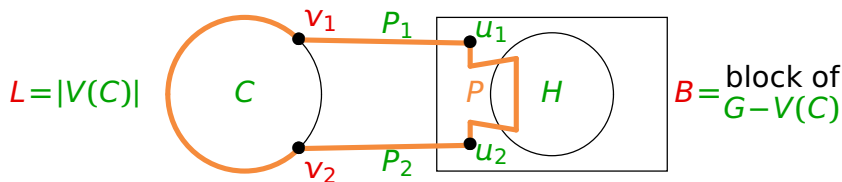
If $d_C(v_1, v_2) \leq L/k$, build cycle C' using P from Path Lem.

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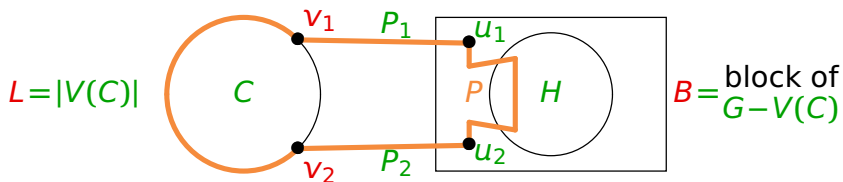
If $d_C(v_1, v_2) \leq L/k$, build cycle C' using P from Path Lem.
 Let $\{c_1, \dots, c_m\}$ be arrival points on C of paths from B .

Path Lemma \Rightarrow Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$,
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If $d_C(v_1, v_2) \leq L/k$, build cycle C' using P from Path Lem.
 Let $\{c_1, \dots, c_m\}$ be arrival points on C of paths from B .
 Strict paths from $b, b' \in B$ reaching c_i, c_{i+1} are disjoint.

Path Lemma \Rightarrow Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$,
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Pf. $\therefore t = |\{i: c_i \& c_{i+1} \text{ reached from diff verts of } B\}| < k$.

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But, Menger $\Rightarrow k$ paths from $b \in B$ to C .

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At least $k - t$ " b -segments" of C meet paths from no other $b' \in B$.

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Over all $b \in B$, total length $< L - t(L/k) = L(k - t)/k$.

Path Lemma \Rightarrow Cycle Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$,
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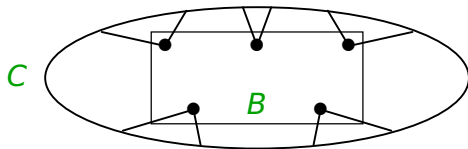
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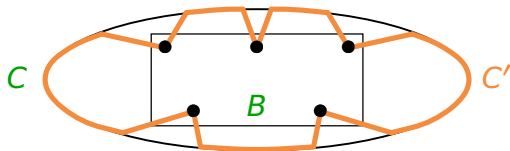
Pf. $\therefore t = |\{i: c_i \text{ \& } c_{i+1} \text{ reached from diff verts of } B\}| < k$.

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At least $k - t$ "b-segments" of C meet paths from no other $b' \in B$.

Over all $b \in B$, total length $< L - t(L/k) = L(k - t)/k$.

Pick a shortest $\forall b \in B$; total length $< L/k$. Form C' .



Proof of Path Lemma

Lem. (Path) Given $H \subseteq G$, $\kappa(G) \geq k$, and $u, v \in V(G)$,
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Pf. For each u, v -path P , let F_P be a smallest component of $G - V(P)$ that intersects H . Choose P so that:

- (i) $\alpha(H - V(P))$ is smallest;
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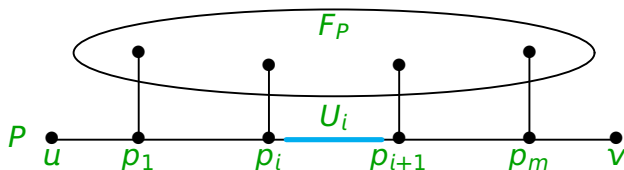
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If $V(H) \not\subseteq V(P)$, then let p_1, \dots, p_m have neighbors in F_P .



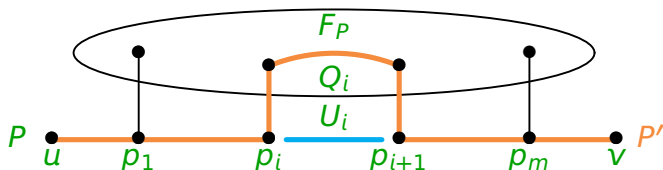
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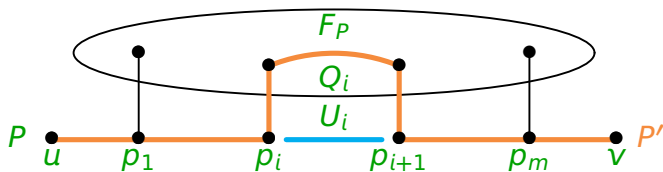
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- $V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H - V(P')) < \alpha(H - V(P))$.

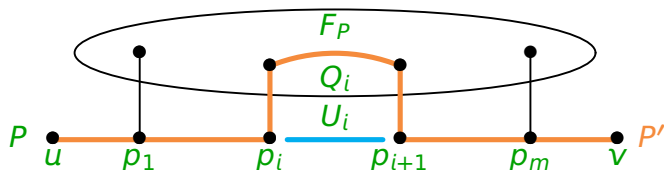
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- $V(F_P) \cap V(H) \not\subseteq V(P')$, else $\alpha(H - V(P')) < \alpha(H - V(P))$.
- $\therefore F_P - V(P')$ has a component intersecting H .

