

Spanning Cycles through Specified Edges in Bipartite Graphs

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Abstract

Pósa proved that if G is an n -vertex graph in which any two nonadjacent vertices have degree sum at least $n + k$, then G has a spanning cycle containing any specified family of disjoint paths with a total of k edges. We consider the analogous problem for a bipartite graph G with n vertices and parts of equal size. Let F be a subgraph of G whose components are nontrivial paths. Let k be the number of edges in F , and let t_1 and t_2 be the numbers of components of F having odd and even length, respectively. For $n \geq 9k + 4$, there is a spanning cycle in G containing F if any two nonadjacent vertices in opposite partite sets have degree-sum at least $n/2 + \tau(F)$, where $\tau(F) = \lceil k/2 \rceil + \epsilon$ (here $\epsilon = 1$ if $t_1 = 0$ or if $(t_1, t_2) \in \{(1, 0), (2, 0)\}$, and $\epsilon = 0$ otherwise). The threshold on the degree-sum is sharp.

1 Introduction

In a graph, a cycle through all the vertices is a *spanning cycle* or *Hamiltonian cycle*, and a graph with such a cycle is a *Hamiltonian graph*. The study of sufficient conditions for Hamiltonian graphs is a classical topic in graph theory. Dirac's Theorem [1] states that every n -vertex graph with minimum degree at least $n/2$ is Hamiltonian. Ore [2] observed a stronger statement: it suffices for each two nonadjacent vertices to have degree-sum at least n . Further refinements have studied sufficient conditions for spanning cycles containing specified edges.

We consider analogues of these results for bipartite graphs. Our graphs have no loops or multiple edges. An X, Y -*bigraph* is a bipartite graph with partite sets X and Y . It is *balanced* if $|X| = |Y|$. For an X, Y -bigraph G , let $\sigma(G)$ denote the smallest sum of the degrees of two vertices $x \in X$ and $y \in Y$ such that $xy \notin E(G)$. Gould [5] has used $\sigma_{1,1}(G)$ for this quantity to distinguish it from the related quantity in Ore's Theorem, which is the smallest degree-sum over all nonadjacent pairs and is usually written as $\sigma_2(G)$. Since we study only

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bipartite graphs in this paper, we use the simplified notation $\sigma(G)$ for the bipartite analogue. Always n will denote the number of vertices in G .

The bipartite analogue of Ore's Theorem was proved by Moon and Moser [4]: if G is an X, Y -bigraph with $\sigma(G) \geq n/2 + 1$, then G is Hamiltonian. The disjoint union of the complete bipartite graphs $K_{a,a}$ and $K_{n/2-a, n/2-a}$ shows that the result is sharp (see Figure 1(a)).

Researchers also studied degree thresholds for the existence of spanning cycles through a specified set F of edges, calling a graph F -Hamiltonian when such a cycle exists. Of course, F must be a *linear forest*, meaning that every component of F is a path and we require all the paths to be nontrivial (positive length). For general graphs, Häggkvist [6] proved that if $\sigma_2(G) \geq n + 1$, then G is F -Hamiltonian whenever F is a perfect matching. Las Vergnas [7] proved the analogue when G is bipartite, showing that $\sigma(G) \geq n/2 + 2$ suffices when F is a perfect matching. Again the threshold is sharp.

More generally, we seek a spanning cycle through a linear forest with k edges. For general graphs, Pósa [8] proved $\sigma_2(G) \geq n + k$ suffices. Our objective is to obtain the smallest threshold on $\sigma(G)$ to guarantee that if G is an n -vertex balanced bipartite graph, and F is a linear forest in G consisting of k edges, then G is F -Hamiltonian. In most cases, when n is sufficiently large, $\sigma(G) \geq n/2 + \lceil k/2 \rceil$ is sufficient. However, the threshold is larger by 1 for some arrangements of k edges.

In particular, suppose that G is an n -vertex balanced bipartite graph, and F is a linear forest in G whose k edges form t_1 components of odd length and t_2 components of even (positive) length. Let

$$\epsilon(t_1, t_2) = \begin{cases} 1 & t_1 = 0 \\ 1 & (t_1, t_2) \in \{(1, 0), (2, 0)\} \\ 0 & \text{otherwise} \end{cases} ,$$

and let $\tau(F) = \lceil k/2 \rceil + \epsilon(t_1, t_2)$. Our main result is that if $n \geq 9k + 4$, and $\sigma(G) \geq n/2 + \tau(F)$, then G is F -Hamiltonian. Furthermore, this threshold on $\sigma(G)$ is sharp for all k, t_1, t_2 . Our sufficiency proof needs $n \geq 9k + 4$, and the sharpness examples usually need $n > 3k$; when $k = n/2$ Las Vergnas' result yields $n/2 + 2$ as the threshold.

We will abuse notation slightly by often viewing F as a specified set of edges rather than a subgraph, but this will be clear from context. For example, when P is a path (or cycle) in G , we say that P *passes through* F or P *is a path through* F if $F \subseteq E(P)$.

Pósa's result for linear forests in general graphs does not depend on the number of components in F . The result for $t < k$ follows easily from the case $t = k$, where F is a matching. In the bipartite analogue the linear forest case reduces analogously to the case of paths with lengths 1 and 2. Paths of odd and even lengths behave differently in the bipartite setting because traversing them does or does not switch partite sets.

In the next section we present sharpness constructions for all cases and reduce the sufficiency argument to the case where all components in the linear forest have length at most 2. That case then occupies the bulk of the paper.

2 Sharpness Constructions

In this section we introduce needed terminology and provide constructions showing that the results are sharp. We use $V(G)$ and $E(G)$ for the vertex set and edges set of a graph G . Let $G + H$ denote the disjoint union of graphs G and H , and let $G[A]$ denote the subgraph of G induced by vertex set A .

We begin by presenting sharpness constructions for the case where all paths in the linear forest have length at most 2. Since this will be the main case in the sufficiency proof, we introduce special terminology for it.

Definition 2.1. A (t_1, t_2) -linear forest is the disjoint union of t_1 isolated edges and t_2 paths of length 2.

Always $k = t_1 + 2t_2$ when F is a (t_1, t_2) -linear forest, and this includes all cases with $k \leq 2$. The situation when $\epsilon(t_1, t_2) = 1$ is also special, so our first constructions cover only that. The constructions in Figure 1(a) and Figure 1(b) prove that the threshold is sharp for $k \leq 1$; for $k = 0$ this is the sharpness example for the Moon–Moser result [4]. Since the graph in Figure 1(b) has a perfect matching containing xy , that construction also proves sharpness of Las Vergnas’s result. Note that the (t_1, t_2) -linear forests for which $\epsilon(t_1, t_2) = 1$ are those with $(t_1, t_2) \in \{(0, t), (1, 0), (2, 0)\}$, where t is any nonnegative integer.

Lemma 2.2. Let n be an even number greater than $2(t_1 + 2t_2 + 1)$. If $\epsilon(t_1, t_2) = 1$, then there is an n -vertex balanced X, Y -bigraph G and a (t_1, t_2) -linear forest F in G such that $\sigma(G) = n/2 + \tau(F) - 1$ and G has no spanning cycle through F .

Proof. Since $\epsilon(t_1, t_2) = 1$ and $k = t_1 + 2t_2$, we have $\tau(F) - 1 = \lceil t_1/2 \rceil + t_2$.

