Three Variations on Edge-Coloring

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Proper Path-Factors and Interval Edge-Coloring of (3,4)-Biregular Bigraphs

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Joint work with
Armen S. Asratian, Carl Johan Casselgren, Jennifer Vandenbussche

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Necessary condition (Asratian-Kamalian [1994]):
$\chi'(G) = \Delta(G)$ (reduce colors modulo $\Delta(G)$)

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Recognizing whether $(3, 6)$-biregular bigraphs have interval 6-colorings is NP-complete.

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Neither result implies the other.

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Henceforth let G be a $(3, 4)$-biregular X, Y-bigraph.
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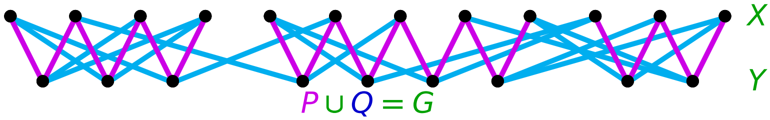

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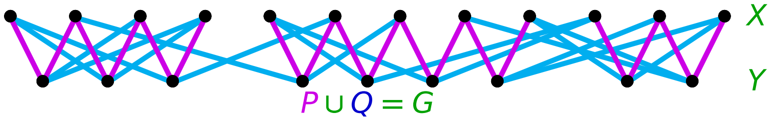

Prop. Every component of Q is an even cycle or is a path with endpoints in X.

Pf. Always $d_Q(y) = 2$ for $y \in Y$ and $d_Q(x) \in \{1, 2\}$ for $x \in X$. ∎

The auxiliary graph G_P

Def. For a proper path-factor P of G, let G_P be the graph with vertex set $\{x \in X\colon\ d_P(x) = 2\}$ having x_i and x_j adjacent when any condition below holds:

(a) x_i and x_j have degree 2 in one copy of P_7 in P, or

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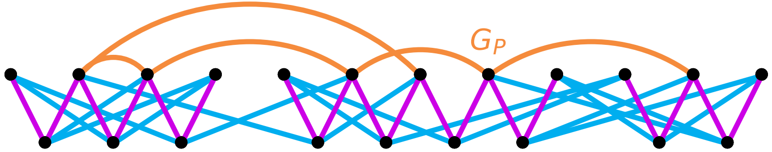

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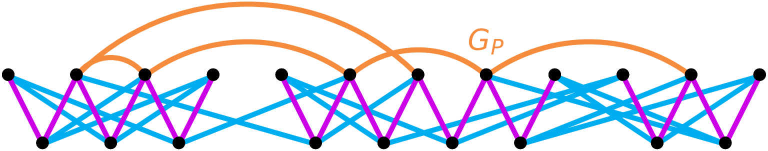

Lem. If P is a proper path-factor, then G_P is bipartite.

Pf. Every vertex of G_P has one incident type (c) edge; some have another of type (a) or (b). Hence $\Delta(G_P) \le 2$ and no odd cycle. ∎

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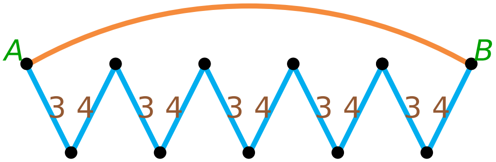

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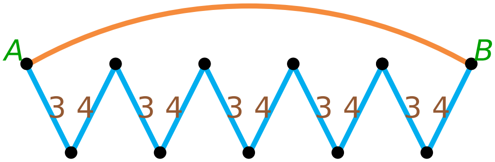
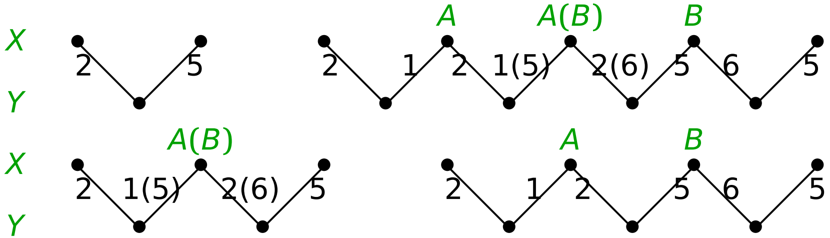


Every vertex of G with degree 2 in Q gets 3 and 4.

