Implications among linkage properties in graphs

Qi Liu[*], Douglas B. West[†], Gexin Yu[‡]
Department of Mathematics, University of Illinois
Urbana, IL 61801

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Abstract

Given a graph H with vertices $w_1, \ldots, w_m$, a graph G with at least m vertices is H*-linked* if for every choice of vertices $v_1, \ldots, v_m$ in G, there is a subdivision of H in G such that v_i is the branch vertex representing w_i (for all i). This concept generalizes the notions of k-linked, k-connected, and k-ordered graphs. For graphs H_1 and H_2 with the same order that are not contained in stars, the property of being H_1-linked implies that of being H_2-linked if and only if $H_2 \subseteq H_1$. The implication also holds when H_1 is obtained from H_2 by replacing an edge xy with an edge from y to a new vertex x'. Other instances of non-implication are obtained, using a lemma that the number of vertices appearing in minimum vertex covers of a graph G is at most the vertex cover number plus the size of a maximum matching.

1 Introduction

Many applications require measures of the "connectedness" of a graph. A graph is k-*connected* if it has more than k vertices and deletion of any $k-1$ vertices leaves a connected subgraph. By a fundamental result (Menger's Theorem [10]), this is equivalent to the existence of k pairwise internally disjoint paths joining any pair of vertices.

More restrictive conditions have been studied. A graph is k*-linked* if for every list $(v_1, \ldots, v_{2k})$ of vertices, there are pairwise internally disjoint paths $P_1, \ldots, P_k$ such that each P_i is a path joining v_i and v_{i+k}. A graph is k*-ordered* if for every list of k distinct vertices, there is a cycle that visits those vertices in the given order; this again specifies k pairwise internally disjoint paths.

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[*] qiliu@math.uiuc.edu

[†] west@math.uiuc.edu. Work supported in part by the NSA under Award No. MDA904-03-1-0037

[‡] gexinyu@math.uiuc.edu.

An H*-subdivision* of a graph H is obtained from H by replacing each edge with a path whose internal vertices (if any) are new vertices of degree 2. The *branch vertices* in the H-subdivision are the original vertices of H. An H*-subdivision* in a graph G is a subgraph of G that is an H-subdivision. The concept of H-subdivision leads to a natural generalization of k-linked graphs and k-ordered graphs.

With $\{w_1, \dots, w_m\}$ denoting the vertices of H, a graph G with at least m vertices is H*-linked* if every injective mapping $f\colon V(H) \to V(G)$ extends to an H-subdivision in G; that is, for each choice of distinct vertices $v_1, \dots, v_m$ in G, there is an H-subdivision in which v_i is the branch vertex representing w_i, for all i. A graph G has the H*-linkage property* if G is H-linked. The notion of H-linked graphs was introduced independently in [6] and [3].

In the definition of k-linked graphs, it is sufficient to study only lists of $2k$ distinct vertices. Thus G is k-linked if and only if G is M_k-linked, where M_k is a matching of size k. Similarly, G is k-ordered if and only if G is C_k-linked, where C_k is a cycle of order k.

Several papers have been devoted to H-linkage properties, such as [3, 4, 6, 7, 8, 9]. These all focus on sufficient degree conditions (Dirac-type or Ore-type) for a graph to be H-linked. In contrast, here we compare H-linkage properties for distinct H. For example, C_k-linked implies C_{k-1}-linked, since $(k-1)$-ordered is a weaker condition than k-ordered. Also, various such properties are equivalent to k-connectedness, using a trivial observation.

Fact A *If $H_2 \subseteq H_1$, then every H_1-linked graph is H_2-linked.*

Let S_k be the star with k edges and I_k be the graph with $k+1$ vertices and one edge.

Proposition 1 *If H is a graph such that $I_k \subseteq H \subseteq S_k$, then a graph G is H-linked if and only if it is k-connected.*

Proof. The definitions of k-connected and I_k-linked are essentially the same. The S_k-linkage property is that for any set U of k vertices and a vertex $v \notin U$, there are k paths from v to U that pairwise share only v. This is the "Fan Condition" that Dirac [2] proved equivalent to being k-connected. Since $I_k \subseteq H \subseteq S_k$, Fact A now yields

$$k\text{-connected} \Leftrightarrow S_k\text{-linked} \Rightarrow H\text{-linked} \Rightarrow I_k\text{-linked} \Leftrightarrow k\text{-connected}. \qquad \square$$

The equivalence of I_k-linked with k-connected motivates the requirement that G have at least as many vertices as H to be H-linked.