For $t_1 = t_2 = 0$, the graph G in Figure 1(a) is $K_{a,a} + K_{n/2-a, n/2-a}$. It is disconnected and hence has no spanning cycle, but $\sigma(G) = n/2$.

For $t_2 = 0$ and $t_1 \in \{1, 2\}$, where $\tau(F) - 1 = 1$, we construct the graph G in Figure 1(b) from $K_{a-1, a-1} + K_{n/2-a, n/2-a}$ by adding x to X and y to Y such that xy is the required edge for the cycle, x is adjacent to all Y , and y is adjacent to all X . Note that $\sigma(G) = n/2 + 1$, but there is no spanning cycle through xy . If $t_1 = 2$, then there is another required edge not incident to x or y , but there is still no cycle through xy .

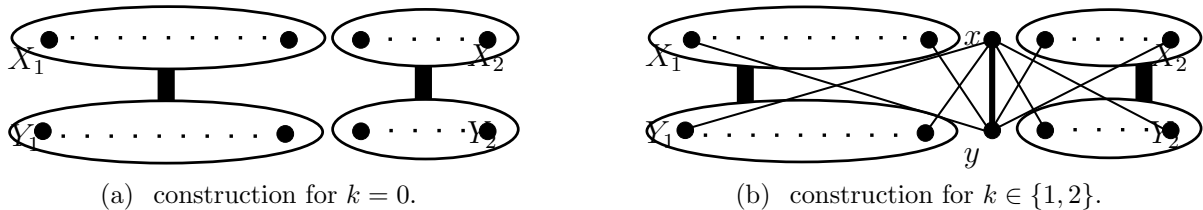


Figure 1: Sharpness constructions for (t_1, t_2) -linear forests with $t_2 = 0$ and $t_1 \leq 2$.

The remaining case is $t_1 = 0$ and $t_2 > 0$. Figure 2 shows the construction for this case. The partite sets are $X_1 \cup X_0 \cup X_2$ and $Y_1 \cup Y_0 \cup Y_2$, with $|X_1| = |Y_1| = |X_2| = |Y_2| = m$ and $|X_0| = |Y_0| = t_2$. Let G contain all the edges joining the partite sets except those from X_1 to

Y_2 and from X_2 to Y_1 . Let F be a perfect matching in $G[X_0 \cup Y_0]$ plus a matching of size t_2 in $G[X_2 \cup Y_0]$; note that F is the edge set of a $(0, t_2)$ -linear forest. If G has a spanning cycle C through F , then removing all the vertices incident to F cuts C into t_2 paths. Covering the remainder of G needs at least $t_2 + 1$ paths, so no such cycle exists. \square

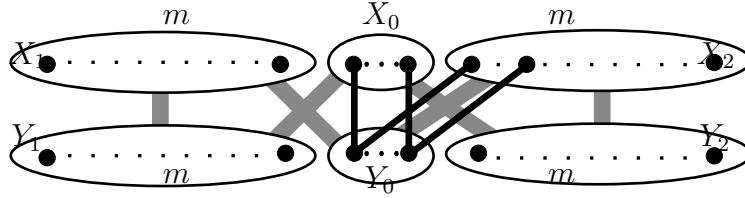


Figure 2: Sharpness when $t_1 = 0$.

Lemma 2.3 presents a sharpness construction for all (t_1, t_2) -linear forests such that $\epsilon(t_1, t_2) = 0$. These are all (t_1, t_2) such that $t_1 \geq 3$ or $t_1 t_2 > 0$. The construction differs from those of Lemma 2.2 in that here $|X_1| \neq |Y_1|$.

Lemma 2.3. Given t_1 and t_2 such that $\epsilon(t_1, t_2) = 0$, let $k = t_1 + 2t_2$. If n is an integer such that $n \geq 2\lceil \frac{k+1}{2} \rceil + 2k$ and $n \equiv 2\lceil \frac{k}{2} \rceil - 2 \pmod{4}$, then there is an n -vertex balanced bipartite graph G and a (t_1, t_2) -linear forest F in G such that $\sigma(G) = n/2 + \tau(F) - 1$ and G has no spanning cycle through F .

Proof. Since $\epsilon(t_1, t_2) = 0$, we have $\tau(F) = \lceil k/2 \rceil$. Let m be an integer with $m \geq \lfloor t_1/2 \rfloor + t_2 + 1$. Build a graph G as in Figure 3 such that $|X_0| = |Y_0| = t_1 + t_2$, $|X_2| = |Y_1| = m - \lfloor t_1/2 \rfloor - 1$, and $|X_1| = |Y_2| = m$. Let $E(G)$ consist of all edges joining the partite sets except those from X_1 to Y_2 and from X_2 to Y_1 . Let F consist of a perfect matching in $G[X_0 \cup Y_0]$ plus a matching of size t_2 in $G[X_2 \cup Y_0]$; we have specified a (t_1, t_2) -linear forest. For $x \in X_1$ and $y \in Y_2$, we have $d_G(x) + d_G(y) = 2(m - \lfloor t_1/2 \rfloor - 1 + t_1 + t_2) = n/2 + \lceil k/2 \rceil - 1$, and such a pair has the smallest degree-sum. For $m \geq \lfloor t_1/2 \rfloor + t_2 + 1$ the construction exists, yielding all values of n specified in the hypothesis.

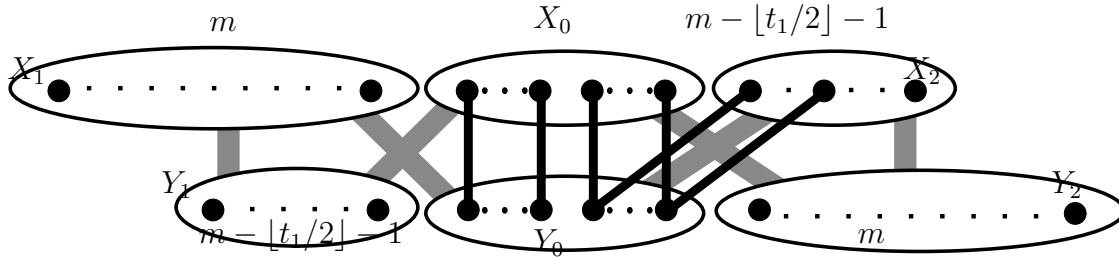


Figure 3: Sharpness construction.

Suppose that G has a spanning cycle C through F . Let U be the set of vertices in F . Since F consists of $t_1 + t_2$ paths, deleting U cuts C into at most $t_1 + t_2$ paths. Since $|X_1| - |Y_1| = \lfloor t_1/2 \rfloor + 1$, covering $G[X_1 \cup Y_1]$ needs at least $\lfloor t_1/2 \rfloor + 1$ paths; similarly, covering

$G[(X_2 - U) \cup Y_2]$ needs at least $\lfloor t_1/2 \rfloor + t_2 + 1$ paths. Thus covering $G - U$ needs more than $t_1 + t_2$ paths, so no such cycle exists. \square

Lemma 2.2 and Lemma 2.3 together provide sharpness constructions whenever $k = t_1 + 2t_2$. If F has more than $t_1 + 2t_2$ edges, then F has a longer path. From the sharpness constructions for (t_1, t_2) -linear forests, we obtain constructions for linear forests with longer paths.

Lemma 2.4. Let F be a linear forest with k edges in an n -vertex bipartite graph G with $\sigma(G) = n/2 + \tau(F) - 1$. If G has no spanning cycle through F , then there is a bipartite graph G' with $n + 2$ vertices and a linear forest F' with $k + 2$ edges in G' such that G' has no spanning cycle through F' and the following properties hold: F' has the same numbers of components of odd and even lengths as F , and $\sigma(G') = |V(G')|/2 + \tau(F') - 1$.

