Forbidden Subposets for Fractional Weak Discrepancy at Most k

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Abstract

The *fractional weak discrepancy* of a poset P, written $\mathrm{wd}_F(P)$, is the least k such that some $f\colon P \to \mathbb{R}$ satisfies $f(y) - f(x) \geq 1$ for $x \prec y$ and $|f(y) - f(x)| \leq k$ for $x \| y$. We determine the minimal forbidden subposets for the property $\mathrm{wd}_F(P) \leq k$ when k is an integer.

1 Introduction

In any company, the employees form a partially ordered set (poset), putting $x \prec y$ if y is more valuable to the company than x (we use $\preceq$ or $\preceq_P$ for the order relation in a poset P and $\|$ for incomparability, with $\prec$, $\succ$, $\succeq$ having the natural meanings). Such a poset may have incomparable pairs, because sometimes it is impossible to tell which of two employees is more valuable. Clearly y should be paid more than x if $x \prec y$. Also, to be "fair", one wants salaries of incomparable employees not to differ by much.

Using this (and other examples) as motivation, Tanenbaum, Trenk, and Fishburn [7] introduced "linear discrepancy" of posets, which models the case where the salary values form an arithmetic progression. The salaries place the elements of P in a linear order that preserves the order relations in P. The problem is to choose this order to minimize the maximum difference between the positions of incomparable elements.

Requiring the salaries to to form an arithmetic progression seems unrealistic, and later models addressed this issue. A *consistent labeling* of a poset P is a real-valued function f on P such that $f(x) + 1 \leq f(y)$ when $x \prec y$ in P. A *k-weak labeling* of P is a consistent labeling f such that $|f(x) - f(y)| \leq k$ whenever $x \parallel y$. The *weakness* of a consistent labeling

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is the least k such that it is k-weak. The *weak discrepancy* of P, denoted $\mathrm{wd}(P)$, is the least integer k such that P has an integer-valued k-weak labeling. The *fractional weak discrepancy* denoted $\mathrm{wd}_F(P)$, is the least weakness of any consistent labeling of P. A consistent labeling with that weakness is an *optimal* labeling.

In a 0-weak labeling, elements receive the same label if and only if they are incomparable. Thus the distinct values of f define levels, with elements on the same level if and only if they are incomparable. Such posets are called *weak orders*, which motivates the terminology for weak discrepancy.

We consider these parameters for finite posets. Weak discrepancy was introduced by Gimbel and Trenk [1]; fractional weak discrepancy later by Shuchat, Shull, and Trenk [6]. The latter noted that wd_F can be formulated as a linear program. The dual maximization problem bounds $\mathrm{wd}_F(P)$ from below using special subposets of P.

A *forcing cycle* C in a poset P is a list $x_1, \dots, x_m$ of elements of P, such that for each i (with indices modulo m) either $x_i \prec x_{i+1}$ or $x_i \parallel x_{i+1}$. When C is a forcing cycle, let $u(C)$ be the number of indices i such that $x_i \prec x_{i+1}$, and let $s(C)$ count those such that $x_i \parallel x_{i+1}$. That is, $u(C)$ counts the "up" steps and $s(C)$ counts the "side" steps. Since the up steps force f to increase by at least $u(C)$ along the cycle, the pigeonhole principle yields $\mathrm{wd}_F(P) \geq u(C)/s(C)$. Call $u(C)/s(C)$ the *ratio* of the cycle C. Like consistent labelings with minimum weakness, forcing cycles with maximum ratio are *optimal* cycles.

If P has two incomparable elements, then a forcing cycle exists and provides a lower bound on $\mathrm{wd}_F(P)$. Strong duality is expressed by the following results.

Theorem 1.1 (Shuchat, Shull, and Trenk [6]). *For a poset P that is not a chain,* $\mathrm{wd}_F(P) = \max_C \frac{u(C)}{s(C)}$*, where the maximum is taken over all forcing cycles C in P.*

Theorem 1.2 (Gimbel and Trenk [1]). *For a poset P that is not a chain,* $\mathrm{wd}(P) = \max_C \lceil \frac{u(C)}{s(C)} \rceil$*, where the maximum is taken over all forcing cycles C in P.*

These results imply for all P that $\mathrm{wd}(P) = \lceil \mathrm{wd}_F(P) \rceil$ and that $\mathrm{wd}_F(P)$ is rational. In [3], the possible values of $\mathrm{wd}_F(P)$ are explored (it can be any value at least 1, or $r/(r+1)$ for integer r). Shuchat et al. ([4, 5]) studied the values of $\mathrm{wd}_F(P)$ within various families described by forbidden subposets.

