Linear Discrepancy and Products of Chains

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Abstract

The *uncertainty* of a linear extension of a poset P is the maximum difference between the positions of incomparable elements. The *linear discrepancy* of P, denoted $\mathrm{ld}(P)$, is the minimum uncertainty over all linear extensions. We prove that $\mathrm{ld}(\mathbf{k}\times\mathbf{k}\times\mathbf{k}) = \frac{3}{4}k^3 + O(k^2)$ and $\mathrm{ld}(\mathbf{k}\times\mathbf{k}\times\mathbf{k}\times\mathbf{k}) = \frac{7}{8}k^4 + O(k^3)$, where $\mathbf{k}$ is a k-element chain (the upper bound generalizes to $\mathbf{k}^d$). Also, $\mathrm{ld}(P) \le n - 1 - \lfloor h/2 \rfloor$ when P has n elements and height h. If every element is incomparable to at most t others, then $\mathrm{ld}(P) \le t$ if P is an interval order, and $\mathrm{ld}(P) \le \lfloor (3t-1)/2 \rfloor$ when P has width 2. All bounds are sharp. Finally, we show that $\mathrm{ld}(P)$ can be approximated quickly within a factor of 3 by computing the uncertainty of any linear extension.

1 Introduction

In any company, the employees form a poset, with $x \prec y$ if y is more valuable to the company than x. The relation is a partial order, because it may be impossible to tell which of two given employees is more valuable. Clearly y should be paid more than x if $x \prec y$. Also, to be "fair", one wants salaries of incomparable employees not to differ by much.

Using this (and other examples) as motivation, Tanenbaum, Trenk, and Fishburn [4] introduced "linear discrepancy" of posets, which corresponds to the situation where the salary values form an arithmetic progression. The salaries place the elements of P in a linear order that preserves the order relations in P. The problem is to choose such a linear order minimizing the maximum difference between the positions of incomparable elements.

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More precisely, we use $\preceq$ or $\preceq_P$ for the order relation in a poset P, with $\prec$, $\succ$, $\succeq$ having the natural meanings. All our posets are finite. A *linear extension* L of P is an order-preserving linear ordering on the elements of P; if $x \prec y$ in P, then x appears before y in L. The *uncertainty* of a linear extension L of P, written $\mathrm{uncert}(L)$, is the maximum difference between the positions in L of incomparable elements of P (by convention, we set $\mathrm{uncert}(\mathbf{k}) = 0$, where $\mathbf{k}$ denotes a chain of k elements). Finally, the *linear discrepancy* of P, written $\mathrm{ld}(P)$, is the minimum uncertainty over all linear extensions of P. The name reflects the fact that $\mathrm{ld}(P)$ is a measure of how far P is from being a linear order.

Linear discrepancy is related to bandwidth of graphs. A linear extension is an order-preserving bijection from P to $\{1, \ldots, |P|\}$. Like a linear extension of a poset, a *numbering* of an n-vertex graph G is a bijection from $V(G)$ to $\{1, \ldots, n\}$. The *bandwidth* of G, denoted $B(G)$, is the minimum, over all numberings of G, of the maximum difference between the labels on adjacent vertices. Thus the uncertainty of any linear extension of P is an upper bound on the bandwidth of its incomparability graph G, which yields $\mathrm{ld}(P) \geq B(G)$. For linear discrepancy, we are restricted to using the numberings of $V(G)$ that are linear extensions of P, so the parameters might be unequal.

Nevertheless, Fishburn, Tanenbaum, and Trenk [2] proved that always $\mathrm{ld}(P) = B(G)$ (and hence two posets with isomorphic incomparability graphs have the same linear discrepancy). For a disjoint union of chains of sizes $r_1, \ldots, r_t$, the linear discrepancy is thus the bandwidth of the complete multipartite graph $K_{r_1,\ldots,r_t}$. The value is $n - 1 - \lfloor r_1/2 \rfloor$, where $n = \sum r_i$ and $r_1 = \max\{r_i\}$; this is easy to prove in either the linear discrepancy or the bandwidth context. Often, computing the linear discrepancy involves determining how much can be saved over the trivial bound $n - 1$, which is the value for antichains.

In this paper, we first discuss general bounds on linear discrepancy. The *height* and *width* of a poset are the maximum sizes of a chain and an antichain, respectively. We show that $\mathrm{ld}(P) \leq n - 1 - \lfloor h/2 \rfloor$ when P has n elements and height h. Equality is achievable for all n and h using disjoint unions of chains, in fact for any width between $\lceil n/h \rceil$ and $n - h + 1$.