Proof. Let t_1 and t_2 be the numbers of components with odd and even (positive) lengths in F . Let xy be an edge in F with $x \in X$. Form G' from G by adding two new vertices x' and y' and making y' adjacent to all X and x' adjacent to all Y . Note that $\sigma(G') = \sigma(G) + 2$.

Form F' by adding to $F - \{xy\}$ the edges $\{xy', y'x', x'y\}$. Since this does not change the parity of the length of any path, $\tau(F') = \tau(F) + 1$. Hence $\sigma(G') = |V(G')|/2 + \tau(F') - 1$.

Also, any spanning cycle through F' in G' can be converted to a spanning cycle through F in G by replacing the path through x, y', x', y with the edge xy . Therefore, there is no spanning cycle through F' in G' .

Thus F' in G' provides the desired larger sharpness example. \square

By repeating this construction, we can build sharpness examples for any desired list of path-lengths in the linear forest. The following lemma shows that also to prove sufficiency of $\sigma(G) \geq n/2 + \tau(F)$, it suffices to study the case where paths in the linear forest have length at most 2. In essence, the proof reverses the enlargement construction of Lemma 2.4.

Lemma 2.5. Let G be an n -vertex balanced X, Y -bigraph, and let F be the edge set of a selected linear forest in G . If $\sigma(G) \geq n/2 + \tau(F)$ suffices for a spanning cycle through F when each path in F has length at most 2, then it also suffices without the length restriction.

Proof. Let F have k edges forming t_1 paths of odd length and t_2 paths of even (positive) length. We use induction on $k - (t_1 + 2t_2)$; when this value is 0 there is nothing to prove. Otherwise, some path in F has length at least 3. Let x, y', x', y be consecutive vertices along such a path. Form G' from $G - \{x', y'\}$ by adding the edge xy . Since each vertex that remains loses at most one neighbor in $\{x', y'\}$, we have $\sigma(G') \geq \sigma(G) - 2 = |V(G')|/2 + \tau(F) - 1$.

The specified linear forest F' in G' is the same as F except for replacing the portion of the path through x, y', x', y with the edge xy . Since $\tau(F') = \tau(F) - 1$, the induction hypothesis yields a spanning cycle C' through F' in G' . Obtain the desired cycle C in G by replacing xy in C' with the path through x, y', x', y . \square

3 Odd, Even, and Full Edges

In this section we introduce concepts needed for the sufficiency proof and establish some structural conditions that allow a spanning path through the specified forest F to be transformed into a spanning cycle through F . The more difficult arguments in the subsequent sections can be viewed as saying that in all cases one of these conditions will apply.

By convention, in the X, Y -bigraph G all uses of x, x', x_i indicate vertices in X and those of y, y', y_i indicate vertices in Y , without further explicit specification.

When vertices u and v are adjacent in G , we write $u \leftrightarrow v$; otherwise we write $u \nleftrightarrow v$. Given a vertex v in a graph G , we write $d_G(v)$ for the degree of v in G and $N_G(v)$ for the set of vertices adjacent to v in G (the *neighborhood* of v). For a vertex set A and a vertex v not in A , we define $d_A(v) = |N_G(v) \cap A|$. We write $G - uv$ to denote the graph obtained from a graph G by deleting the edge uv ; similarly, if $uv \notin E(G)$, then we write $G + uv$ for the graph obtained by adding uv to the edge set.

A path with endpoints x and y is an x, y -path. Given vertices u and v on a path P , we write $P(u, v)$ for the ordered list of vertices along P from u to v . Given a cycle C and an edge uv on C , we write $C(u, v)$ for the list of vertices along the path $C - uv$. When L is a list of vertices along a path, $\langle L \rangle$ denotes the path through the vertices of L in the specified order; this designates only the path, not the subgraph induced by L , which we would write as $G[L]$. Analogously, when L is the list of vertices along a cycle, $[L]$ denotes the cycle through the vertices of L in the specified order; the square brackets suggest “closing” the path.

Definition 3.1. Let G be an X, Y -bigraph containing a path P of odd length. Index the edges of P in order along P . An edge of P is an *odd edge* or *even edge* (with respect to P) when it has *odd position* or *even position* along P in this indexing. We write $E_{\text{odd}}(P)$ for the set of all odd edges on P and $E_{\text{even}}(P)$ for the set of all even edges on P .

An edge uv on P is *full* with respect to P when its endpoint in X is adjacent to y and its endpoint in Y is adjacent to x . It is *half-full* if exactly one of these adjacencies is present.

To illustrate the use of full edges and introduce the ideas and notation in our proof, we prove the Moon-Moser result here. This is necessary to make our result self-contained, since we later use it as the base step of an induction on the number of edges in F .

Proposition 3.2. [4] If G is an n -vertex balanced X, Y -bigraph and $\sigma(G) \geq n/2 + 1$, then G has a spanning cycle.

Proof. Adding edges does not destroy the condition $\sigma(G) \geq n/2 + 1$, so a maximal counterexample has a spanning path P joining nonadjacent vertices x and y . Since $d_G(x) + d_G(y) \geq \sigma(G) \geq n/2 + 1$ and P has $n/2$ odd edges, some odd edge uv (with $u \in X$ and $v \in Y$) is full. Now $[P(x, u), P(y, v)]$ is a spanning cycle (see Figure 4(a)). \square

By Lemma 2.5, it suffices to prove the sufficiency of the threshold on $\sigma(G)$ when all paths in F have length at most 2. In the rest of the proof, F always satisfies this condition.

Definition 3.3. We refer to the edges of a specified (t_1, t_2) -linear forest F as the *selected* edges. We let F_1 denote the set of all isolated edges in F and let F_2 denote the set of edges in paths of length 2.

The spanning paths we study in a graph G with a selected forest F will be spanning paths through F . We will frequently use the next statement, an easy fact about the edges joining an edge on a spanning path P to the endpoints of P .

Remark 3.4. Let G be an n -vertex balanced X, Y -bigraph, and let P be a spanning x, y -path in G . Since each endpoint of an edge along P has at most one neighbor in $\{x, y\}$, the pigeonhole principle implies that if $x \leftrightarrow y$ and $d_G(x) + d_G(y) \geq n/2 + p$, then there are at least p full odd edges and at least $p+1$ full even edges along P . Moreover, if $d_G(x) + d_G(y) = n/2 + p$ and there are exactly p full odd edges on P , then all other odd edges on P are half-full.

Definition 3.5. Let P be an x, y -path of odd length, and let uv be a full odd edge with $u \in X$. We write $\pi(P, uv)$ for the cycle $[P(u, x), P(v, y)]$ (see Figure 4(a)). If a full odd edge uv is not an end-edge of P , then using the four consecutive vertices v', u, v, u' along P , we write $\pi'(P, uv)$ for the path $\langle P(u', y), u, v, P(x, v') \rangle$ (see Figure 4(b)). Note that every edge belonging to both P and $\pi'(P, uv)$ has the same parity on P and $\pi'(P, uv)$.



Figure 4: $\pi(P, uv)$ and $\pi'(P, uv)$.

Lemma 3.6. Let G be an n -vertex balanced X, Y -bigraph, and let F be a selected (t_1, t_2) -linear forest in G . Let P be a spanning path through F in G . If $\sigma(G) \geq n/2 + p$ and there are fewer than p selected full odd edges along P , then G has a spanning cycle through F .