Choosing Colors in P

A component H of P is in $\{P_3, P_5, P_7, P_9\}$. If $x \in V(G_P)$ and $c(x) = A$, use 1 and 2 at x; if $c(x) = B$, use 6 and 5.



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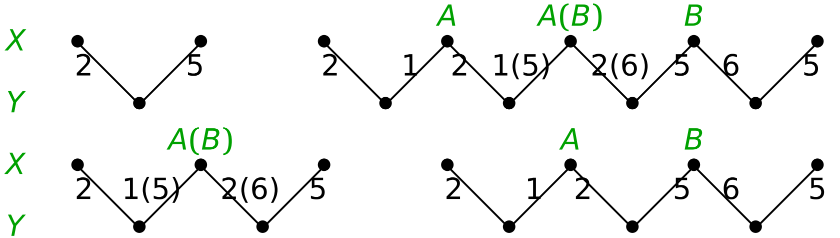

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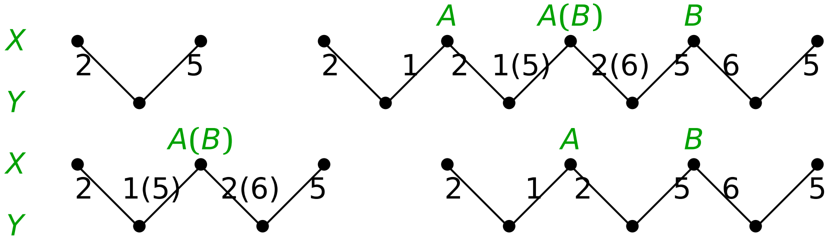

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Each Y-vertex gets $\{3, 4\}$ from Q and $\{2, 5\}$ or $\{1, 2\}$ or $\{5, 6\}$ from P. Each leaf in P gets $\{3, 4\}$ from Q and 2 or 5 from P. Non-leaves get 3 from Q and $\{1, 2\}$ from P if colored A; 4 from Q and $\{5, 6\}$ from P if colored B. ∎

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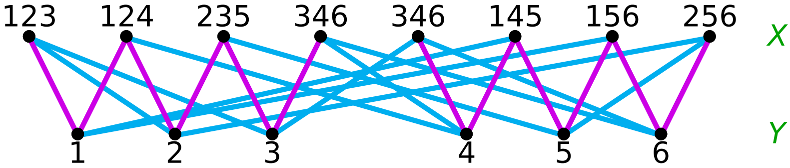

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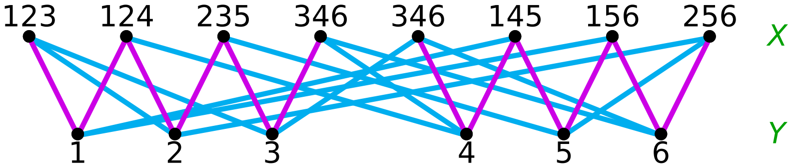

Combining such examples generates bigraphs having P_7-factors, but no full 3-regular subgraph, and order that is any nontrivial multiple of 7.

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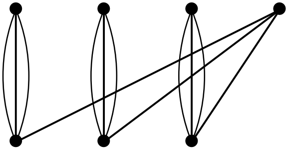

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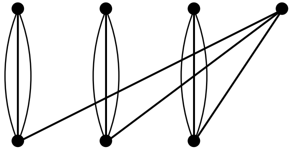

Conj. Every simple $(3, 4)$-biregular X, Y-bigraph has a proper path-factor.

Parity Edge-Coloring of Graphs

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(Joint with David Bunde, Kevin Milans, Hehui Wu)

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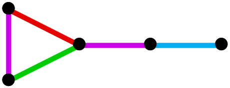

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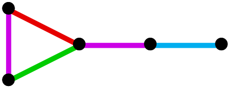

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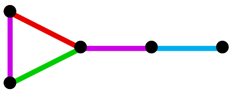

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Obs. $p(G) \geq \chi'(G)$, and $H \subseteq G \Rightarrow p(H) \leq p(G)$.

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Conj. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n. (Known for $n \leq 16$.)

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- Embeddability in hypercubes is NP-complete for trees (Wagner–Corneil [1990]), so computing $p(G)$ is also.

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Each remaining edge e completes a cycle. When $e = uv$, the color on e is the only color with odd usage on the u, v-path in T. Hence $f(u) \leftrightarrow f(v)$ in Q_k. ■

n-vertex Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$, with equality for paths and even cycles.

