Ore, Berge–Tutte, and Gallai–Edmonds

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Dedicated to Jack Edmonds on the occasion of his 75th birthday

Abstract

We present a short proof of the Berge–Tutte Formula and the Gallai–Edmonds Structure Theorem based on Ore's Defect Formula and Anderson's proof of Tutte's 1-Factor Theorem from Hall's Theorem.

The fundamental theorems on matchings in graphs have been proved in many ways, and generally they all imply each other. The most well-known is Hall's Theorem [7] characterizing when a bipartite graph has a matching that covers one partite set. Anderson [1] used Hall's Theorem to prove Tutte's 1-Factor Theorem [10], characterizing when a graph has a perfect matching. Meanwhile, Ore [8] extended Hall's Theorem to give a min-max formula for the maximum size of a matching in a bipartite graph. Similarly, Berge [2] extended Tutte's 1-Factor Theorem to give a min-max formula (known as the Berge–Tutte Formula) for the maximum size of a matching in a general graph.

We show that Anderson's approach proves the Berge–Tutte Formula from Ore's Defect Formula as easily as it proves Tutte's 1-Factor Theorem from Hall's Theorem. The same approach then yields the Gallai–Edmonds Structure Theorem [3, 5, 6], which describes all the maximum matchings in a given graph. This is shorter than earlier proofs by not needing a characterization of factor-critical graphs or a "Stability Lemma".

For a set of vertices S in a graph G, let $N_G(S)$ or $N(S)$ denote the set of vertices having at least one neighbor in S. An X,Y-*bigraph* is a bipartite graph with partite sets X and Y. A *matching* is a set of pairwise non-incident edges. An obvious necessary condition for the existence of a matching that covers X in an X,Y-bigraph is that $|N(S)| \geq |S|$ for all $S \subseteq X$. This is *Hall's Condition*, and Hall's Theorem states that it is also sufficient.

The *defect* $\mathrm{df}(S)$ of a set $S \subseteq X$ in an X,Y-bigraph is $|N(S)| - |S|$. The *matching number* $\alpha'(G)$ is the maximum size of a matching in G. By applying Hall's Theorem to the graph obtained from an X,Y-bigraph G by adding $\max\{\mathrm{df}(S)\}$ vertices to Y that are adjacent to all of X, Ore [8] observed that $\alpha'(G) = |X| - \max\{\mathrm{df}(S)\}$.

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A 1-*factor* is a spanning 1-regular subgraph; its edge set is a *perfect matching*. Let $o(H)$ denote the number of components of odd order (odd number of vertices) in a graph H. An obvious necessary condition for a 1-factor in a graph G is that $o(G-S) \le |S|$ whenever $S \subseteq V(G)$. This is *Tutte's Condition*; Tutte proved that it is also sufficient.

The *deficiency* $\mathrm{def}(S)$ of a set $S \subseteq V(G)$ in a graph G is $o(G-S) - |S|$. Covering an odd component of $G-S$ in a matching requires matching one of its vertices with a vertex of S, so in any matching at least $\mathrm{def}(S)$ such components contain uncovered vertices. A *Tutte set* is a vertex subset with positive deficiency. Let $\mathrm{def}(G) = \max_{S\subseteq V(G)} \mathrm{def}(S)$. By applying Tutte's Theorem to the graph obtained from G by adding $\mathrm{def}(G)$ vertices with no non-neighbors, Berge observed that $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(G))$, where $n = |V(G)|$.

Anderson [1] proved Tutte's Theorem by applying Hall's Theorem to a bipartite graph derived from a maximal set of maximum deficiency. We note that by applying Ore's Defect Formula to the same graph, one obtains the Berge–Tutte Formula instead. Another recent direct proof of the Berge–Tutte Formula is due to Schrijver[9].

With a few more observations, the same derived graph yields the Gallai–Edmonds Structure Theorem that describes all the maximum-sized matchings in a graph G. We state it after proving the relevant lemmas and obtaining the Berge–Tutte Formula.