Proof. Let x and y be the endpoints of P . By the pigeonhole principle there are at least p full odd edges along P . Since there are fewer than p selected odd edges on P , there is an unselected full odd edge uv on P . Now $\pi(P, uv)$ is a spanning cycle through F in G . \square

Lemma 3.7. Let G be an n -vertex balanced X, Y -bigraph, and let F be a selected (t_1, t_2) -linear forest in G . Let P be a spanning path through F in G . If $\sigma(G) \geq n/2 + \lceil t_1/2 \rceil + t_2$ and F_1 contains a full even edge of P , then G has a spanning cycle through F .

Proof. Let x and y be the endpoints of P ; we may assume $x \leftrightarrow y$, and thus $d_G(x) + d_G(y) \geq n/2 + \lceil t_1/2 \rceil + t_2$. By Remark 3.4, there is a full even edge. Since the end-edges of P are odd edges, we find four consecutive vertices u, y', x', v in order on P such that $y'x'$ is a full even edge. Let $Q = \langle P(u, x), y', x', P(y, v) \rangle$. Since $y'x' \in F_1$, we have $uy', x'v \notin F$, and hence Q passes through F . Hence we may assume $u \leftrightarrow v$, and therefore $d_G(u) + d_G(v) \geq n/2 + \lceil t_1/2 \rceil + t_2$. Since every edge except $y'x'$ has different parity on P and Q , one of P and Q has fewer than $\lceil t_1/2 \rceil + t_2$ selected odd edges, and therefore Lemma 3.6 guarantees a spanning cycle through F . \square

4 Paths Splitting F by Parity

Given two spanning paths P and Q through a (t_1, t_2) -linear forest F in G such that every selected edge has opposite parities in P and Q , one of $\{P, Q\}$ has at most $\lfloor t_1/2 \rfloor + t_2$ selected odd edges. Given $\sigma(G) \geq n/2 + \lceil t_1/2 \rceil + t_2$, Lemma 3.6 applies to guarantee a spanning cycle through F when t_1 is odd. When t_1 is even, this observation is not sufficient. For this reason, the case when t_1 is even is harder, and we need an additional structural lemma.

Definition 4.1. Let G be a graph, F be a subset of $E(G)$, and P be a spanning path in G . Path P *splits* F if $F \subseteq E(P)$ and $|F \cap E_{\text{odd}}(P)| = |F \cap E_{\text{even}}(P)|$.

Lemma 4.2. Let G be an n -vertex balanced X, Y -bigraph, and let F be the edge set of a selected $(2r, t_2)$ -linear forest in G . If G has a spanning path that splits F , and $\sigma(G) \geq n/2 + \tau(F)$, then G has a spanning cycle through F .

Proof. We first show that if G has no spanning cycle through F , then the end edges of any spanning path P that splits F are not in F . Let x and y be the endpoints of P . We may assume that $x \leftrightarrow y$, and so $d_G(x) + d_G(y) \geq \sigma(G) \geq n/2 + \tau(F)$. Hence there are at least $r + t_2 + \epsilon(2r, t_2)$ full odd edges along P . If there is an unselected full odd edge uv on P , then $\pi'(P, uv)$ is the desired cycle, so we may assume that every full odd edge on P is in F . Since P splits F and $|F| = 2r + 2t_2$, we have exactly $r + t_2$ odd edges of P in F . Hence we have exactly $r + t_2$ full odd edges along P , and $d_G(x) + d_G(y) = n/2 + r + t_2$. By Remark 3.4, all the unselected odd edges on P are half-full. Since all selected odd edges on P are full and $x \leftrightarrow y$, the first and the last edges of P are unselected.

Given a path P that splits F , suppose there are consecutive vertices u_1, v_1, u_2, v_2 on P such that u_1v_1 and u_2v_2 are selected odd edges and at least one of them is in F_1 . By symmetry, we may assume $u_1v_1 \in F_1$; now $\pi'(P, u_1v_1)$ also splits F because it contains all of F and does not change the parity of any edge of F . The end edge u_2v_2 of $\pi'(P, u_1v_1)$ is selected, which contradicts the conclusion of the preceding paragraph. Hence we may assume that P any two successive selected odd edges are both in F_2 .

If $r = 0$, or if $(r, t_2) = (1, 0)$, then $\epsilon(2r, t_2) = 1$. Because P splits F , there are $r + t_2$ selected odd edges on P . Since $\sigma(G) \geq n/2 + r + t_2 + 1$, Lemma 3.6 implies that G has a spanning cycle through F . In all remaining cases, $\epsilon(2r, t_2) = 0$.

Let x_1y_1 and x_2y_2 be the odd edges of P in F_1 that are nearest to x and y , respectively. Since P splits F , we have r edges of F_1 in odd position and r edges of F_1 in even position on P . Since $r \geq 2$, edges x_1y_1 and x_2y_2 are different.

Case 1: $r \geq 3$. Since $r \geq 3$, there is an odd edge x_3y_3 in $F_1 - \{x_1y_1, x_2y_2\}$; it is between x_1y_1 and x_2y_2 on P . Let $\langle u_1, v_1, x_3, y_3, u_2, v_2 \rangle$ be the 6-vertex portion of P with $\{x_3, y_3\}$ in the center (see Figure 5). Since $x_3y_3 \in F_1$ and any two successive selected odd edges are both in F_2 , none of $u_1v_1, v_1x_3, y_3u_2, u_2v_2$ is selected.

Let x' be next to y_2 between y_2 and y . If $u_2 \leftrightarrow y$, then let $Q = \langle P(x', y), P(u_2, y_2), P(x, y_3) \rangle$. All edges in $E(Q) \cap E(P)$ have the same parity in both, so Q splits F but has a selected end-edge. We have forbidden this, so we may assume $u_2 \leftrightarrow y$. Symmetrically, we may assume $v_1 \leftrightarrow x$. Since all unselected odd edges on P are half-full, $v_2 \leftrightarrow x$ and $u_1 \leftrightarrow y$.

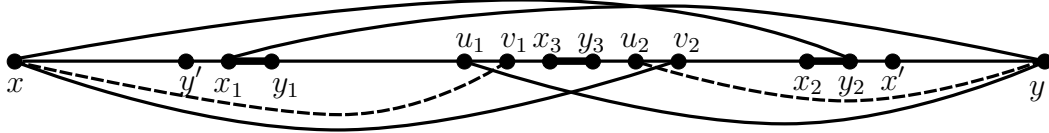


Figure 5: Three selected odd edges.

Now consider the two paths $\langle P(x, u_1), P(y, v_1) \rangle$ and $\langle P(u_2, x), P(v_2, y) \rangle$ through F . Every edge of F_1 except x_3y_3 has different parity on these two paths. Since x_3y_3 is even on both paths and $|F| = 2r + 2t_2$, one of the two paths has fewer than $r + t_2$ selected odd edges. By Lemma 3.6, G has a spanning cycle through F .