To study the relation between H_1-linked and H_2-linked for general H_1 and H_2, let $\mathcal{G}_H$ denote the family of H-linked graphs. We write $H_2 \le H_1$ when $\mathcal{G}_{H_1} \subseteq \mathcal{G}_{H_2}$. This means that every H_1-linked graph is H_2-linked; equivalently, the H_1-linkage property implies the H_2-linkage property. We have defined a partial order on the isomorphism classes of graphs. Fact A states that $H_2 \subseteq H_1$ implies $H_2 \le H_1$.

The resulting poset $\mathcal{P}$ is the containment poset among the families $\mathcal{G}_H$ for all H. At the 2005 conference in Denver celebrating Joan Hutchinson's 60th birthday, Ron Gould described the subposet of $\mathcal{P}$ formed by the graphs with three vertices. Here we determine the subposets corresponding to the k-vertex graphs, for all k.

We also study whether $H_2 \le H_1$ when $|V(H_1)| \ne |V(H_2)|$. Fact A applies when H_1 or H_2 contains the other, but otherwise the problem is more difficult. Fact B below holds because $K_{|V(H_1)|}$ is H_1-linked but not H_2-linked, or similarly for the graph obtained by deleting one edge from $K_{|V(H_2)|}$.

Fact B *If H_2 has more vertices than H_1, then $H_2 \not\le H_1$.*

On the other hand, in Section 2 we prove that $H_2 \le H_1$ in some cases where H_1 has more vertices than H_2 but does not contain H_2. A *splinter* operation on H deletes an edge xy and replaces it with an edge $x'y$, where x' is a new vertex of degree 1. Theorem 5 states that if H' is obtained from H by a splinter operation, then $H \le H'$.

Applying successive splinter operations thus yields a spectrum of successively stronger properties between H-linked and k-linked, where H has k edges and no isolated vertices. In particular, M_k is the unique maximal element of the subposet of $\mathcal{P}$ on the graphs with k edges. Most of this subposet is unknown, but not for $k = 4$.

Example 2 Among graphs with four edges, the results we prove in this paper determine all of the relations except for five pairs of graphs. Four of them are the pairs of *consecutive* graphs in the following list (where "+" means disjoint union and $K'_{1,3}$ arises from a claw by subdividing one edge):

$$K'_{1,3}, \quad C_4, \quad K_2 + K_{1,3}, \quad K_2 + K_3, \quad 2P_3.$$

We also do not know whether $C_4 \le K_2 + K_3$ is true. This example is interesting partly because we do know that $C_4 \not\le 2K_1 + K_3$. To see this, consider a 5-connected plane graph with a face of length 4. It is well known that such a graph is not 2-linked, and hence it is not C_4-linked, since $M_2 \subset C_4$. On the other hand, every 5-connected graph is $(2K_1 + K_3)$-linked, since deleting any two vertices leaves a 3-connected graph, and Dirac [2] proved that every k-connected graph has a cycle through any k vertices, which for $k = 3$ is equivalent to 3-ordered. Fact B yields $2K_1 + K_3 \not\le C_4$.

Subposets of $\mathcal{P}$ formed by fixing the number of vertices are easier to analyze than those where the number of edges is fixed. The job is completed by Theorem 7 in Section 3: when H_1 and H_2 have the same number of vertices and H_2 is not contained in a star, $H_2 \le H_2$ if and only if $H_2 \subseteq H_1$. Theorem 7 and Fact B yield the corollary that if H is not contained in a star, then the H-linkage and H'-linkage properties are equivalent only if $H \cong H'$.

Theorem 10 in Section 4 generalizes the negative part of Theorem 7 to yield instances where H_1-linked does not imply H_2-linked even though H_1 may have more vertices and edges than H_2. This and Theorem 5 provide the first steps toward analyzing the rest of the poset $\mathcal{P}$. A lemma of interest in its own right (Lemma 9) states that in any graph G, the number of vertices belonging to minimum-sized vertex covers is at most the vertex cover number plus the maximum size of a matching.