In particular, let $\mathbf{r}_1 + \cdots + \mathbf{r}_t$ denote the poset consisting of chains of sizes $r_1, \dots, r_t$ and no additional relations. The weak orders described earlier are characterized by forbidding $\mathbf{2}+\mathbf{1}$ as a subposet. Similarly characterized families include *interval orders* (no $\mathbf{2}+\mathbf{2}$) and *semiorders* (no $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$). Semiorders are the posets for which wd_F can be less than 1. Suchat et al. [5] studied the possible values of wd_F for interval orders, non-interval orders, posets containing $\mathbf{2}+\mathbf{2}$ but not $\mathbf{3}+\mathbf{1}$, and posets with no $\mathbf{t}+\mathbf{1}$ for each $t \geq 3$.

Forbidden subposet characterizations of hereditary families of posets have both structural and algorithmic value. Minimal forbidden subposets are often called *obstructions*. For linear discrepancy 1, Tanenbaum, Trenk, and Fishburn [7] determined the list of obstructions to be $\{\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1}, \mathbf{1}+\mathbf{1}+\mathbf{1}\}$. Howard, Keller, and Young [2] determined the obstructions for linear discrepancy at most 2.

In this paper, we study the obstructions for the poset families defined by $\mathrm{wd}_F(P) \le k$ or $\mathrm{wd}_F(P) < k$ for a given natural number k. Analogous results for (integer) weak discrepancy follow as corollaries. In this language, our remark about weak orders was that $\mathbf{2}+\mathbf{1}$ is the unique obstruction to $\mathrm{wd}_F(P) \le 0$. Shuchat et al. [3] solved one of the problems for $k = 1$.

Theorem 1.3 (Shuchat, Shull, and Trenk [3]). *A poset P satisfies $\mathrm{wd}_F(P) < 1$ if and only if P is a semiorder; that is, the obstructions to $\mathrm{wd}_F(P) < 1$ are $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$.*

We characterize the obstructions completely for $\{P\colon\ \mathrm{wd}_F(P) \le k\}$, where k is an integer. In Section 2, we construct the forbidden subposets, show that their fractional weak discrepancy exceeds k, and show that none contains another. Unfortunately, the list is infinite. In Section 3, we prove that $\mathrm{wd}_F(P) \le k$ when P contains none of these posets. In Section 4, we consider the analogous problem for the family where $\mathrm{wd}_F(P) < k$. Although the list is finite when $k = 1$, we provide infinitely many examples for larger k, but not a full description.

The following result is a tool for our characterizations. It holds simply because an optimal consistent labeling must meet with equality the bounds on the difference between the values of successive elements that are imposed by the pigeonhole principle.

Proposition 1.4 (Shuchat, Shull, and Trenk [3]). *Let P be a poset with $\mathrm{wd}_F(P) = r$. If C is a forcing cycle with elements $x_1, \ldots, x_m$ and $\frac{u(C)}{s(C)} = r$, and f is an r-weak labeling of P, then $f(x_{i+1}) = f(x_i) + 1$ whenever $x_i \prec x_{i+1}$ and $f(x_{i+1}) = f(x_i) - r$ whenever $x_i \parallel x_{i+1}$.*

2 Obstructions to $\mathrm{wd}_F(P) \le k$

Consider a composition of $(k+1)q+1$ with q parts, written as a q-tuple $(b_1, \ldots, b_q)$ of positive integers with sum $(k+1)q+1$. We view two q-tuples as equivalent if one is a cyclic shift of the other; this defines an equivalence relation on the q-tuples. We abuse notation by writing $(b_1, \ldots, b_q)$ to refer to the class containing $(b_1, \ldots, b_q)$. From each class, we construct a family of posets with fractional weak discrepancy $k + 1/q$.