Case 2: $r = 2$. Since $r = 2$, there is no odd edge in $F_1 - \{x_1y_1, x_2y_2\}$; the two members of $F_1 - \{x_1y_1, x_2y_2\}$ are even edges of P . Let $\langle y', x_1, y_1, u_1, v_1 \rangle$ be the 5-vertex portion of P centered at y_1 , and let $\langle u_2, v_2, x_2, y_2, x' \rangle$ be the 5-vertex portion of P centered at x_2 (see Figure 6). Again since $x_1y_1, x_2y_2 \in F_1$, edges $y'x_1, y_1u_1, u_1v_1, u_2v_2, v_2x_2, y_2x'$ are unselected. As before, if $u_1 \leftrightarrow y$, then $\langle P(x', y), P(u_1, y_2), P(x, y_1) \rangle$ has a selected end edge, so $u_1 \leftrightarrow y$ and $x \leftrightarrow v_1$. Similarly, $x \leftrightarrow v_2$ and $y \leftrightarrow u_2$, which also implies that $u_1v_1 \neq u_2v_2$.



Figure 6: Exactly two selected odd edges.

If either even member of F_1 lies in $\langle P(v_1, u_2) \rangle$, then $\langle P(u_1, x), P(v_1, u_2), P(y, v_2) \rangle$ has at most $1 + t_2$ selected edges in odd position, and Lemma 3.6 completes the proof. Therefore, we may assume that each even member of F_1 lies in $\langle P(x, u_1) \rangle$ or $\langle P(v_2, y) \rangle$. By symmetry, we may assume that $\langle P(x, u_1) \rangle$ contains such an edge e . Now in $\langle P(u_1, x_1), P(y, v_1), P(x, y') \rangle$ the edges y_1x_1, y_2x_2 , and e all have even position. Hence this path through F has at most $1 + t_2$ selected odd edges, and again G has a spanning cycle through F .

Case 3: $r = 1$ and $t_2 > 0$. Since P splits F , there are exactly $t_2 + 1$ selected odd edges. By Lemma 3.6 and Remark 3.4, we may assume that $\sigma(G) = n/2 + t_2 + 1$, every odd selected edge is full, and every odd unselected edge is half-full. Let x_1y_1 be the odd edge of F_1 . By symmetry, we may assume that the even edge of F_1 is in $P(x, x_1)$. Let $\langle x_0, y_0, x_1, y_1 \rangle$ be the 4-vertex portion of P centered at $\{y_0, x_1\}$. Since $x_1y_1 \in F_1$, the edge y_0x_1 is unselected. Since any two successive selected odd edges lie in F_2 , also x_0y_0 is unselected.

If $x_0 \leftrightarrow y$, then the path $\langle P(x, x_0), P(y, y_0) \rangle$ has exactly t_2 selected odd edges, so Lemma 3.6 guarantees a spanning cycle through F . Hence we may assume $x_0 \leftrightarrow y$. Since unselected odd edges are half-full, we obtain $x \leftrightarrow y_0$. Since $t_2 > 0$, the set F_2 is nonempty. Hence P has a subpath $\langle d, a, b, c \rangle$ such that $ab, bc \in F_2$; we consider two subcases.

Case 3a: $d, a, b, c \in P(x, x_0)$ (see Figure 7). If $a \notin X$, then we can use $\pi'(P, x_1y_1)$ instead of P and interchange X and Y . Hence may assume that $a \in X$. Now ab is a selected odd edge, hence full, hence $a \leftrightarrow y$. Let x' be the neighbor of y on P . Now $\langle P(x', x_1), y, P(a, y_0), P(x, d) \rangle$ has exactly t_2 selected odd edges and Lemma 3.6 completes the proof.

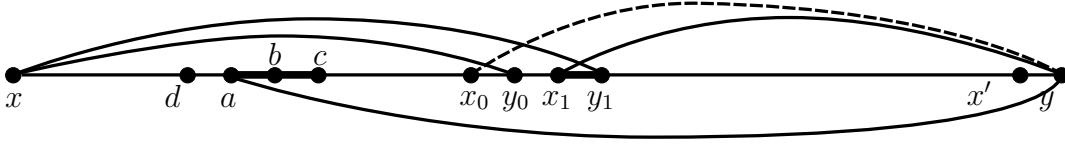


Figure 7: The edge of F_1 and $\langle d, a, b, c \rangle$ on the same side of $x_1 y_1$.

Case 3b: $d, a, b, c \in P(y_0, y)$ (see Figure 8). If $a \in X$, then Case 3a applies to $\pi'(P, x_1 y_1)$ to finish the proof. Hence we may assume that $a \in Y$. If $d \leftrightarrow y$, then $\langle P(a, y), P(d, x) \rangle$ is a path that splits F and has a selected end edge; so we may assume $d \leftrightarrow x$. Since unselected odd edges are half-full, $x \leftrightarrow a$. Now $\langle P(d, x_1), P(y, a), P(x, y_0) \rangle$ has exactly t_2 selected odd edges and Lemma 3.6 completes the proof. \square

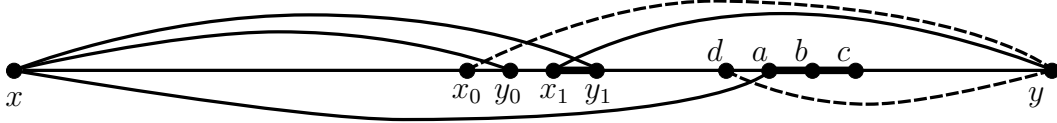


Figure 8: The edge of F_1 and $\langle d, a, b, c \rangle$ on opposite sides of $x_1 y_1$.

5 Paths with Unselected End Edges

The main result of this section completes the proof whenever there is a path through the selected edges whose first and last edges are unselected. The first two lemmas are tools.

Lemma 5.1. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F and a spanning path P through F . If $\sigma(G) \geq n/2 + \tau(F)$ and there are at most $\lfloor t_1/2 \rfloor + t_2$ selected odd edges along P , then G has a spanning cycle through F .

Proof. Lemma 3.6 applies when there are fewer than $\lfloor t_1/2 \rfloor + t_2$ selected odd edges, and Lemma 4.2 applies when equality holds. \square

Lemma 5.2. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F and a spanning path P through F whose end edges are unselected. Suppose that P has two unselected full even edges, and let Q be the portion of P consisting of the edges between them. If $\sigma(G) \geq n/2 + \tau(F)$ and the inequality below holds, then G has a spanning cycle through F .

$$|E_{\text{odd}}(Q) \cap F_1| - |E_{\text{even}}(Q) \cap F_1| \leq 2\lfloor t_1/2 \rfloor - t_1 + 1$$

Proof. Let $y_i x_i$ and $y_j x_j$ be the two given edges. By symmetry, we may assume that $y_i x_i$ comes first along P from x to y , and then $Q = \langle P(x_i, y_j) \rangle$. Let y' be the neighbor of x and x' the neighbor of y on P . Let $R = \langle P(x', x_j), y, P(x_i, y_j), x, P(y_i, y') \rangle$. For edges in both P and R , their parities in R and P differ, except for those in Q . Therefore

$$|E_{\text{odd}}(R) \cap F_1| + |E_{\text{odd}}(P) \cap F_1| = t_1 + |E_{\text{odd}}(Q) \cap F_1| - |E_{\text{even}}(Q) \cap F_1| \leq 2\lfloor t_1/2 \rfloor + 1.$$

We conclude that P or R has at most $\lfloor t_1/2 \rfloor$ edges of F_1 in odd position. Since $y_i x_i, y_j x_j$, and the end edges of P are unselected, R and P both pass through F . One of them has ath through F with at most $\lfloor t_1/2 \rfloor + t_2$ selected odd edges, so Lemma 5.1 yields a spanning cycle through F . \square

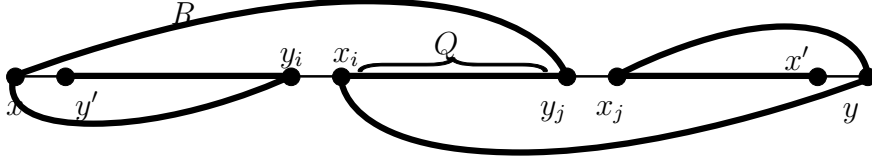


Figure 9: Horizontal path P and modified path R .