Lemma 1 *(Parity Lemma) If G is an n-vertex graph and $S \subseteq V(G)$, then $o(G-S) - |S| \equiv n$ mod 2. In particular, if S is a Tutte set and n is even, then $o(G-S) \ge |S| + 2$.*

Proof. Counting vertices shows that $o(G-S) + |S| \equiv n \mod 2$. □

Lemma 2 *Let T be a maximal set among the vertex sets of maximum deficiency in a graph G. If x is a vertex of an odd component C of $G-T$, then $C-x$ satisfies Tutte's Condition. Also, $G-T$ has no even components.*

Proof. We use subscripts on def to denote the relevant graph. For $S \subseteq V(C-x)$, the odd components of $C-x-S$ are in $C-x$. Thus

$$\begin{aligned}\mathrm{def}_G(T \cup x \cup S) &= o(G-T-x-S) - (|T| + 1 + |S|)\\ &= o(G-T) - 1 + o(C-x-S) - |T| - 1 - |S|\\ &= \mathrm{def}_G(T) - 2 + \mathrm{def}_{C-x}(S)\end{aligned}$$

The choice of T yields $\mathrm{def}_G(T \cup x \cup S) < \mathrm{def}_G(T)$. By the Parity Lemma, they have the same parity. Hence $\mathrm{def}_{C-x}(S) \le 0$. Since S is arbitrary, $C-x$ satisfies Tutte's Condition.

If $G-T$ has an even component C, then adding to T any leaf of a spanning tree of T creates a larger set with the same deficiency as T. □

For $S \subseteq V(G)$, we define an auxiliary bipartite graph $H(S)$ by contracting each component of $G - S$ to a single vertex and deleting edges within S.

Lemma 3 *If T is a maximal set of maximum deficiency in a graph G, then $H(T)$ contains a matching that covers T.*

Proof. Let X be the set of components of $G - T$, so that $H(T)$ is an X, T-bigraph. For $S \subseteq X$, let $R = N_H(S)$. The elements of S are odd components of $G - R$, so $|S| \le o(G - R)$. Since $o(G - R) - |R| \le \mathrm{def}(G)$, we have $\left|N_{H(T)}(S)\right| = |R| \ge o(G - R) - d \ge |S| - d$.

Since $\left|N_{H(T)}(S)\right| \ge |S| - d$ for all $S \subseteq X$, Ore's Defect Formula yields $\alpha'(H(T)) \ge |X| - d = o(G - T) - d = |T|$. Thus $H(T)$ has a matching M that covers T. □

Edges of G corresponding to M match T into vertices of distinct odd components of $G - T$. Anderson observed that with Lemma 2 this yields Tutte's 1-Factor Theorem; with Ore's Defect Formula instead it yields the Berge-Tutte Formula. As noted, Tutte's 1-Factor Theorem is the special case $d = 0$.

Theorem 4 *(Berge–Tutte Formula; Berge [2]). If G is an n-vertex graph, then the maximum number of vertices in a matching is $n - \mathrm{def}(G)$; that is, $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(G))$.*

Proof. Let $d = \mathrm{def}(G)$, and let T be a maximal set with deficiency d. Since covering the vertices of odd components of $G - T$ requires using distinct vertices of T, at least d vertices remain uncovered in any matching.

Using induction on n, we build a matching in G that covers all but d vertices. The claim is trivial for $n = 0$; consider $n > 0$. By Lemma 2, $G - T$ has no even components, and $C - x$ satisfies Tutte's Condition whenever x is a vertex in an odd component C of $G - T$. Since $C - x$ has fewer vertices than G, the induction hypothesis yields a perfect matching in $C - x$.

Since this holds for every vertex x in every component C of $G - T$, and there are $|T| + d$ such components, it suffices to cover T using edges to distinct components of $G - T$. This follows immediately from Lemma 3. □

Gallai [4] introduced the notion of a *factor-critical* graph, one in which every subgraph obtained by deleting one vertex has a 1-factor. A matching in G is *near-perfect* if it covers all but one vertex of G. By Tutte's Theorem, a connected graph is factor-critical if and only if the graph obtained by deleting any one vertex satisfies Tutte's Condition.