Finally, the special role of M_k leads us to introduce a problem that generalizes the well-known problem of finding the minimum value $f(k)$ such that every $f(k)$-connected graph is k-linked [1, 5, 11] (it is known that $2k \le f(k) \le 10k$). Since I_r-linked is equivalent to r-connected, $f(k)$ is one more than the number of isolated vertices that must be added to K_2 to obtain a graph H such that H-linked implies M_k-linked. Since $H_2 \le M_k$ for every graph H with k edges and no isolated vertices, and $I_r \le H_1$ for every nontrivial graph H_1 with $r+1$ vertices, we may define $f(H_1, H_2)$ to be the least p such that $H_2 \le H_1 + pK_1$. We offer this problem as a subject for future study but provide no results on it.

2 Establishing $H_2 \le H_1$

We begin with easy observations. We use $G + v$ to denote the graph obtained from G by adding an isolated vertex v, and we use $G \vee v$ to denote the graph obtained from G by adding a vertex v with neighborhood $V(G)$.

Lemma 3 *A graph G is H-linked if and only if $G \vee u$ is $(H + v)$-linked.*

Proof. *Necessity.* Assume that G is H-linked and consider any injective $f\colon V(H + v) \to V(G \vee u)$. Let $w = f(v)$. Since $(G \vee u) - w$ contains a copy of G, this subgraph contains an H-subdivision with the desired branch vertices to complete the $(H + v)$-subdivision in $G \vee u$.

Sufficiency. Assume that $G \vee u$ is $(H + v)$-linked and consider any injective $f\colon V(H) \to V(G)$. Extend f to $V(H + v)$ by setting $f(v) = u$. Since $G \vee u$ is $(H + v)$-linked, f extend to an $(H + v)$-subdivision in $G \vee u$, and deleting u yields the desired extension of f on $V(H)$ to an H-subdivision in G. □

Lemma 4 *Graphs H_1 and H_2 satisfy $H_2 \le H_1$ if and only if $H_2 + v \le H_1 + v$.*

Proof. *Necessity.* Suppose that $H_2 \le H_1$. If G is $(H_1 + v)$-linked, then Lemma 3 implies that $G - u$ is H_1-linked, for any $u \in V(G)$. Hence $G - u$ is also H_2-linked. Given $f\colon V(H_2+v) \to V(G)$, let $u = f(v)$. Since $G-u$ is H_2-linked, the desired subdivision exists.

Sufficiency. Suppose that $H_2+v \le H_1+v$. If G is H_1-linked, then $G\vee u$ is (H_1+v)-linked, by Lemma 3, and hence $G\vee u$ is (H_2+v)-linked. Now Lemma 3 implies that G is H_2-linked. □

A *splinter* operation at a vertex x in a graph H forms a new graph H' by deleting an edge xy and introducing a new vertex x' adjacent only to y. In H', the new vertex x' has degree 1, and x has degree $d_H(x) - 1$. The new graph has the same number of edges as H.

Theorem 5 *If H' is obtained from a graph H by a splinter operation, then $H \leq H'$.*

Proof. Let x' be the new vertex in H', and let x be the vertex from which its edge was splintered. Let h and h' be the numbers of vertices in H and H'; note that $h' = h + 1$.

Let G be an H'-linked graph. We need to show that G is also H-linked. Since H' has an edge and has $h+1$ vertices, $I_h \subseteq H'$. Hence by Fact 0 and Proposition 1, G is I_h-linked and h-connected.

Consider an injective mapping $f\colon V(H) \to V(G)$ with $u = f(x)$. Since G is h-connected, $d_G(u) \geq h$. Choose $v \in N_G(u) - f(V(H))$. Define $f'\colon V(H') \to V(G)$ by $f'(x') = v$ and $f'(y) = f(y)$ for $y \neq x'$. From the extension of f' to an H'-subdivision in G, we obtain the desired H-subdivision in G by adding the edge vu to the path with endpoint v. □

Corollary 6 *A graph is k-linked if and only if it is H-linked for every graph H with k edges and no isolated vertices.*

Proof. From a graph H with k edges and no isolated vertices, we can obtain M_k by a succession of splinter operations. Hence Theorem 5 implies that every k-linked graph is H-linked for every such H. The converse is immediate, since M_k is such a graph. □

3 Pairs of Graphs with Equal Order

For graphs with n vertices, we show that the corresponding subposet of $\mathcal{P}$ is given by graph containment except for the subgraphs of S_{n-1}. Let $\beta(H)$ denote the vertex cover number of a graph H, the minimum size of a vertex subset incident to every edge.