Lemma 5.3. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F . If $\sigma(G) \geq n/2 + \tau(F)$ and G has a spanning path through F whose end edges are unselected, then G has a spanning cycle through F .

Proof. Let P be such a path, with endpoints x and y . We may assume that $x \leftrightarrow y$, so $d_G(x) + d_G(y) \geq \sigma(G) \geq n/2 + \tau(F)$. Let $p = \lceil t_1/2 \rceil + \epsilon(t_1, t_2)$, so $\sigma(G) \geq n/2 + t_2 + p$. By the pigeonhole principle, there are at least $p + t_2 + 1$ full even edges along P . Let q be the number of even edges in F_2 that are not full, so $t_2 - q$ even edges in F_2 are full. Hence there is a set S of $p + q + 1$ full even edges in $E(P) - F_2$; index them as $y_1 x_1, \dots, y_{p+q+1} x_{p+q+1}$ in order along P from x to y . By Lemma 3.7, if any edge of S is in F_1 , then G has a spanning cycle through F . Hence, we may assume that $S \cap F = \emptyset$.

If $p \geq 2$ or $p = 1$ and $q \geq 1$, then $|S| \geq 3$. Let $Q_j = \langle P(x_j, y_{j+1}) \rangle$ for $1 \leq j \leq p + q$.

If $t_1 = 0$, then there are exactly t_2 selected odd edges along P . Since $\sigma(G) \geq n/2 + \tau(F)$, Lemma 5.1 completes the proof.

If $t_1 \in \{1, 2\}$ and $t_2 = 0$, then $\epsilon(t_1, t_2) = 1$. Hence $p = 2$, and Q_1 or Q_2 has at most $\lfloor t_1/2 \rfloor$ edges of F_1 . Since $t_1 \in \{1, 2\}$, we have $\lfloor t_1/2 \rfloor = 2\lfloor t_1/2 \rfloor - t_1 + 1$. Now Lemma 5.2 completes the proof.

If $t_1 \in \{1, 2\}$ and $t_2 > 0$, then $\epsilon(t_1, t_2) = 0$. Hence $p = 1$, and $\sigma(G) \geq n/2 + 1 + t_2$. Now Lemma 5.1 guarantees a spanning cycle through F unless all edges of F_1 are odd edges. Also, if every edges in F_1 is not full, then Lemma 3.6 yields a spanning cycle through F . Hence we may assume that some odd edge $x_1 y_1$ in F_1 is full.

If some even edge in F_2 is not full, then $q \geq 1$, and the paths Q_1 and Q_2 exist. One of them has at most $\lfloor t_1/2 \rfloor$ edges of F_1 . Again $t_1 \in \{1, 2\}$ and $\lfloor t_1/2 \rfloor = 2\lfloor t_1/2 \rfloor - t_1 + 1$, so Lemma 5.2 completes the proof. Hence we may assume that all even edges in F_2 are full.

Since $t_2 > 0$, we know that F_2 is nonempty, so by symmetry we may assume that that F_2 contains some edge of $\langle P(y_1, y) \rangle$. Let d, a, b, c be four successive vertices along $\langle P(y_1, y) \rangle$ in order such that $ab, bc \in F_2$ (as in Figure 8). If $a \notin Y$, then we can apply the same argument to $\pi(P, x_1 y_1)$ instead of P , with X and Y interchanged; hence we may assume $a \in Y$. Now ab is a full even edge in F_2 , so $x \leftrightarrow a$. The path $\langle P(d, x), P(a, y) \rangle$ has at most $\lfloor t_1/2 \rfloor$ selected odd edges, so Lemma 5.1 yields a spanning cycle through F .

The remaining case is $t_1 \geq 3$, and hence $\epsilon(t_1, t_2) = 0$. For $1 \leq j \leq p + q$, let $R_j = \langle P(x', x_{j+1}), y, P(x_j, y_{j+1}), x, P(y_j, y') \rangle$. If $|E_{\text{odd}}(Q_j) \cap F_1| - |E_{\text{even}}(Q_j) \cap F_1| \leq 2\lfloor t_1/2 \rfloor - t_1 + 1$ for any j , then Lemma 5.2 completes the proof. When t_1 is even, we may thus assume that $|E_{\text{odd}}(Q_j) \cap F_1| \geq 2$ for all j . Now $p = t_1/2$ requires that all edges of F_1 have odd position in P (and $q = 0$), and every Q_j contains exactly two of them. Therefore exactly two members of F_1 (those in Q_j) have odd position in R_j ; now Lemma 5.1 completes the proof.

Hence we may assume that t_1 is odd. Lemma 5.2 completes the proof unless

$$|E_{\text{odd}}(Q_j) \cap F_1| - |E_{\text{even}}(Q_j) \cap F_1| \geq 1 \quad \text{for } 1 \leq j \leq p. \quad (*)$$

Since $|F_1| = t_1 < 2p$, for some j we have $|E_{\text{odd}}(Q_j) \cap F_1| = 1$ and $|E_{\text{even}}(Q_j) \cap F_1| = 0$. For this j , the assumption in Lemma 5.2 yields $|E_{\text{odd}}(R_j) \cap F_1| + |E_{\text{odd}}(P) \cap F_1| = t_1 + 1$. If R_j or P has at most $\lfloor t_1/2 \rfloor$ odd edges in F_1 , then it contains at most $\lfloor t_1/2 \rfloor + t_2$ selected odd edges, and Lemma 5.1 completes the proof. Hence we may assume that F_1 has exactly p odd edges and $p - 1$ even edges of P . Now $(*)$ requires $|E_{\text{odd}}(Q_j) \cap F_1| = 1$ and $|E_{\text{even}}(Q_j) \cap F_1| = 0$ for $1 \leq j \leq p$. Therefore each even edge of F_1 is in $P(x, y_1)$ or $P(x_{p+1}, y)$. By symmetry, we may assume that $P(x_{p+1}, y)$ has at most $\lfloor (p - 1)/2 \rfloor$ even edges in F_1 . Now $\langle P(x', x_2), y, P(x_1, y_2), P(x, y_1) \rangle$ has at most $\lfloor (p - 1)/2 \rfloor + 1 + t_2$ selected odd edges. Since $\lfloor (p - 1)/2 \rfloor + 1 + t_2 = \lceil p/2 \rceil + t_2 \leq \lfloor t_1/2 \rfloor + t_2$ for $t_1 \geq 3$, Lemma 5.1 completes the proof. \square

6 The Role of Vertex Degrees

Lemma 6.1. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F . If $\sigma(G) \geq n/2 + \tau(F)$, and G has spanning paths P and Q through F such that all edges of F_1 have opposite parity in P and Q , then G has a spanning cycle through F .

Proof. Lemma 5.1 applies to P or Q . \square

Lemma 6.2. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F such that $\sigma(G) \geq n/2 + \tau(F)$. Let P be a spanning path through F . Let y_1x_1 be an unselected full even edge on P , and let x_2y_2 be an unselected odd edge on P . Let U be the vertex set of the component of $P - y_1x_1$ that does not contain x_2y_2 . If $d_U(x_2) > |F|$ or $d_U(y_2) > |F|$, then G has a spanning cycle through F .