In terms of other parameters, Tanenbaum et al. [4] asked whether $\mathrm{ld}(P) \leq \lfloor (3t - 1)/2 \rfloor$ holds whenever every element of P is incomparable to at most t other elements (equality holds for $\mathbf{t} + \mathbf{t}$). We show that it does when P has width 2, and we prove the stronger bound $\mathrm{ld}(P) \leq t$ when P is an interval order (an *interval order* is a poset representable by assigning each element an interval on the real line so that $x \prec y$ if and only if the interval for x is entirely less than the interval for y).

We also show that the uncertainty of any two linear extensions of a given poset cannot differ by a factor of more than 3. That is, $\mathrm{ld}(P) \leq \mathrm{uncert}(L) \leq 3\mathrm{ld}(P)$ whenever L is a linear extension of P. This yields a very fast approximation algorithm for linear discrepancy (construct any linear extension and compute its uncertainty), which is of some interest since the relation to bandwidth implies that computing $\text{ld}(P)$ is NP-complete.

Our most difficult result is the asymptotic computation of linear discrepancy for products of three or four identical chains. The *product* $P \times Q$ of posets P and Q is a poset whose elements are the ordered pairs of elements from P and Q, with $(p, q) \preceq_{P\times Q} (p', q')$ if and only if $p \preceq_P p'$ and $q \preceq_Q q'$. The containment poset on the subsets of an n-element set is the product of n chains of size 2, so we write it as $\mathbf{2}^n$.

Tanenbaum, Trenk, and Fishburn [4] determined $\text{ld}(\mathbf{2}^n)$. For $n \geq 2$, the value is $2^n - 2^{(n+1)/2} - 1$ when n is odd, and it is $2^n - 3 \cdot 2^{n/2}$ when n is even. The value also is known for products of two chains of any sizes; Hong, Hyun, Kim, and Kim [3] proved that $\text{ld}(\mathbf{m} \times \mathbf{n}) = \lceil mn/2 \rceil - 2$ when m and n both exceed 1 and are not both 2 (obviously, $\text{ld}(\mathbf{2} \times \mathbf{2}) = 1$).

We study products of chains of equal size, writing $\mathbf{k}^d$ for the product of d chains that each have k elements. Our construction shows in general that $\text{ld}(\mathbf{k}^d) \leq (1 - 2^{-(d-1)})k^d + O(k^{d-1})$. This agrees with the result in [3] for $d = 2$. We prove also that if $d \in \{3, 4\}$, then $\text{ld}(\mathbf{k}^d) \geq (1 - 2^{-(d-1)})k^d - O(k^{d-1})$. Finally, when $d = 3$ and k is even, we obtain the exact value: $\text{ld}(\mathbf{k}^3) = \frac{3}{4}k^3 - \frac{1}{2}k^2 - 1$.

2 General Bounds on Linear Discrepancy

Our first easy lemma describes the usual way of proving upper bounds on $\text{ld}(P)$. We use it for an easy upper bound involving height. Then we prove special cases of an upper bound conjectured by Tanenbaum et al. [4]. Finally, we compare the uncertainties of linear extensions of a given poset to obtain a fast approximation algorithm for $\text{ld}(P)$.

An *ideal* in a poset P is a set $I \subseteq P$ such that $y \in I$ and $x \prec_P y$ implies $x \in I$. A *dual ideal* is a set $I' \subseteq P$ such that $y \in I$ and $x \succ_P y$ implies $x \in I$.

Lemma 2.1. *Let I be an ideal and I' a dual ideal in a poset P, both of size at least r. If $x \prec y$ for all $x \in I$ and $y \in I'$, then* $\text{ld}(P) \leq |P| - 1 - r$.

Proof. Let L_1, L_2, L_3 be linear extensions of the subposets $I, P - I - I', I'$, respectively. A linear extension L of P formed by concatenating L_1, L_2, L_3 satisfies the desired bound on the uncertainty, since no element is incomparable to an element more than $|P| - 1 - r$ positions before or after it in L. □

Proposition 2.2. *If P is an n-element poset with height h, then* $\text{ld}(P) \leq n - 1 - \lfloor h/2 \rfloor$*. For each w with $n/h \leq w \leq n - h + 1$, the bound is sharp for posets having width w.*

Proof. As suggested in the introduction, equality holds for the poset consisting of $\lfloor n/h \rfloor$ chains of size h and possibly one shorter chain; this is the smallest possible width for an n-vertex poset with height h. To increase the width without changing the linear discrepancy, simply remove one element from a chain of size at least two that is not the only chain of size h and make it an isolated element. The result is still a disjoint union of chains, and the linear discrepancy remains the same until width $n-h+1$ is reached and the process stops.