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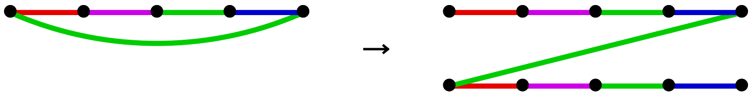

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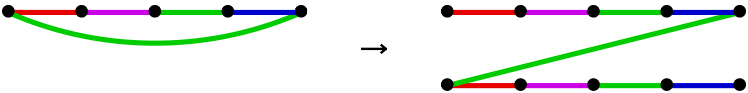

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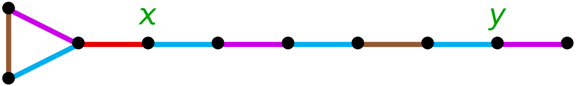
Example Showing $p \neq \hat{p}$

- Unrolling technique (like lower bound for odd cycle)

G



$p(G) \leq 4$

not spec

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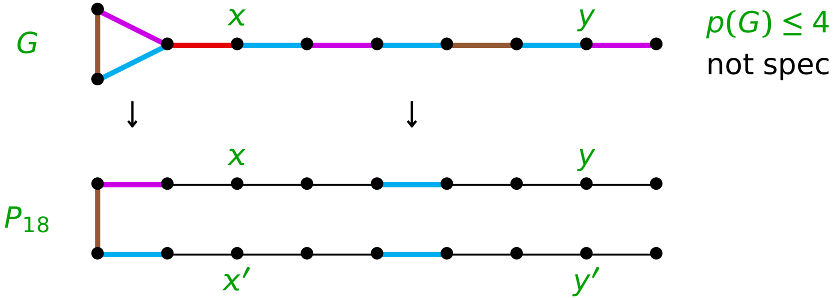
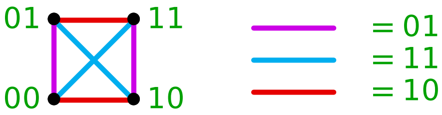


Obs. $\hat{p}(G) \geq p(P_{18}) = 5$.

Pf. Copy a spec of G onto P_{18} (path edges doubled).
An x, y'-subpath of P_{18} comes from an open walk in G.
An x, x'-subpath of P_{18} comes from an odd walk in G. ■

Complete Graphs, $n = 2^k$

Def. canonical coloring of K_{2^k} = edge-coloring f defined by $f(uv) = u + v$, where $V(K_{2^k}) = \mathbf{F}_2^k$.



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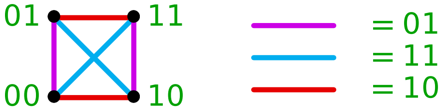

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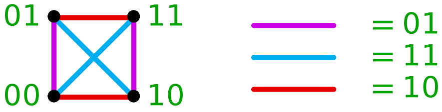

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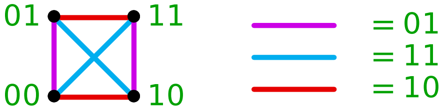

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Cor. $\hat{p}(K_n) \le 2^{\lceil \lg n \rceil} - 1 \le 2n - 3$.

Conj. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$. (**Thm.** $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$.)

Just Above the Threshold: K_2, K_3, K_5, K_9

- It suffices to prove $p(K_{2^k+1}) = 2^{k+1} - 1$.

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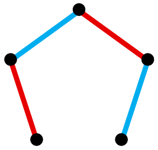

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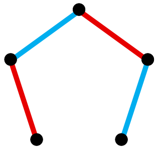

Thm. $p(K_9) = 15$. (Longer ad hoc argument.)

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Conj. $\hat{p}(K_{r,s}) = r \circ s$. (Would strengthen Yuzv. & ours.)

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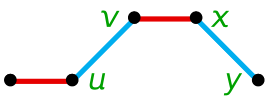

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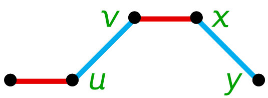

2) Binary vector spaces (vector of color usage, mod 2, along a walk).

