

# Equitable Hypergraph Orientations

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Joint work with  
Yair Caro and Raphael Yuster

## A Classical Exercise

**Thm.** Every graph can be oriented so that the indegree and outdegree differ by at most **1** at each vertex.

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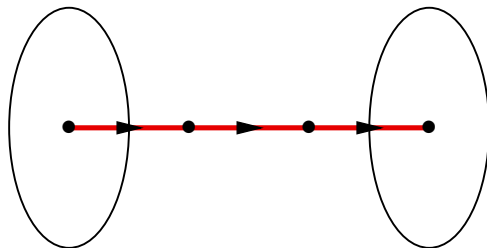
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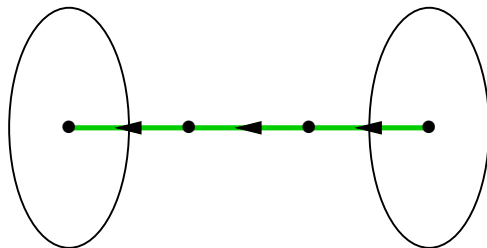
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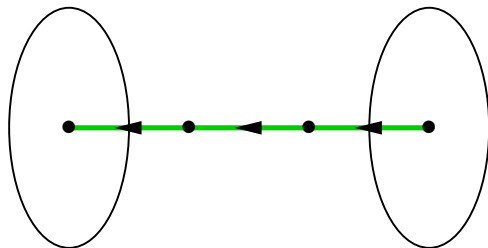


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**Aim:** Generalize this result to uniform hypergraphs.

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**Ex.** Equitable orientation of graphs

= 1-equitable orientations of 2-uniform hypergraphs.

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The  $p$ -set multigraph  $L_p(H)$  of a hypergraph  $H$  has vertex set  $E(H)$ , and the multiplicity of the edge joining  $e$  and  $e'$  is the number of  $p$ -sets in  $e \cap e'$ .

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Each  $U$  is put in each of the  $r$  possible positions at most  $\lceil d_B(U)/r \rceil$  and at least  $\lceil d_B(U)/r \rceil - 1$  times. ■

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**Thm.** If  $1 < p < r - 1$  and  $\binom{r}{p}$  divides  $\binom{n-p}{r-p}$ , then a  $p$ -equitable orientation of  $K_n^r$  requires  $r$  to divide  $\binom{n-1}{r-1}$ .

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$\therefore z$  occurs equally in each column:  $r$  divides  $\binom{n-1}{r-1}$ . ■

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## Sparse $p$ -set Multigraphs

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Since there are fewer than  $\binom{r}{p}$  such pairs, some ordering among the  $r!$  orderings remains available. ■

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**Def.** An orientation of an  $r$ -uniform hypergraph  $H$  is **nearly  $p$ -equitable** if  $|d_S(U) - d_T(U)| \leq 2$  whenever  $U$  is a  $p$ -set of vertices and  $S$  and  $T$  are  $p$ -sets of positions.

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For a  $p$ -set  $W \subset e$ , at most  $k-1$  other edges contain  $W$ , so at most  $\binom{k-1}{2}$  events have the form  $A_{W, \{e, x, y\}}$ , and there are  $\binom{r}{p}$  choices for  $W$ . Considering also  $f$  and  $g$ , fewer than  $3\binom{k-1}{2}\binom{r}{p}$  other events are excluded from  $F$ .

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**Ques.** For fixed  $p, r \in \mathbf{N}$ , is  $f(p, r, n)$  bounded by a value independent of  $n$ ?