For the upper bound, we produce an ideal I and dual ideal I' as in Lemma 2.1, with $|I|, |I'| \geq \lfloor h/2 \rfloor$. Let C be a largest chain, having elements $x_1, \ldots, x_h$ from bottom to top. For $1 \leq i < \lfloor h/2 \rfloor$, let S_i be the set of elements that are below x_{i+1} and not below x_i; note that $x_1 \in S_1$. Similarly, for $1 \leq i < \lceil h/2 \rceil$, let T_i be the set of elements that are above x_{h-i} and not above x_{h+1-i}.

Form I by taking elements from $S_1, S_2, \ldots$ and so on until $\lfloor h/2 \rfloor$ elements are reached, adding $x_{\lfloor h/2 \rfloor}$ if $S_i = \{x_i\}$ for $1 \leq i < \lfloor h/2 \rfloor$. Choose the elements taken from the last subset used so that they form an ideal in it. Similarly, form I' by taking elements from $T_1, T_2, \ldots$ and so on until $\lceil h/2 \rceil$ elements are reached, with the elements from the last subset used forming a dual ideal in it.

By construction, I is an ideal, and I' is a dual ideal. Since C has size h, every element of I is less than every element of I', as desired. □

Tanenbaum et al. [4] asked whether $\mathrm{ld}(P) \leq \lfloor (3t-1)/2 \rfloor$ always holds when every element of P is incomparable to at most t other elements. The bound holds with equality for $P = \mathbf{t} + \mathbf{t}$. We show that the bound holds when P has width 2, and we prove the stronger bound $\mathrm{ld}(P) \leq t$ when P is an interval order.

We note first that always $\mathrm{ld}(P) \leq 2t-1$. For any linear extension, if incomparable elements x and y are at least $2t$ positions apart, then at most $2t-2$ of the $2t-1$ elements between them can be incomparable to either of them. This leaves an element z between them that is comparable to both, contradicting the incomparability of x and y.

Proposition 2.3. *If P is an interval order in which every element is incomparable to at most t others, then* $\mathrm{ld}(P) \leq t$*. Equality holds when P is an antichain.*

Proof. Antichains are interval orders and have the desired linear discrepancy.

To prove the bound, consider a linear extension L placing the elements in the same order as the left endpoints of the intervals in a representation of P as an interval order. If x and y are incomparable in P with x before y in L, then the interval for x extends to the beginning of the interval for y. Hence every element between x and y in L is incomparable to x. Since also y is incomparable to x, there are at most $t-1$ elements between them, and $\mathrm{uncert}(L) \leq t$. □

When an element is incomparable to t others, no matter where it is placed we have $\mathrm{uncert}(L) \geq t/2$. Any value r between $t/2$ and t is achievable as the linear discrepancy of the interval order P consisting of one chain of size $2t-2r$ and $2r-t+1$ isolated elements. An isolated element is incomparable to t others, and P is an interval order. Since P is a disjoint union of chains, the computation using size and height yields $\mathrm{ld}(P) = t+1-1-(2t-2r)/2 = r$.

Now consider posets of width 2. In a poset P, let $I(x) = \{z \in P\colon\ z\|x\}$, where we write $a\|b$ to mean that a and b are incomparable.

Theorem 2.4. *If P is a poset of width 2 in which every element is incomparable to at most t others, then $\mathrm{ld}(P) \le \lfloor (3t-1)/2 \rfloor$.*

Proof. We may assume that $I(z) \neq \varnothing$ for all $z \in P$, since if $I(z) = \varnothing$, then in any linear extensioni, all elements after z are comparable to all elements before z, and deleting z would change neither the linear discrepancy nor t.

By Dilworth's Theorem [1], P is covered by disjoint chains C_0 and C_1. Let C_0 consist of elements $x_1, \ldots, x_p$, indexed in increasing order in P, and similarly let C_1 consist of $y_1, \ldots, y_q$. Each $I(z)$ is a nonempty interval on the chain C_i not containing z.