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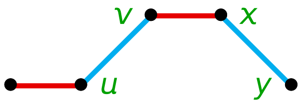

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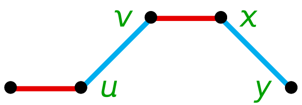

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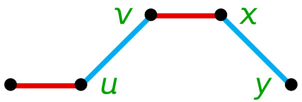

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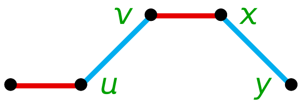

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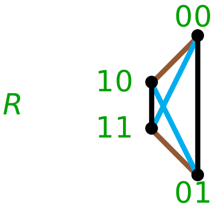
With $|R| = 2^{j-1}$, we are given a bijection from R to $\mathbf{F}_2^{j-1}$ under which f is the canonical coloring.

Expanding the Canonical Portion

f canonical on R $\Rightarrow$ any color used within R pairs up R.



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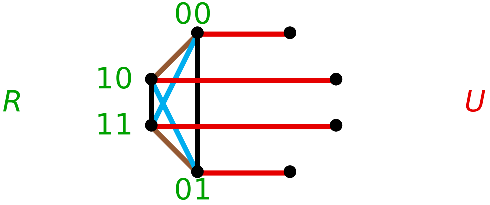

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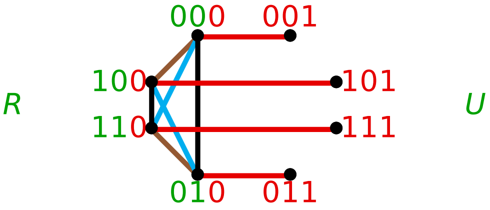

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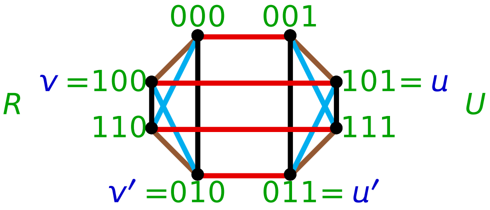

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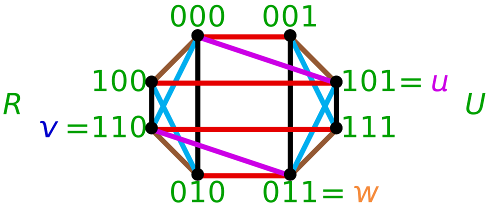

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Use u to name the color on $0^j u$, so $f(0^j u) = u = 0^j + u$.
The rest: $v \in R$ & $w = u + v \in U$ $\Rightarrow$ $f(v0^j) = f(uw) = v$;
4-constraint $\Rightarrow$ $f(vw) = f(0^j u) = u = v + w$. ∎

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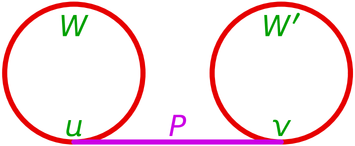
Pf. When W is a u,u-walk and W' is a v,v-walk, let P be a u,v-path, with P' its reverse. Now W, P, W', P' is a u,u-walk with parity vector $\pi(W) + \pi(W')$. ■



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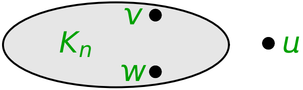

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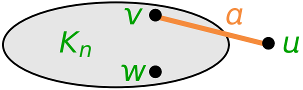

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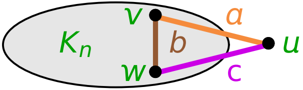

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- $p(G) \ge t(G) \ge \chi'(G)$.

Open Problems

Conj.1 $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.
Known for $n \le 16$; proved $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.

Conj.2 $p(K_{n,n}) = \hat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$. ($\hat{p}(K_{r,s}) = r \circ s$?)

Conj.3 $\hat{p}(G) = p(G)$ for every bipartite graph G.

Ques.4 What is $\max \hat{p}(G)$ (or $\max c(G)$) for $p(G) = k$?

Ques.5 What is $\max p(T)$ when T is an n-vertex tree with maximum degree k? (That is, what cube contains all n-vertex trees with maximum degree k?)

Ques.6 When does $p(G)$ equal $\lceil \lg n(G) \rceil$?

Ques.7 Is $p(T)$ NP-hard on trees w. bounded degree?

Ques.8 $\hat{p}(G \,\square\, H)$ . . . Digraphs . . .

Circular Chromatic Index of Cartesian Products of Graphs

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Traffic Lights and Coloring

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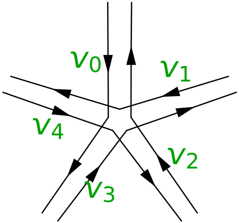
Aim: Minimize weighting time.