Our next theorem is a special case of Theorem 10, but its proof is much less complicated. We give it here to motivate the proof for Theorem 10.

Theorem 7 *If H_2 and H_1 both have n vertices, and $\beta(H_2) \geq 2$, then $H_2 \leq H_1$ if and only if $H_2 \subseteq H_1$.*

Proof. Sufficiency is by Fact A. For the converse, if H_2 is contained neither in H_1 nor in a star, then we construct an H_1-linked graph that is not H_2-linked. Let $m = |E(H_2)|$.

Let $G = K_{n+m-1} - E(H_2)$; that is, G is the complement of the disjoint union of H_2 with $m-1$ isolated vertices. When we map $V(H_2)$ into $V(G)$ by sending the vertices to their natural images with respect to the missing edges, extension to an H_2-subdivision requires an added vertex on the path representing each edge of H_2, but there are only $m-1$ added vertices available. Hence G is not H_2-linked.

It remains to prove that G is H_1-linked. Consider an injective mapping $f\colon V(H_1) \to V(G)$. Let x_iy_i denote the ith edge of H_1. Let $u_i = f(x_i)$ and $v_i = f(y_i)$ for each i. Let s be the number of edges of H whose endpoints are mapped into nonadjacent pairs in G by f. We may index the edges so that $u_iv_i \notin E(G)$ for $1 \leq i \leq s$ and $u_iv_i \in E(G)$ for $i > s$.

To complete the extension to an H_1-subdivision, we must find pairwise internally disjoint paths to represent the s missing edges. There are also $m-s$ other nonadjacent pairs in G. Since $H_2 \nsubseteq H_1$, the nonadjacent pairs in G cannot all be edges desired for the H_1-subdivision, and hence $s \le m-1$. Let $S = \bigcup_{i=1}^{s} \{u_i, v_i\}$.

Let $W = f(V(H_1))$, so $S \subseteq W$. Let $T = V(G) - W$, so $|T| = m-1$. Let $T_2 = \{v \in T\colon\ S \subseteq N_G(v)\}$, and let $T_1 = T - T_2$ (see Fig. 1). Let $t_1 = |T_1|$ and $t_2 = |T_2|$, so $t_1 + t_2 = m-1$. There are exactly m nonadjacent pairs of vertices in $V(G)$. Since each vertex of T_1 forms such a pair with some vertex of S, we have $s + t_1 \le m$, and hence $t_2 \ge s-1$.

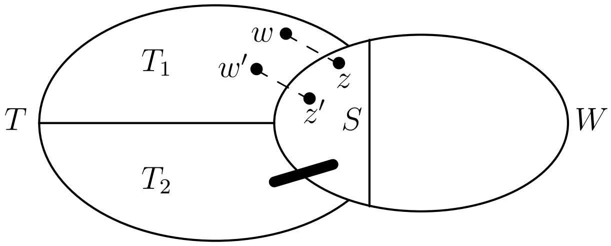


Fig. 1. Vertex partition of G

If $t_2 \ge s$, then s distinct vertices of T_2 can be used to complete the s needed paths, all with length 2. Thus we may assume that $t_2 = s-1$. Now $s + t_1 = m$ (and also $t_1 \ge 1$). This means that every vertex of T_1 is nonadjacent to exactly one vertex in S, and together with the s pairs corresponding to $E(H_2)$ these are all the nonadjacent pairs in G.

Let w be a vertex of T_1, with unique nonneighbor z in S. If there exists $i \le s$ such that $z \notin \{u_i, v_i\}$, then we use the path $\langle u_i, w, v_i \rangle$ to link u_i and v_i. Otherwise, $\beta(H_2) > 1$ requires another vertex w' in T with unique nonneighbor z' in S such that $z' \ne z$. If z' avoids some pair $\{u_i, v_i\}$, then we proceed as above; otherwise, z and z' are both contained in all such pairs, so $s = 1$ and we use the path $\langle z, w', w, z' \rangle$ to link u_1 and v_1.

In each case, the remaining $s-1$ needed paths each use one vertex of T_2. $\square$

Corollary 8 *If H_1 is not contained in a star, then the H_1-linkage and H_2-linkage properties are equivalent if and only if $H_1 \cong H_2$.*

Proof. By Fact B, we may assume that $|V(H_1)| = |V(H_2)|$. If H_2 is not contained in a star, then the conclusion is stated by Theorem 7. If H_2 is contained in a star but H_1 is not, then Theorem 7 implies that $H_1 \not\leq H_2$. $\square$

We add two remarks about the proof of Theorem 7. First, the proof gives an algorithm for extending f into an H_1-subdivision in G. Second, the construction of a graph that is H_1-linked but not H_2-linked extends easily to provide an infinite family of such examples. We can successively add any number of vertices joined to all vertices not incident to the deleted copy of H_2.