Define a_j and b_j by $I(y_j) = \{x_{a_j}, \ldots, x_{b_j}\}$. The q-tuples a and b are nondecreasing, and hence $\frac{a_{j+1}+b_{j+1}}{2} \ge \frac{a_j+b_j}{2}$ for $1 \le j < q$. We thus can form a linear extension L of P by inserting y_j between x_r and x_{r+1}, where $r = \left\lfloor \frac{a_j+b_j}{2} \right\rfloor$. By hypothesis, $b_j - a_j \le t-1$. Let $g(z)$ be the position of z on L.

Every incomparable pair consists of some x_i and y_j with $a_j \le i \le b_j$. Let m_i be the number of elements of C_i between x_i and y_j on L, so $|g(x_i) - g(y_j)| = m_0 + m_1 + 1$. It suffices to show that $m_0 \le \left\lfloor \frac{b_j - a_j}{2} \right\rfloor$ and $m_1 \le t-1$, because then

$$\mathrm{ld}(P) \le \mathrm{uncert}(L) \le \max_{x_i\|y_j}\{m_0 + m_1 + 1\} \le \left\lfloor \frac{b_j - a_j}{2} \right\rfloor + t - 1 + 1 \le \left\lfloor \frac{3t-1}{2} \right\rfloor .$$

Since $m_0 \le b_j - r - 1$ if $g(y_j) < g(x_i)$ and $m_0 \le r - a_j$ if $g(x_i) < g(y_j)$, $m_0 \le \left\lfloor \frac{b_j-a_j}{2} \right\rfloor$.

For $g(y_j) < g(y_k) < g(x_i)$, having $y_j\|x_{i-1}$ (even when $g(x_{i-1}) < g(y_j)$) prevents $y_k \prec x_{i-1}$. If $y_k \succ x_{i-1}$, then $a_k \ge i$, but this yields $g(y_k) > g(x_i)$. Hence $y_k\|x_{i-1}$. Since also $y_j\|x_{i-1}$, we have $m_1 \le t-1$.

For $g(x_i) < g(y_k) < g(y_j)$, having $y_j\|x_i$ prevents $y_k \succ x_i$. If $y_k \prec x_i$, then $b_k < i$, but this yields $g(y_k) < g(x_i)$. Hence $y_k\|x_i$. Since also $y_j\|x_i$, again $m_1 \le t-1$. □

Finally, we compare $\mathrm{ld}(P)$ with the uncertainly of an arbitrary linear extension L. By definition, $\mathrm{uncert}(L)$ is an upper bound on $\mathrm{ld}(P)$. Surprisingly, it also provides a lower bound. It cannot be wrong by more than a factor of 3, and the ratio 3 is sharp.

Theorem 2.5. *If L is a linear extension of a poset P, then $\mathrm{ld}(P) \ge \frac{1}{3}\mathrm{uncert}(L)$, and the inequality is sharp for infinitely many posets.*

Proof. Sharpness. We construct an infinite family of posets and a linear extension L for each such that the uncertainty of L is 3 times the uncertainty of another linear extension L'

Fix k with $k \equiv 1 \mod 3$. Form P from a chain of $2k$ elements by adding two elements u and v such that u is below the top half of the chain, v is above the bottom half of the chain, and there are no other relations.

Linear extensions of P are formed by inserting u and v within the chain; we need only ensure that u is below the top half and v is above the bottom half of the chain. The linear extension L with u on the bottom and v on the top has uncertainty $2k+1$. Define L' by inserting u just above $(2k+1)/3$ elements of the chain and v just below $(2k+1)/3$ elements of the chain. The distances from u to the bottom of the chain, from u to v, and from v to the top of the chain all equal $(2k+1)/3$. Hence $\mathrm{uncert}(L) = 3\mathrm{uncert}(L')$, as desired.

The bound. Fix a linear extension L of P. Let x and y be incomparable elements that are farthest apart in L. Let Q be the subposet of P obtained by deleting the elements below x or above y in L, so $\mathrm{uncert}(L) = |Q| - 1$. Let L' be a linear extension of Q with $\mathrm{uncert}(L') = \mathrm{ld}(Q)$. Note that $Q \subseteq P$ implies $\mathrm{ld}(Q) \le \mathrm{ld}(P)$. For $x \in Q$, let $g(x)$ denote the position of x on L'.