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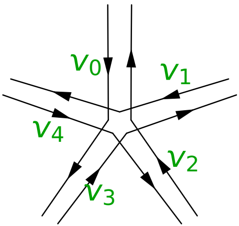

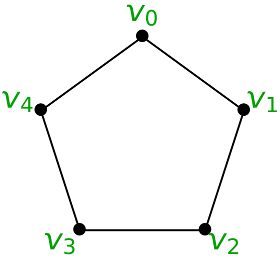

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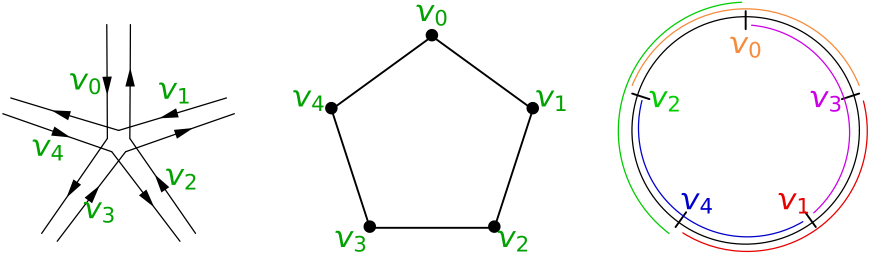

More efficient cycle: 2.5 units.

A Circular Coloring Model

Many equivalent definitions, this one convenient:

Def. r-coloring of G, for real r: a fcn $f\colon V(G) \to [0, r)$ such that $1 \le |f(x) - f(y)| \le r - 1$ when $xy \in E(G)$.

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- (Vince [1988]) infimum is achieved, χ_c is rational.
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- $\kappa'(G) \geq 2$ & $\Delta(G) = d$ & large girth $\Rightarrow$ $\chi'_c(G)$ near d ($d = 3$ Kaiser–Král–Škrekovski [2004]; general d K–K–Š–Zhu [2007]).

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- $G \,\square\, H$ is Class 2 when G and H are regular graphs of odd order (no perfect matching!).

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- Always $\chi'_c(H \square C_{2m+1})$ descends to a limit as m grows.

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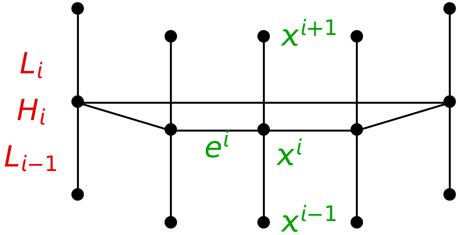
We need graphs G_i with $\lambda(G_i) = \lambda(H)$.

Layers and Links



ith link, vertical edges

ith layer, horizontal edges

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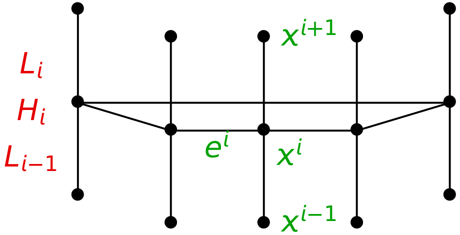

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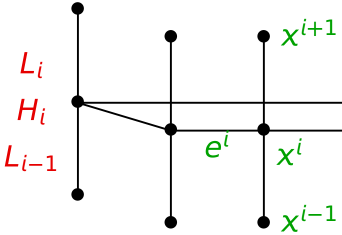

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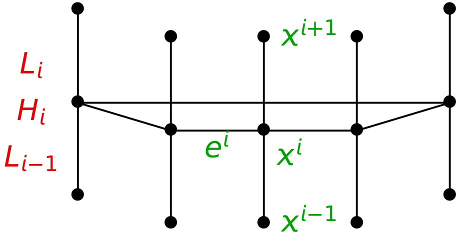

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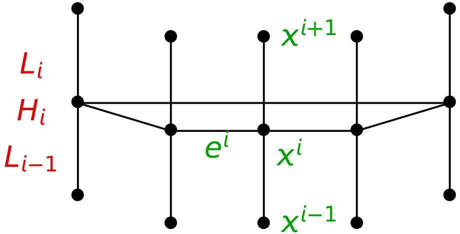
Lem. Given an r-coloring f of G, with $r < s + \lambda^*(G, s)$, fix a vertex $v^* \in V(G)$. Let $g(x) = \lfloor (f(x) - f(v^*))_r \rfloor$; this is a proper s-coloring of G, with $g(x) = g(x') \iff x$ gets to x' by steps of length 2 in G that change f by $< 1/2$.