From $(b_1, \ldots, b_q)$, first form a disjoint union of chains with sizes $b_1, \ldots, b_q$. Let x_i^j denote the jth element on the ith chain, for $1 \le j \le b_i$, and let B_i denote the chain with elements $x_i^1, \ldots, x_i^{b_i}$ in order. To $B_1 + \cdots + B_q$, we will add additional relations to form P. Order the elements by the indices (i, j) in lexicographic order, running through all $B_1, \ldots, B_q$

successively. The added relations will make this list an optimal forcing cycle C in the resulting poset. The top element of each chain B_i will remain incomparable to the bottom element of the next chain (cyclically), and hence we will have $u(C) = kq + 1$ and $s(C) = q$.

We will want an optimal labeling f to match C as in Proposition 1.4. Let

$$f(x_i^j) = (j-1) - (i-1)\cdot\frac{kq+1}{q} + \sum_{r=1}^{i-1}(b_r - 1).$$

Note that $f(x_i^{j+1}) = f(x_i^j) + 1$ for $1 \le j \le b_i - 1$. Also $f(x_i^{b_i}) - f(x_{i+1}^1) = \frac{kq+1}{q}$ for all i, taking subscripts modulo q. Hence we have defined f to match C, but to ensure that C is an optimal cycle we still must add relations to prevent the occurrence of forcing cycles with larger ratio. In particular, if property (3) in the definition below does not hold, then C may not be optimal and f will have weakness larger than $k + 1/q$.

Definition 2.1. Let $\mathcal{F}(b_1, \ldots, b_q)$ denote the family of all posets with elements $\{x_i^j \colon 1 \le i \le q, 1 \le j \le b_i\}$ that satisfy the following properties.

(1) $x_i^j \prec x_i^{j+1}$ for $1 \le j \le b_i - 1$.
(2) $x_i^{b_i} \parallel x_{i+1}^1$ for all i (viewing indices modulo q).
(3) $x_i^j \prec x_{i'}^{j'}$ whenever $f(x_{i'}^{j'}) - f(x_i^j) > \frac{kq+1}{q}$.
(4) $x_i^j \parallel x_{i'}^{j'}$ whenever $|f(x_{i'}^{j'}) - f(x_i^j)| < 1$.
(5) $f(x_i^j) < f(x_{i'}^{j'})$ if $x_i^j \prec x_{i'}^{j'}$.

The family $\mathcal{F}(b_1, \ldots, b_q)$ may be large, since we have not specified whether $x_i^j \prec x_{i'}^{j'}$ when $1 \le f(x_i^j) < f(x_{i'}^{j'}) \le k + 1/q$. We may add any set of relations of that form (not putting x_i^j above $x_{i+1}^{j'}$) before requiring transitivity. We may add none, so $\mathcal{F}(b_1, \ldots, b_q)$ is nonempty.

Lemma 2.2. *The posets in $\mathcal{F}(b_1, \ldots, b_q)$ have weakness $k+1/q$, and f is an optimal labeling.*

Proof. Since all relations $x_i^j \prec x_{i'}^{j'}$ that are added satisfy $|f(x_{i'}^{j'}) - f(x_i^j)| \ge 1$, there is no cycle of implications, and under transitive closure of the relation the set becomes a poset on which f is a consistent labeling. The requirement of $x_i^{b_i} \parallel x_{i+1}^1$ implies that no element of B_i is placed above any element of B_{i+1}. Together, (1) and (3) imply that f has weakness $k + 1/q$. Also C is a forcing cycle, with $u(C) = kq + 1$ and $s(C) = q$. Hence each such poset has weakness $k + 1/q$, and f is an optimal labeling. □

In the definition, each poset P in $\mathcal{F}(b_1, \ldots, b_q)$ is built from a particular forcing cycle C and optimal labeling f. We call these the *fundamental cycle* and *fundamental labeling* of P.