4 Forbidding $H_1 \leq H_2$

In this section, we compare H_1-linkage and H_2-linkage when H_1 and H_2 do not have the same number of vertices. Note that if H_2 has more vertices, then always $H_2 \not\leq H_1$, since deleting an edge from a complete graph with $|V(H_2)|$ vertices then yields a graph that is H_1-linked but not H_2-linked.

We will generalize the construction of Theorem 7 to apply to pairs where H_1 has $n+k$ vertices, and H_2 has n vertices and m edges. If the number of common edges is not too big, and the vertex cover number is not too small, then again $K_{n+m-1} - E(H_2)$ will be a graph that is H_1-linked but not H_2-linked. The precise statement is in Theorem 10.

We will need a lemma about vertex covers; it is of independent interest. Let $\alpha'(G)$ denote the maximum size of a matching in G. The edges of a matching require distinct vertices in a cover, so $\alpha'(G) \leq \beta(G)$. Hence the lemma implies that the total number of vertices in minimum vertex covers of G is at most $2\beta(G)$.

An X, Y*-bigraph* is a bipartite graph with partite sets X and Y. For a graph G and $S, T \subseteq V(G)$, let $G[S, T]$ denote the maximal S, T-bigraph contained in G.

Lemma 9 *In a graph G, the total number of vertices that belong to minimum vertex covers of G is at most $\beta(G) + \alpha'(G)$.*

Proof. Let S be the set of all vertices belonging to minimum vertex covers; suppose that $|S| > \beta(G) + \alpha'(G)$. Hence there are at least two minimum vertex covers, C_1 and C_2. Let $A = C_1 \cap C_2$, $B_1 = C_1 - C_2$, $B_2 = C_2 - C_1$, and $D = S - (C_1 \cup C_2)$ (see Fig. 2). Since $|C_1| = |C_2|$, also $|B_1| = |B_2|$. Let $b = |B_1| = |B_2|$, so $|A| = \beta(G) - b$.

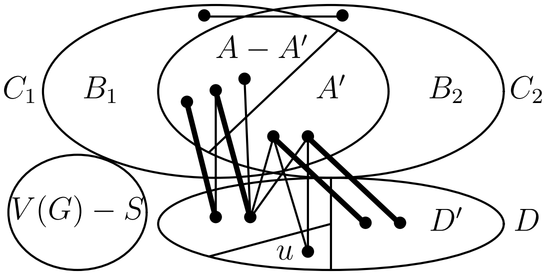


Fig. 2. Vertex covers in G

Let $G_B = G[B_1, B_2]$ and $G_A = G[A, D]$. Since deleting a vertex cover leaves an independent set, the only edges in G not incident to A are those of G_B. Hence $\beta(G) \leq |A| + \beta(G_B)$. Thus $\beta(G_B) \geq b$, which yields $\alpha'(G_B) \geq b$, by the König–Egerváry Theorem. Since G_B and G_A are disjoint, $\alpha'(G_B) + \alpha'(G_A) \leq \alpha'(G)$, and hence $\alpha'(G_A) \leq \alpha'(G) - b$.

In G_A, let Q be a minimum vertex cover and M be a maximum matching (edges drawn in Fig. 2 show G_A, with M bold). Since $|Q| = |M|$ (by the König–Egerváry Theorem), each edge of M is incident to exactly one vertex of Q. Let $A' = Q \cap A$, and let D' be the subset of D matched to A' by M; we have $Q \cap D' = \varnothing$. Thus no edges join D' and $A - A'$.