Given a graph G, let B be the set of vertices in G that are covered by every maximum matching in G. The *Gallai–Edmonds Decomposition* of G is the partition of of $V(G)$ into three sets A, C, D such that A is the subset of B consisting of vertices with at least one neighbor outside B, and $C = B - A$, and $D = V(G) - B$ (see Figure 1).

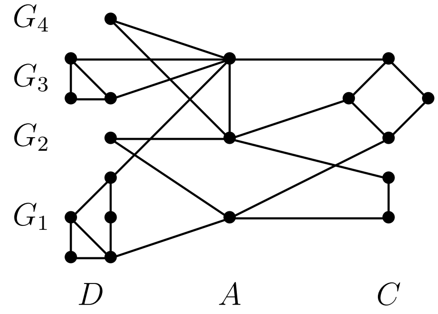

Figure 1: Example of Gallai–Edmonds decomposition

Theorem 5 *(Gallai–Edmonds Structure Theorem) Let A, C, D be the sets in the Gallai–Edmonds Decomposition of a graph G. Let $G_1, \ldots, G_k$ be the components of $G[D]$. If M is a maximum matching in G, then the following properties hold.*
a) M covers C and matches A into distinct components of $G[D]$.
b) Each G_i is factor-critical and has a near-perfect matching in M.
c) If $\varnothing \neq S \subseteq A$, then $N(S)$ intersects at least $|S| + 1$ of $G_1, \ldots, G_k$.
d) $\operatorname{def}(A) = \operatorname{def}(G) = k - |A|$.

Proof. Let $d = \operatorname{def}(G)$, and let T be a maximal set of deficiency d, so M must leave d vertices uncovered. To achieve this bound, M must match T to vertices in distinct components of $G - T$. By Lemma 3, such matchings exist. By Lemma 2, each component of $G - T$ is factor-critical. Since each component of $G - T$ is factor-critical, each matching of T into distinct components of $G - T$ can be augmented with a near-perfect matching in each component of $G - T$ to form a matching leaving exactly d vertices uncovered. Hence M must have that form.

We must use T to find the sets of the Gallai–Edmonds Decomposition. We have observed that $H(T)$ has a matching that covers T. Hence Hall's Condition holds in H for subsets of T. That is, $\operatorname{df}_{H(T)}(S) \leq 0$ for all $S \subseteq T$. Since $\operatorname{df}(\varnothing) = 0$, we may let R be a maximal subset of T with defect 0.

We claim that $C = R \cup R'$, where R' consists of all vertices of components of $G - T$ in $N_H(R)$. Since $|N_H(R)| = |R|$, in M there is a matching of R into R', using all components of $N_H(R)$. We have also observed that M covers the rest of R'. Since no vertex of R or R' has a neighbor in the other odd components of $G - T$, we conclude that $R \cup R' = C$.

Next we claim that $A = T - R$. We must show that any vertex in $V(G) - T - R'$ may be uncovered and hence does not lie in A, and every vertex of $T - R$ has a neighbor in that set. Let H' be the subgraph of $H(T)$ formed by deleting $R \cup N_H(R)$. For $S \subseteq T - R$ with S nonempty, we have $|N_H(S)| > |S|$, since otherwise S could be added to R to obtain a larger set with defect 0. Therefore, deleting any element of $N_{H'}(T-R)$ from H' leaves a subgraph of H' satisfying Hall's Condition. We conclude that each vertex in $V(G)-T-R'$ is avoided by some maximum matching. Hence $D=V(G)-T-R'$ and $A=T-R$. We have proved (a), (b), and (c).

For statement (d), since $D=V(G)-T-R'$ and $o(G[D])=k$, we have $\mathrm{def}(T)=o(G-T)-|T|=k+|R|-|A\cup R|=k-|A|$. Since $G[C]$ has a perfect matching, its components all have even order, yielding $o(G-A)=k$. Hence A is another set with maximum deficiency. □

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