Layers and Links



ith link, vertical edges
ith layer, horizontal edges
$(i-1)$th link, vertical edges

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- The color ptn of $V(G)$ under g does not depend on v^*.

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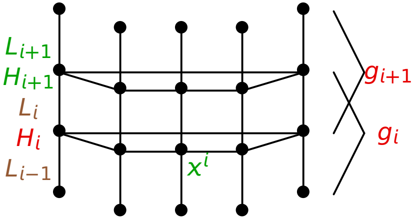

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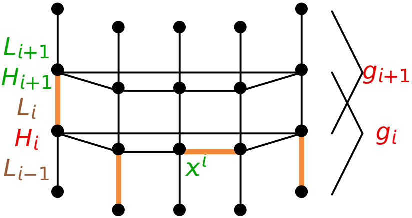

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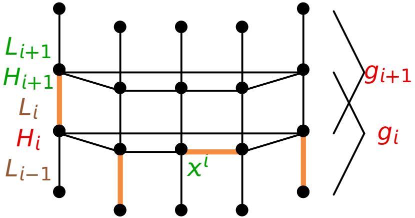

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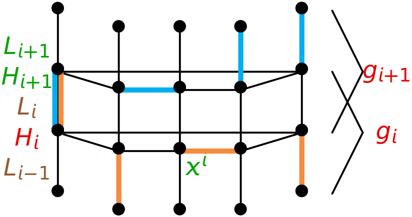

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Same # has odd usage under g_{i+1} (same ptn of L_i).

Odd usage in L_i $\Leftrightarrow$ even usage in L_{i+1}, so $c_{i+1} = s - c_i$.

Now $4 \mid s$ $\Rightarrow$ $c_i \ne s/2$, so c_i alternates values but can't!

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Pf. Step 1: Modify f to multiples of $1/q$.

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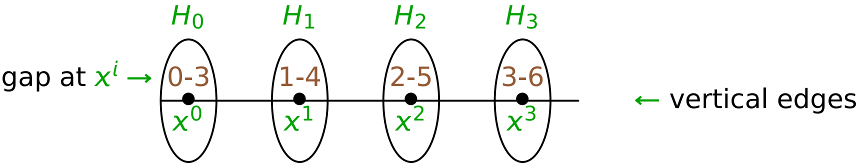
Difference between $f'(x)$ and $f'(y)$ is bounded by the same multiples of $1/q$ as between $f(x)$ and $f(y)$.

Color gaps of length ≥ 3 are preserved.

Construction of Coloring

Step 2: The case $2m + 1 = p$.
Use f on each layer H_i, shifted cyclically by i.
Gaps $(a{-}1, a{+}2)$ at x^i and $(a, a{+}3)$ at x^{i+1} share $[a, a{+}1]$.



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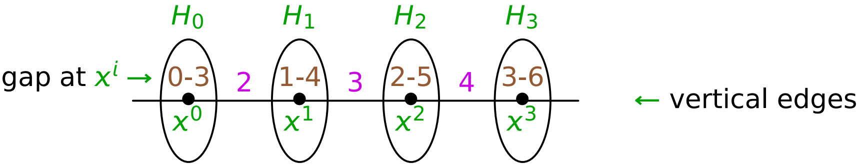

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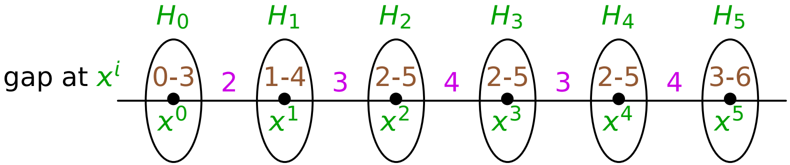

Step 3: The case $2m+1>p$.
Augment the case $2m-1$: insert two new layers next to a fixed H^i with the same r-edge-coloring as H^i. Go down once between the new layers and then continue up.

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To get $\chi'_c(C_{2k+1} \square C_{2m+1}) \leq 4+1/q$, the theorem requires $2m+1 \geq p$, where $r = p/q$. Thus we want

$$2m+1 \geq p = 4q+1 \geq 6k+1,$$

so $m \geq 3k+1$ suffices. ■