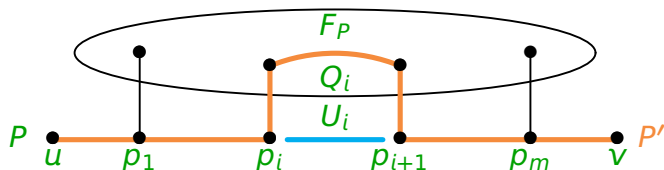
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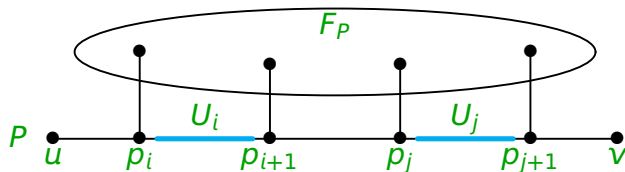
$\therefore F_P - V(P')$ has a component intersecting H .

Thus $\alpha(H - V(P - U_i)) \geq \alpha(H - V(P')) > \alpha(H - V(P))$.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

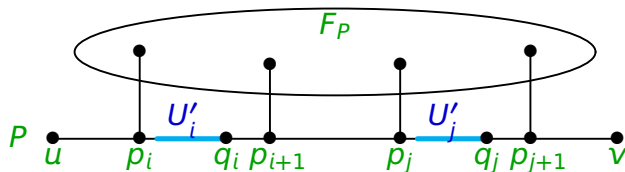
We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i .



Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
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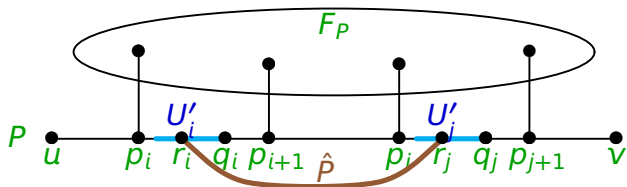


Restore U_i from the left; q_i is where α first increases.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
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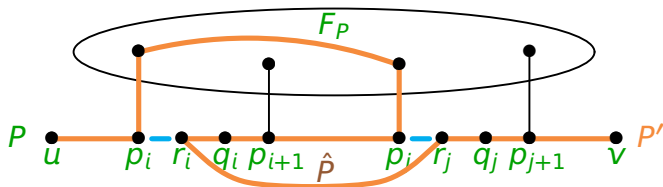
Restore U_i from the left; q_i is where α first increases.

A path \hat{P} outside P joining U'_i and U'_j can't visit F_P .

Proof of Path Lemma, cont.

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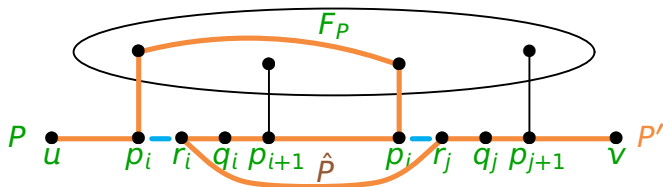
A path \hat{P} outside P joining U'_i and U'_j can't visit F_P .

Pick leftmost such r_i and form new path P' .

Proof of Path Lemma, cont.

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We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i .



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A path \hat{P} outside P joining U'_i and U'_j can't visit F_P .

Pick leftmost such r_i and form new path P' .

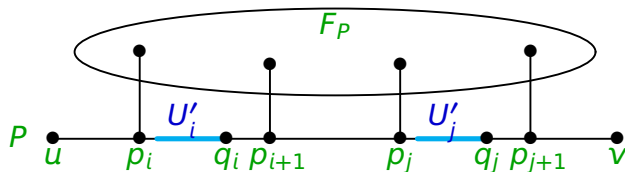
Restoring \blacksquare can't increase α : again

$\alpha(H - V(P')) \leq \alpha(H - V(P))$, contradicting choice of P .

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i .



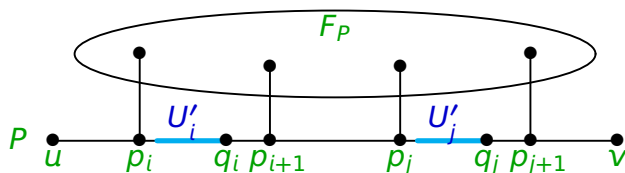
Restore U_i from the left; q_i is where α first increases.

Sets U'_1, \dots, U'_{m-1} are in diff comps of $G - V(P - U)$.

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
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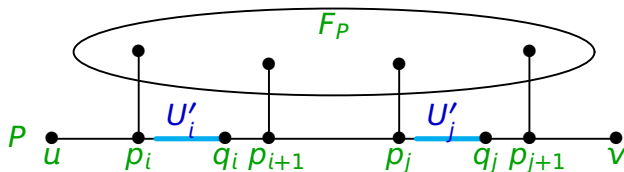
Sets U'_1, \dots, U'_{m-1} are in diff comps of $G - V(P - U)$.

$\therefore \alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1 \geq \alpha(H - V(P)) + k - 1.$

Proof of Path Lemma, cont.

Recall: P is chosen so that: (i) $\alpha(H - V(P))$ is smallest;
(ii) subject to (i), F_P has the fewest vertices.

We proved: $\alpha(H - V(P - U_i)) > \alpha(H - V(P))$ for all i .



Restore U_i from the left; q_i is where α first increases.

Sets U'_1, \dots, U'_{m-1} are in diff comps of $G - V(P - U)$.

$$\therefore \alpha(H - V(P - U)) \geq \alpha(H - V(P)) + m - 1 \geq \alpha(H - V(P)) + k - 1.$$

$$\therefore \alpha(H) \geq \alpha(H - V(P)) + k - 1. \quad \blacksquare$$