Lemma 2.3. *If $P \in \mathcal{F}(b_1, \ldots, b_q)$ has fundamental cycle C, fundamental labeling f, and weakness r, then $|f(x) - f(y)| \in \{1, r\}$ if and only if x and y are consecutive on C.*

Proof. For P, we have $r = (kq+1)/q$. By Proposition 1.4, the condition is sufficient. For necessity, choose $x, y \in P$ arbitrarily. As we follow C from x to y, we have some number a of up-steps and some number b of side-steps. Since the condition is sufficient, $f(y) - f(x) = a - br$. Multiplying by q yields $|aq - b(kq+1)| \in \{q, kq+1\}$. Since q and $kq+1$ are relatively prime, there is exactly one integer solution (a,b) to the equation $aq - b(kq+1) = p$ for each integer p. For $p = \pm q$, the solution is $(a,b) = (\pm 1, 0)$. For $p = \pm kq + 1$, the solution is $(0, \mp 1)$. Hence the specified differences occur only when x and y are consecutive along C. □

Let $\mathcal{F}_k$ be the union of all families $\mathcal{F}(b_1, \ldots, b_q)$ such that $\sum_{i=1}^{q} b_i = (k+1)q + 1$.

Lemma 2.4. *The fundamental cycle and labeling for a poset $P \in \mathcal{F}_k$ are well-defined. That is, each such P arises from exactly one equivalence class of compositions defining a fundamental cycle, and in exactly one way from the lexicographically least element of the equivalence class.*

Proof. Since the number of elements depends on q, we need only consider posets generated by fundamental cycles with the same number of chains. Suppose that $P' \cong P$, but P' arises from a fundamental cycle C' and fundamental labeling f'. If P' arises from a different class of compositions of $(k+1)q+1$, then under the isomorphism $\phi\colon\ P \to P'$, some $x, y \in P$ that are consecutive along C are mapped by ϕ to elements $\phi(x)$ and $\phi(y)$ that are not consecutive along C'. Since $\mathrm{wd}_F(P) = \mathrm{wd}_F(P') = r$, Lemma 2.3 implies that $|f(x) - f(y)| \in \{1, r\}$, but $|f'(\phi(x)) - f'(\phi(y))| \notin \{1, r\}$.

This contradicts Proposition 1.4, because elements consecutive along C must be mapped by any automorphism to elements that are consecutive along some optimal cycle. Hence their values under any optimal labeling must differ by 1 or r.

Hence ϕ can do nothing other than shift along C. After this shift, we have chosen a canonical element of the equivalence class of compositions, the fundamental cycles are the same, and the relations added to form P and P' are the same. □

Lemma 2.5. *No poset in $\mathcal{F}_k$ is contained in another poset in $\mathcal{F}_k$.*

Proof. By Lemma 2.4, $P \subseteq P'$ only if P is smaller than P'. This requires $q < q'$, where q and q' are the numbers of chains in the fundamental cycles of P and P'. Now $\mathrm{wd}_F(P) = k + 1/q > k + 1/q' = \mathrm{wd}_F(P')$. However, $\mathrm{wd}_F(P) \le \mathrm{wd}_F(P')$ when $P \subseteq P'$, since every forcing cycle in P is also a forcing cycle in P'. □

Note that $\mathcal{F}_k$ is an infinite family; we construct distinct members of $\mathcal{F}_k$ for each value of q. The fact that the posets in $\mathcal{F}_k$ are minimal posets with fractional weak discrepancy greater than k will follow from the proof in the next section that every poset containing no poset in $\mathcal{F}_k$ has fractional weak discrepancy at most k.

3 Characterization of posets with $\mathrm{wd}_F(P) \leq k$

Consider $P \in \mathcal{F}(b_1, \ldots, b_q)$. When we treat the fundamental cycle C of P as a forcing cycle, we consider only the comparabilities along the q special chains and the incomparabilities involving the top of one chain and the bottom of the next. In this section, we will also view C as a subposet. When we write $\mathrm{wd}_F(C)$, we are considering the subposet of P whose set of elements is C. Since the cyclic interpretation of C is in fact a forcing cycle in the subposet C, we have $\mathrm{wd}_F(C) \geq \mathrm{wd}_F(P)$. Equality holds, since C is a subposet of P.

The next lemma is the crux of the sufficiency argument for the characterization. We consider an arbitrary optimal cycle in an arbitrary poset P.

Lemma 3.1. *Let C be an optimal cycle in a poset P, with $s(C) = q$ and $u(C) = qk + r$ for some positive k. If $r \geq 2$, then P contains a forcing cycle C' such that the subposet C' has weak discrepancy $k + r'/q'$ for some pair (r', q') with $1 \leq r' < r$ and $1 \leq s' \leq s$.*

Proof. We may assume that C is a shortest optimal cycle, since otherwise any shorter optimal cycle in P serves as the desired C'. Since C is a shortest optimal cycle, every subposet of C has smaller fractional weak discrepancy than C.