Since M covers $\alpha'(G_A)$ vertices in D, and $|D| = |S - (C_1 \cup C_2)| > \alpha'(G) - b$, some vertex u in D is not covered by M. Nevertheless, $N_G(u) \subseteq A'$. Let R be a minimum vertex cover of G containing u. Let $R' = A' \cup (R - D')$. Since all edges incident to D' are covered by A, R' is a vertex cover of G. Since M yields $|A'| = |D'|$, we have $|R'| = |R| = \beta$. However, since $N_G(u) \subseteq A'$, also $R' - \{u\}$ is a vertex cover, which contradicts the minimality of R. □

Our general construction of a graph that is H_1-linked but not H_2-linked fails when H_2 lies in a special family. A *double-star* is a tree with exactly two non-leaf vertices. Let $\mathcal{H}_{r,k}$ be the class of graphs with $k+1$ components consisting of a double-star with $r-k$ edges plus k isolated edges. Such graphs have vertex cover number $k+2$. It should be noted that the main part of the proof suffices when $\beta(H_2) \geq k+3$ and ignores the special family. The last page of the proof is needed only for the case $\beta(H_2) = k+2$.

Theorem 10 *Let H_1 and H_2 be simple graphs, where H_1 has $n+k$ vertices and H_2 has n vertices and m edges. If $|E(H_1 \cap H_2)| \leq m-k-1$ and*
(1) $\beta(H_2) \geq k+3$, or
(2) $\beta(H_2) = k+2$ and ($H_2 \notin \mathcal{H}_{m,k}$ or $m > 2k+2$),
then $K_{n+m-1} - E(H_2)$ is H_1-linked but not H_2-linked.

Proof. Let $G = K_{n+m-1} - E(H_2)$. The construction is the same as in Theorem 7, not dependent on H_1, so the argument that G is not H_2-linked is the same.

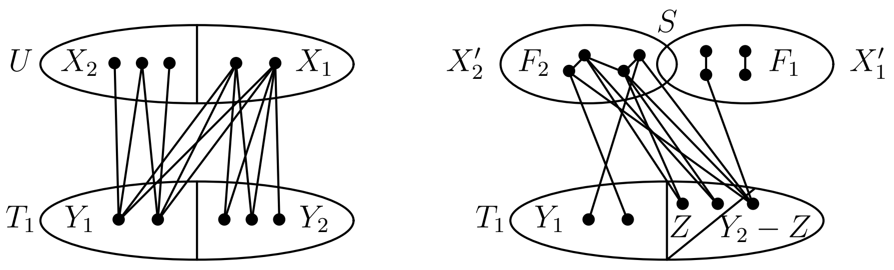
It remains to prove that G is H_1-linked. Again, consider $f\colon V(H_1) \to V(G)$, and let $x_i y_i$ denote the ith edge of H_1, with $u_i = f(x_i)$ and $v_i = f(y_i)$ for each i. Again let s be the number of edges of H_1 mapped into nonadjacent pairs, and index them so that $u_i v_i \notin E(G)$ for $1 \leq i \leq s$ and $u_i v_i \in E(G)$ for $i > s$.

Again we need paths for the s missing edges and G has $m-s$ other nonadjacent pairs. The restriction on $|E(H_1) \cap E(H_2)|$ means that f cannot map more than $m-k-1$ edges of H_1 onto missing edges. Thus $s \leq m-k-1$. Define $S, W, T, T_1, T_2, t_1, t_2$ as in Theorem 7 (see Fig. 1). Since H_1 has $n+k$ vertices, $t_1 + t_2 = m-k-1$.

Let $U = \{1, \ldots, s\}$. Let B be an auxiliary U, T_1-bigraph defined by putting i adjacent to w if and only if w is a common neighbor of u_i and v_i in G. If $\alpha'(B) \geq s - t_2$, then the paths of length 2 (in G) that correspond to a matching of size $s - t_2$ combine with paths of length 2 through vertices of T_2 to complete the desired H_1-subdivision.

Hence we may assume that $\alpha'(B) < s - t_2$, and hence also $\beta(B) < s - t_2$. Let $X_1 \cup Y_1$ be a minimum vertex cover of B, where $X_1 \subseteq U$ and $Y_1 \subseteq T_1$ (see Fig. 3). Let $x_1 = |X_1|$ and $y_1 = |Y_1|$, so $x_1 + y_1 \leq s - t_2 - 1$. Let $X_2 = U - X_1$ and $Y_2 = T_1 - Y_1$, so every $u \in Y_2$ is adjacent to at most one vertex of each pair in S corresponding to a vertex of X_2.

We have $\beta(B) < s - t_2$ under the assumption that the desired H_1-subdivision with branch vertices specified in W does not exist in G. Will we obtain a contradiction to this assumption in most cases by using the vertex cover of B to build a cover of $E(\overline{G})$ with fewer than $\beta(H_2)$ vertices.