Proof. Let x and y be the endpoints of P ; by symmetry we may assume $x_2 \in P(x, y_1)$. First consider $d_U(x_2) > |F|$ (Figure 10(a)). Each neighbor of x_2 in U is incident to exactly one odd edge of P . Since $d_U(x_2) > |F|$, there is a neighbor y' of x_2 in U such that $x'y'$ is an unselected odd edge. Let $Q = \langle P(x', x_1), P(y, y'), P(x_2, x), P(y_1, y_2) \rangle$. Every edge of F has opposite parity in Q and P , so Lemma 6.1 completes the proof.

The argument is similar when $d_U(y_2) > |F|$ (Figure 10(b)). \square

Lemma 6.3. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F . Suppose that G has a spanning path through F and that every such path has an endpoint of degree at most $t_1 + 2t_2$. If $n \geq 9|F| + 2$ and $\sigma(G) \geq n/2 + \tau(F)$, then G has a spanning cycle through F .

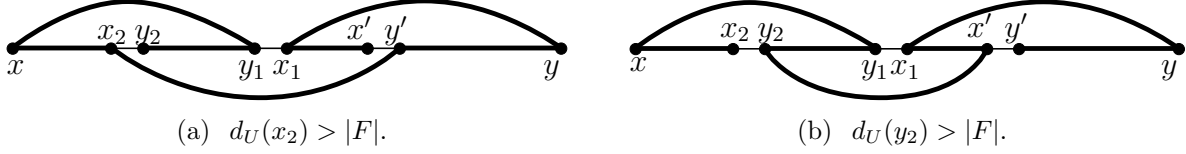


Figure 10: $d_U(x_2) > |F|$ or $d_U(y_2) > |F|$.

Proof. Let $k = t_1 + 2t_2$, and suppose that G has no spanning cycle through F . If $t_1 = 0$, then Lemma 5.1 applies, so we may assume $t_1 \geq 1$. Let P' be a spanning x, v -path through F in G . By symmetry, we may assume $d_G(x) \leq t_1 + 2t_2 = k$. Since G has no spanning cycle through F , we have $x \leftrightarrow v$, and hence $d_G(v) \geq \sigma(G) - d_G(x) \geq n/2 + \lceil k/2 \rceil + \epsilon(t_1, t_2) - k \geq 2k + 1$. We seek consecutive vertices u, y, u', v' in order on $P'(x, v)$ such that $v \leftrightarrow u$ and $uy, yu', u'v' \notin F$. Such a neighbor u of v exists, since $d_G(v) \geq 2k + 1$ and each edge of F eliminates only two neighbors of v as possible choices for u .

Now $P = \langle P'(x, u), P'(v, y) \rangle$ is a path through F whose last two edges are unselected. Let e be the edge of P incident to x . If e is unselected, then Lemma 5.3 guarantees a spanning cycle through F , so $e \in F$. Since G has no spanning cycle through F , we have $x \leftrightarrow y$ and $d_G(x) + d_G(y) \geq n/2 + \tau(F)$. Let $p = \lceil t_1/2 \rceil + \epsilon(t_1, t_2)$. Since $t_1 \geq 1$, the definition of $\epsilon(t_1, t_2)$ yields $p \geq 2$. By the pigeonhole principle, there are at least $p + t_2 + 1$ full even edges in P . Since F_2 has exactly t_2 even edges, we can pick a set S consisting of $p + 1$ such edges; index them as $y_1x_1, \dots, y_{p+q+1}x_{p+q+1}$ in order along P from x to y . By Lemma 3.7, if any edge of S is in F_1 , then G has a spanning cycle through F . Hence, we may assume that $S \cap F = \emptyset$.

Let $L = P(x, y_{p+1})$ and $R = P(x_{p+1}, y)$. If $|L| \geq 3k + 1$, then $d_R(y) \leq |X - L| \leq n/2 - \lceil (3k + 1)/2 \rceil$. Hence

$$d_L(y) \geq d_G(y) - d_R(y) \geq \sigma(G) - d_G(x) - d_R(y) \geq \frac{n}{2} + \left\lceil \frac{k}{2} \right\rceil - k - \left(\frac{n}{2} - \left\lceil \frac{3k + 1}{2} \right\rceil \right) \geq k + 1.$$

Now Lemma 6.2 applies to $y_{p+1}x_{p+1}$ and $u'y$. Hence we may assume that $|L| \leq 3k$.

We next aim to prove that F_1 has at least p odd edges in $\langle L \rangle$. We show first that if $x'y'$ is an unselected odd edge in $\langle L \rangle$, then $x \leftrightarrow y'$ and the other edge in $\langle L \rangle$ incident to x' is selected. If $x \leftrightarrow y'$, then $d_G(y') \geq \sigma(G) - d_G(x) \geq n/2 - \lfloor k/2 \rfloor$. Since $|L| \leq 3k$, we have $d_R(y') \geq k + 1$. Now Lemma 6.2 applies to $y_{p+1}x_{p+1}$ and $x'y'$. Hence we may assume $y' \leftrightarrow x$. Also, if the first edge on $\langle P(x', x), P(y', y) \rangle$ is unselected, then Lemma 5.3 guarantees a spanning cycle through F . Hence we may assume that two edges of $\langle L \rangle$ incident to a single vertex of X are not both unselected.

Consider $y_i x_i$ for $i \leq p$, and let y' be the other neighbor of x_i on P . Since $y_i x_i$ is an unselected even edge in $\langle L \rangle$, the preceding paragraph implies that $x_i y' \in F$. If $x_i y' \in F_2$, then the next edge $y' \hat{x}$ on P is selected and the next odd edge $\hat{x} \hat{y}$ is not selected. For an unselected odd edge $\hat{x} \hat{y}$ in $\langle L \rangle$, we proved above that $x \leftrightarrow \hat{y}$. If $y_i \leftrightarrow \hat{x}$, then $[x_i, y', \hat{x}, P(y_i, x), P(\hat{y}, y)]$ is a spanning cycle through F . Thus $y_i \leftrightarrow \hat{x}$ and $d_G(\hat{x}) + d_G(y_i) \geq n/2 + \lceil k/2 \rceil$. Since $|L| \leq 3k$ and $n \geq 9k + 2$ (this is where we use this restriction), $d_R(\hat{x}) + d_R(y_i) \geq 2k + 1$.

Now $d_R(\hat{x}) > k$ or $d_R(y_i) > k$. If $d_R(\hat{x}) > k$, then Lemma 6.2 applies to $y_{p+1}x_{p+1}$ and $\hat{x} \hat{y}$. If $d_R(y_i) > k$, then y_i has a neighbor x'' in R such that the odd edge $x'' y''$ is unselected. Let

$Q_1 = \langle P(x, y_i), P(x'', x_i), P(y, y'') \rangle$ and $Q_2 = \langle P(\hat{x}, x), P(\hat{y}, y) \rangle$; see Figure 11. Every edge of F_1 has opposite parity in Q_1 and Q_2 (since $x_i y', y' \hat{x} \in F_2$, it is not important that they have the same parity in Q_1 and Q_2), so Lemma 6.1 completes the proof.



Figure 11: Q_1 and Q_2 .