Since $\mathrm{wd}_F(C) > 1$, some chain in C (as a cycle) has at least three elements. Let x be the middle element in a chain of three consecutive elements in C. Let Q be the subposet obtained by deleting x from C. Also Q is a forcing cycle with $u(Q) = qk + r - 1$ and $s(Q) = q$, since the consecutive incomparabilities in C were preserved by deleting x. Since Q is a subposet of C, we conclude that $k + (r-1)/q \leq \mathrm{wd}_F(Q) < \mathrm{wd}_F(C) = k + r/q$.

Thus $\mathrm{wd}_F(Q)$ is a rational number $k + a/b$ with $(r-1)/q \leq a/b < r/q$. Also, $a + b \leq r - 1 + q$, since $(k+1)b + a$ is the size of a forcing cycle C' contained in Q. It suffices to show that $b \leq q$, since than $a/b < r/q$ implies $a < r$. If $b > q$, then $a + b \leq r - 1 + q$ yields $a < r - 1$, which yields $a/b < (r-1)/q$, a contradiction. □

Note that in Lemma 3.1, there is no upper bound on r in the hypothesis. Hence we can apply the lemma for all P such that $\mathrm{wd}_F(P) > k$, not just those with $k < \mathrm{wd}_F(P) \leq k + 1$.

Theorem 3.2. *If $\mathrm{wd}_F(P) > k$, then P contains a poset in $\mathcal{F}_k$, and hence $\mathcal{F}_k$ is the complete list of obstructions for $\{P\colon\ \mathrm{wd}_F(P) \leq k\}$.*

Proof. By repeated application of Lemma 3.1, we may choose a cycle C in P such that $u(C) = qk + 1$ and $s(C) = q$, for some q, and furthermore $\mathrm{wd}_F(C) = k + 1/q$, where C is viewed as a subposet. Note that C need not be an optimal cycle in P.

Let $(b_1, \ldots, b_q)$ be the sizes of the successive chains in C (between side-steps). Since $\mathrm{wd}_F(C) = k + 1/q$, every optimal numbering f has differences along C as specified by Proposition 1.4. Shifting the values of f by a constant, if necessary, yields f satisfying the definition specified before Definition 2.1. Now, since both C and f have been chosen to be optimal, $f(x) - f(y) > k + 1/q$ requires $x \prec y$. Incorporating these relations forces the subposet C to lie in $\mathcal{F}_k$. □

Since these are the obstructions to $\mathrm{wd}_F(P) \le k$, when $P \in \mathcal{F}_k$ we have $\mathrm{wd}_F(P - x) \le k$ and hence $\mathrm{wd}(P - x) \le k$. If $k < \mathrm{wd}_F(P) \le k + 1$, then P contains some poset in $\mathcal{F}_k$ but no poset in $\mathcal{F}_{k+1}$.

For weak discrepancy, the characterization is the same.

Corollary 3.3. *For each integer k, the family of obstructions to $\{P\colon \mathrm{wd}(P) \le k\}$ is $\mathcal{F}_k$.*

Proof. If k is an integer and $\mathrm{wd}(P) > k$, then $\mathrm{wd}_F(P) > k$, since $\mathrm{wd}(P) = \lceil \mathrm{wd}_F(P) \rceil$. By Theorem 3.2, P contains a poset in $\mathcal{F}_k$. Conversely, if P contains such a poset P', then $\mathrm{wd}(P) \ge \mathrm{wd}_F(P) \ge \mathrm{wd}_F(P') > k$. □

Thus the posets with weak discrepancy k are those that contain a poset in $\mathcal{F}_{k-1}$ but no poset in $\mathcal{F}_k$.

4 Obstructions to $\mathrm{wd}_F(P) < k$

It is not surprising that $\{P\colon \mathrm{wd}_F(P) \le k\}$ has infinitely many obstructions, since such obstructions may have fractional weak discrepancy equal to any number exceeding k by the reciprocal of an integer. However, one can hope that every obstruction to $\mathrm{wd}_F(P) < k$ may have fractional discrepancy equal to k and that the number of obstructions is finite. We have remarked that Shuchat et al. [3] proved this for $k = 1$, with $\mathbf{2} + \mathbf{2}$ and $\mathbf{3} + \mathbf{1}$ being the obstructions.