B, from $G[S, T_1]$ $\overline{G}[S, T_1]$, from H_2

Fig. 3. Auxiliary bigraph in G; structure for vertex cover in $\overline{G}$.

Let l be the number of edges in $\overline{G}[S, T_1]$. Since S covers all edges of $\overline{G}$ incident to S, at most $m - s - l$ vertices are needed to cover the rest. For $j \in \{1, 2\}$, let $X_j' = \bigcup_{i \in X_j} \{u_i, v_i\}$ (see Fig. 3), and let F_j be the minimal subgraph of $\overline{G}$ whose edges set is $\{u_i v_i \colon\ i \in X_i\}$ To complete the vertex cover, we will choose vertices to cover the edge sets of F_1, F_2, $\overline{G}[S, Y_1]$, and $\overline{G}[S, Y_2]$.

We cover $\overline{G}[S, Y_1]$ using the y_1 vertices of Y_1. For the edge $u_i v_i$ in F_1, we choose u_i or v_i, thus covering F_1 using at most x_1 vertices of X_1'.

For the remaining edges, let $F = F_2 \cup \overline{G}[S, Y_2]$. Let $c = \beta(F_2)$. For $u \in Y_2$, let $N_u = N_{\bar{G}}(u) \cap S$. By the choice of the cover $X_1 \cup Y_1$ in B, the vertex u is adjacent in G to at most one vertex of edge in F_2. Hence $N_u \cap X_2'$ is a vertex cover of F_2, which yields $|N_u| \ge c$. Let $Z = \{u \in Y_2 \colon\ |N_u| = c\}$ and $z = |Z|$. If $u \in Z$, then $N_u \cap X_2'$ is a minimum vertex cover of F_2. By Lemma 9, $|\bigcup_{u \in Z} N_u| \le 2c$. Thus $(Y_2 - Z) \cup (\sum_{u \in Z} N_u)$ is a vertex cover for F with size at most $(t_1 - y_1 - z) + 2c$.

We have constructed a vertex cover showing $\beta(\overline{G}) \le (m - s - l) + y_1 + x_1 + (t_1 - y_1 - z) + 2c$. We hope to bound this quantity by $k + 2$.

Let $r = |E(\overline{G}[S, Y_2])|$. Since $\overline{G}[S, T_1]$ has l edges, with each vertex of T_1 incident (by definition of T_1) to at least one, $r \le l - y_1$. However, $r \ge \sum_{u \in Y_2} |N_u| \ge c|Y_2| + |Y_2 - Z|$. Together, $c(t_1 - y_1) + (t_1 - y_1 - z) \le l - y_1$, so $t_1 - y_1 - z \le (l - y_1) - c(t_1 - y_1)$. Also, the expressions $m - k - 1 = t_1 + t_2$ and $x_1 + y_1 \le s - t_2 - 1$ yield $x_1 + y_1 \le s - m + k + t_1$. Thus

$$\begin{aligned} \beta(\overline{G}) &\le x_1 + y_1 + m - s - l + (t_1 - y_1 - z) + 2c \\ &\le k + t_1 - y_1 - c(t_1 - y_1) + 2c \\ &= k + 2 - (c - 1)(t_1 - y_1 - 2). \end{aligned} \tag{1}$$

Let $q = (c - 1)(t_1 - y_1 - 2)$. If q is positive, then already we contradict the hypothesis that $\beta(\overline{G}) \ge k + 2$. Consider the ways that q can be nonpositive.

If $c = 0$ or $y_1 = t_1$ (by definition $y_1 \le t_1$), then $X_1 = U$ or $Y_1 = T$, respectively. The former contradicts $\beta(B) < s - t_2$, while the latter yields $s - 1 \ge t_1 + t_2 = m - k - 1 \ge s$, a contradiction. Hence we may assume that $c \ge 1$ and $t_1 > y_1$.

Whenever $z \le c$, we can improve the upper bound on $\beta(F)$ from $(t_1 - y_1 - z) + 2c$ to $t_1 - y_1 + c$ by using all of Y_2 together with a minimum vertex cover of F_2. If also $z > 1$, then we can improve it by one more, by using N_u as that cover for some $u \in Z$, since then we can drop u. Thus we reduce the upper bound by c if $z = 0$ and by $c - z + 1$ if $z > 0$; call this *replacement reduction*. We call reducing the upper bound on $\beta(\overline{G})$ below $k + 2$ a *win*.