Hence we may assume that $x_i y' \in F_1$. The argument is valid for $1 \leq i \leq p$, so we obtain at least p members of F_1 in odd position along $\langle L \rangle$. Let $K = \{y \in Y \cap L: \text{the odd edge of } P \text{ incident to } y \text{ is unselected}\}$ and $K' = \{y_i: 1 \leq i \leq p\}$. For $y_i \in K \cap K'$, let x' be the vertex preceding y_i on P . Since $y_i \in K$, we have $x' y_i \notin F$. Since $u' v', x' y_i \notin F$, Lemma 5.3 obtains from the path $\langle P(u', x_{p+1}), y, P(x_i, y_{p+1}), P(x, y_i) \rangle$ a spanning cycle through F ; hence we may assume $K \cap K' = \emptyset$.

Next we prove that u' has no neighbor in $K \cup K'$. If $y' \in K$ and $y' \leftrightarrow u'$, then let $x' y'$ be the odd edge of P incident to y' , so $x' y' \notin F$. With $Q = \langle P(x', x), P(y_{p+1}, y'), P(u', x_{p+1}), y \rangle$, Lemma 6.1 completes the proof. On the other hand, if $y_i \in K'$ and $y_i \leftrightarrow u'$, then $[P(u', x_{p+1}), y, P(x_i, y_{p+1}), P(x, y_i)]$ is a spanning cycle through F . With u' having no neighbor in $K \cup K'$, and $K \cap K' = \emptyset$, we have $d_G(u') \leq n/2 - |K| - |K'| = n/2 - |K| - p$.

For $y_i \in K'$, we have $y_i \leftrightarrow u'$, and hence $d_G(y_i) \geq \sigma(G) - d_G(u') \geq |K| + t_1 + t_2$. If y_i has a neighbor x'' in R such that the odd edge $x'' y''$ is unselected, then let $Q = \langle P(x_{p+1}, x''), P(y_i, x), P(y_{p+1}, x_i), P(y, y'') \rangle$ (see Figure 12). We have guaranteed at least p selected odd edges of P in $\langle L \rangle$, which are even in Q . Hence at most $|F_1| - p$ edges of F_1 are even in Q . Since $p \geq \lceil t_1/2 \rceil$, Lemma 5.1 guarantees a spanning cycle through F .

We conclude that y_i does not have such a neighbor x'' . Hence at most $|K| + t_1 + t_2$ vertices of X are available as neighbors of y_i . This yields $d_G(y_i) = |K| + t_1 + t_2$, and furthermore y_i is adjacent to one endpoint of every selected edge. Consider $y_1 x_1$ and $y_2 x_2$, which both exist since $p \geq 2$. We saw that x_2 is one endpoint of a member of F_1 , and y_i is adjacent to one endpoint of every such edge, so therefore $y_1 \leftrightarrow x_2$. Now $[P(x, y_1), P(x_2, y), P(x_1, y_2)]$ is a spanning cycle through F . \square

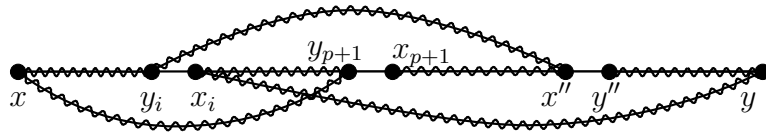


Figure 12: Path Q .

Lemma 6.4. Let G be an n -vertex balanced X, Y -bigraph with a selected (t_1, t_2) -linear forest F . If $n \geq 9|F| + 2$ and $\sigma(G) \geq n/2 + \tau(F)$, then G has a spanning cycle through F .

Proof. Let $k = |F|$; we have $k = t_1 + 2t_2$. We prove the claim by induction on k . The base case where $k = 0$ is just Proposition 3.2.

If $k > 0$, then let uv be some edge of F , and let $F' = F - uv$ and $k' = k - 1$. Note that F' is a (t'_1, t'_2) -linear forest in G for some t'_1, t'_2 with $k' = t'_1 + 2t'_2$. We claim that $\tau(F) \geq \tau(F')$. Since $\tau(F) = \lceil k/2 \rceil + \epsilon(t_1, t_2)$ and $\tau(F') = \lceil k'/2 \rceil + \epsilon(t'_1, t'_2)$, we have $\tau(F) \geq \tau(F')$ unless k is even, $\epsilon(t_1, t_2) = 0$, and $\epsilon(t'_1, t'_2) = 1$. This requires $t_1 = 2$, and then no choice for t_2 is possible. Since $\sigma(G) \geq n/2 + \tau(F')$, the induction hypothesis implies that G has a spanning cycle C through $F - uv$. If $uv \in E(C)$, then C is a spanning cycle through F and we are done, so we may assume $uv \notin E(C)$. Let u', u'' be the two neighbors of u on C and v', v'' be the two neighbors of v on C , with v' and u' on different sides of the chord uv as in Figure 13. Since paths in F have length at most 2, at most one of the edges uu', uu'', vv', vv'' is in F ; if such an edge exists, then by symmetry we may assume it is uu'' . Let Q be the path $C - uu'$. The path $\langle Q(u', v), Q(u, v') \rangle$, is a spanning path through F in G .

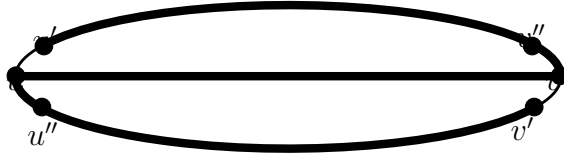


Figure 13: Cycle C .

If every spanning path through F in G has an endpoint of degree at most $2t_2 + t_1$, then Lemma 6.3 implies that G has a spanning cycle through F . Hence we may assume that G has a spanning x', y' -path P through F with $d_G(x') > k$ and $d_G(y') > k$. We have the desired spanning cycle unless $x' \leftrightarrow y'$. Since $d_G(x') > k$, there are consecutive vertices v', x, v along P , with $v \leftrightarrow x'$ and v' between x' and v on P , such that $v'x, xv \notin F$ (each edge of F forbids only one vertex of Y from being v). Let $P_1 = \langle P(x, x'), P(v, y') \rangle$. The endpoint y' of P_1 has high-degree, and the edge incident to x on P is unselected. By applying the argument above to y' , we obtain a neighbor u of y' such that the last two edges $u'y$ and yu before u on P_1 are unselected. Now let $P_2 = \langle P(x, u), P(y', y) \rangle$. Since P_1 and P_2 use the same edge at endpoint x , the path P_2 is a spanning path through F in G with both end edges unselected. Now Lemma 5.3 completes the proof. \square

Theorem 6.5. Let G be an n -vertex balanced X, Y -bigraph, and let F be a linear forest in G with k edges forming t_1 paths of odd length and t_2 paths of even length (and no isolated vertices). If $n \geq k + 8t_1 + 16t_2 + 2$ and $\sigma(G) \geq n/2 + \tau(F)$, then G has a spanning cycle through F .

Proof. Let F' be a (t_1, t_2) -linear forest, and let $k' = t_1 + 2t_2$. Since $k \geq t_1 + 2t_2$, we have $n \geq k' + 8t_1 + 16t_2 + 2 = 9k' + 2$. By Lemma 6.4, $\sigma(G) \geq n/2 + \tau(F')$ is sufficient for existence of spanning cycles in G through linear forests isomorphic to F' . By Lemma 2.5, $\sigma(G) \geq n/2 + \tau(F)$ is sufficient for a spanning cycle in G through F . \square

We remark that when F has long paths, the smallest value of n where $\sigma(G) \geq n/2 + \tau(F)$ becomes sufficient for a spanning cycle through F is smaller than $9k + 2$.

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