Unfortunately, for $k > 1$ the list of obstructions is infinite. We do not provide a complete characterization, but we construct infinitely many obstructions. These examples do not include all obstructions.

If $k \ge 2$, then the fundamental cycle for a poset in $\mathcal{F}_k$ contains a chain of size at least 3. Skipping an interior element of that chain leaves a forcing cycle with ratio k. Hence no poset in $\mathcal{F}_k$ is an obstruction to $\mathrm{wd}_F < k$, and all obstructions have fractional weak discrepancy exactly k. Note that such posets do not lie in $\mathcal{F}_{k-1}$, since those posets have fractional weak discrepancy strictly less than k. We seek minimal posets Q such that $\mathrm{wd}_F(Q) = k$.

Let $M_k = \{\mathbf{r} + \mathbf{s}\colon\ r + s = 2k + 2 \text{ and } r, s \ge 1\}$. For each $P \in M_k$, we have $\mathrm{wd}_F(P) = k$ and $\mathrm{wd}_F(P - x) = k - 1/2$ for each $x \in P$. Hence M_k is a family of obstructions. In fact, there are only $\lceil (k + 1)/2 \rceil$ posets in M_k, so we have not yet disproved finiteness. Note also that M_1 is the complete list of obstructions for $k = 1$.

To construct obstructions, again we start by building a forcing cycle C, but this time the chains in C all have size $k+1$. That is, we start with the composition $(b_1, \ldots, b_q)$ with each $b_i = k+1$. Again let x_i^j denote the jth element on chain i. Minimality requires that the poset have no other elements. We will keep $x_i^{k+1} \parallel x_{i+1}^1$ (modulo q) and have an optimal labeling f with $f(x_i^j) = j$ for all (i, j). This time we add more relations to ensure that none of our examples contains another. We obtain a 3-parameter family, but there are many more examples where the chains in the original forcing cycle do not all have the same length and where the "added relations" are defined in different ways.

Definition 4.1. *Fix parameters k, q, t, all at least 2, and $t \leq k$. Let $P_{k,q,t}$ denote the poset with $(k+1)q$ elements x_i^j for $1 \leq j \leq k+1$ defined by letting $x_i^j \prec x_i^{j+1}$ for all (i, j) such that $1 \leq j \leq k$ and $x_i^1 \prec x_{i'}^j$ for all (i, j) such that $t < j \leq k+1$ and $i' \neq i-1$ (modulo q).*

Note that no further relations in $P_{k,q,t}$ other than those above are implied by transitivity. Furthermore, no two of these posets are isomorphic, except for degeneracy when $q = 2$, since no relations are added regardless of the choice of t. For $q = 2$, the construction reduces to the obstruction $\mathbf{k} + \mathbf{k}$. For $k = 1$, the constructions would generate families of disjoint 2-chains, which all contain $\mathbf{2} + \mathbf{2}$.

Proposition 4.2. *Each poset $P_{k,q,t}$ constructed in Definition 4.1 is a minimal poset with fractional weak discrepancy k.*

Proof. We have preserved $x_i^{k+1} \parallel x_{i+1}^1$, so the original cycle in order is a forcing cycle with ratio k. Letting $f(x_i^j) = j$ defines a consistent labeling with weakness k. Hence f has weakness k, and therefore $\mathrm{wd}_F(P_{k,q,t}) = k$.

To prove minimality, consider deleting x_i^j from $P_{k,q,t}$. By symmetry, we may assume that $i = q$. The remaining elements, in the same order as before, form a forcing cycle with ratio $(qk-1)/q$. We show that this is the fractional weak discrepancy by providing a consistent labeling with weakness $(qk-1)/q$. Throughout the cycle, augment f by 1 with each step up a chain, but decrease f by $(qk-1)/q$ when moving from the top of one chain to the bottom of the next. This is well-defined, since the net change while traversing the cycle is 0. Furthermore, the values of f at incomparable elements differ by no more than $(qk-1)/q$. □

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