If $q < 0$, then $c \geq 2$ and $t_1 = y_1 + 1$, and we obtain $q = -(c - 1)$. In this case, $|Y_2| = 1$ and $z \leq 1$. By replacement reduction, we reduce the upper bound on $\beta(\overline{G})$ by c, a win.

Hence we may assume that $q = 0$, so $c = 1$ or $t_1 = y_1 + 2$. We still have the contradiction if $\beta(H_2) \geq k + 3$, so we may assume that $\beta(H_2) = k + 2$. (That is, the proof is now complete for $\beta(H_2) \geq k + 3$; the remainder is needed only when $\beta(\overline{G}) = k + 2$.)

If $t_1 = y_1 + 2$ and $c \geq 2$, then $|Y_2| = 2$ and $z \leq 2$. Hence $z \leq c$, and replacement reduction reduces the upper bound by c if $z = c = 2$ and by $c - z + 1$ otherwise. This is a win.

Hence we may assume that $c = 1$. This implies that F_2 is a star with at least one edge. If $|X_2| > 1$, then F_2 has a minimum cover $\{w\}$ that covers all of $\overline{G}[X_2, Z]$. In that situation, $\{w\} \cup (Y_2 - Z)$ covers $E(F)$, so $\beta(F) \leq 1 + t_1 - y_1 - z$. Since (1) used $\beta(F) \leq t_1 - y_1 - z + 2c$, this is a win. Hence we conclude that $x_2 = 1$. Furthermore, since we win if $\bigcup_{u \in Z} N_u$ is a single vertex, we have $\bigcup_{u \in Z} N_u = X_2'$.

We win by improving the bounds $\beta(F_1) \leq x_1$ or $\beta(\overline{G}[S, Y_1]) \leq y_1$ unless F_1 is a matching and $x_1 + y_1 = s - t_2 - 1$. With $x_2 = 1$, we have $x_1 = s - 1$, and hence $y_1 = -t_2 \leq 0$. We conclude that $y_1 = t_2 = 0$ and $Y_2 = T_1$.

We claim further that $Y_2 = Z$. We win on $|E(\overline{G}[S, Y_2])|$ unless vertices of Z and $Y_2 - Z$ have degree exactly 1 and 2 in $\overline{G}[S, Y_2]$, and we win on $\beta(\overline{G}[S, Y_2]$ unless $(Y_2 - Z) \cup (\bigcup_{u \in Z} N_u)$ is a minimum vertex cover for $\bar{G}[S, Y_2]$. Since already $\bigcup_{u \in Z} N_u = X_2'$, each vertex $w \in Y_2 - Z$ must have a neighbor $w' \in X_1'$. However, F_1 is a matching, so we can choose the vertex covering the edge containing w' in F_1 to be w'. Now w is not needed in the cover; we win.

Hence $Z = T_1$, and each vertex of T_1 is adjacent to one vertex in X_2'. Since our minimum vertex cover contains all of X_2', we win on $\beta(F_1)$ if $X_2' \cap X_1' \neq \varnothing$. Finally, we win unless $m - s - l$ vertices beyond $\beta(F)$ are needed to cover the edges of $\overline{G}$ not incident to S. This requires that those edges form a matching, and that also no edges join X_1' and X_2'.

We have proved that $\overline{G}$ consists of a double-star (with center X_2' and leaves Z) plus isolated edges and vertices. The double-star has t_1 leaves, and $t_1 = m - k - 1$ (since $t_2 = 0$). Hence the double-star has $m - k$ edges, and there are k other isolated edges in $\overline{G}$, so $H_2 \in \mathcal{H}_{m,k}$.

Since $l = t_1$, the cover uses $k + 1 - s$ vertices outside S to cover those edges. Thus $s \leq k + 1$, and from the beginning we have $s \leq m - k - 1$.

If $s < m - k - 1$, then let $X_2' = \{u, v\}$ and choose $w \in N_{\overline{G}}(u) \cap Z$ and $w' \in N_{\overline{G}}(v) \cap Z$. We have $ww' \in E(G)$. Hence we can use the path $\langle u, w', w, v \rangle$ to link u and v. The maximum matching of size $s - 1$ in B provides paths for the other desired missing edges, so we obtain the desired H_1-subdivision.

Hence we may assume that $s = m - k - 1$. Now $m = k + 1 + s \leq 2k + 2$, which completes the proof. $\square$

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