If $g(x) > g(y)$, then partition $Q - \{x, y\}$ by letting $A = \{z \colon g(z) < g(y)\}$, $B = \{z \colon g(y) < g(z) < g(x)\}$, and $C = \{z \colon g(z) > g(x)\}$. Since x is minimal and y is maximal in Q, we have x incomparable to all of A, and y incomparable to all of B. Therefore, $\mathrm{ld}(Q) = \mathrm{uncert}(L') \ge |A| + |B| + 1$ and $\mathrm{ld}(Q) = \mathrm{uncert}(L') \ge |B| + |C| + 1$. Now

$$\mathrm{uncert}(L) = |Q| - 1 = |A| + |B| + |C| + 1 \le 2\mathrm{ld}(Q) \le 2\mathrm{ld}(P).$$

If $g(y) > g(x)$, then again partition $Q - \{x, y\}$, this time letting $A = \{z \colon g(z) < g(x)\}$, $B = \{z \colon g(x) < g(z) < g(y)\}$, and $C = \{z \colon g(z) > g(y)\}$. Again x is incomparable to all of A and y is incomparable to all of B. Also x is incomparable to y. Hence $\mathrm{ld}(Q)$ is bounded below by $|A|$, by $|C|$, and by $|B| + 1$. We compute

$$\mathrm{uncert}(L) = |Q| - 1 = |A| + |B| + 1 + |C| \le 3\mathrm{ld}(Q) \le 3\mathrm{ld}(P).$$

□

3 Upper Bound on $\mathrm{ld}(\mathbf{k}^d)$

In this section we construct a linear extension providing an upper bound on $\mathrm{ld}(\mathbf{k}^d)$. First we consider the case where k is even. The lower bounds in the next section will show that our construction is optimal when $d = 3$.

Henceforth, let $l = k/2$. Also let $[l] = \{1, \ldots, l\}$, let $[k] = \{1, \ldots, k\}$, and let $[l'] = \{l+1, \ldots, k\}$. Note that $\mathbf{k}^d$ is simply $[k]^d$ under the coordinatewise (product) ordering; the rank or height of an element $(x_1,\dots,x_d)$ is $(\sum_{i=1}^d x_i)-d$. We focus on the bottom and top "orthants", but we also need certain sets outside these orthants. Let

$$\begin{aligned}
A&=[l]^d \qquad \text{and} \qquad A'=[l']^d,\\
\hat{A}&=\{x\in[k]^d\colon\ x_d\ge l+1 \text{ and } x_i\le l \text{ for } i<d\},\\
\hat{A}'&=\{x\in[k]^d\colon\ x_1\le l \text{ and } x_i\ge l+1 \text{ for } i>1\},\\
B_j&=\{x\in[k]^d\colon\ x_j=l+1 \text{ and } x_i\le l \text{ for } i\ne j\},\\
B'_j&=\{x\in[k]^d\colon\ x_j=l \text{ and } x_i\ge l+1 \text{ for } i\ne j\}.
\end{aligned}$$

Each member of $\{A,A',\hat{A},\hat{A}'\}$ has size l^d, while each B_j or B'_j has size l^{d-1}. Elements of the "unprimed" sets have low rank in $\mathbf{k}^d$, while those of the "primed" sets have high rank. By definition, the following properties hold:

(1) Every element of A is below every element of $A'\cup\bigcup_{j=2}^d B'_j$.
(2) Every element of A' is above every element of $A\cup\bigcup_{j=1}^{d-1} B_j$.
(3) If $x\in B_i$ and $y\in B'_j$, then $x\prec y$ if $i\ne j$ and $x\|y$ if $i=j$.
(4) If $d\ge 3$, then the sets $A,A',\hat{A},\hat{A}',B_1,\dots,B_{d-1},B'_2,\dots,B'_d$ are pairwise disjoint.

Note that A is an ideal and A' is a dual ideal. Since all of A is below all of A' and both have size l^d, Lemma 2.1 implies that $\mathrm{ld}(\mathbf{k}^d)\le(1-1/2^d)k^d-1$. Here we are subtracting l^d from the trivial bound k^d-1. Careful use of the positions near A on the bottom and A' on the top will enable use to subtract more than twice as much.

The *lexicographic order* on a subset of $[k]^d$ is defined by $x<y$ if x is smaller in the first coordinate where x and y differ.

Definition 3.1. Let $X=A\cup\hat{A}\cup\bigcup_{j=1}^{d-1}B_j$ and $X'=A'\cup\hat{A}'\cup\bigcup_{j=2}^{d}B'_j$. For $d\ge 3$ and k even, the *modified lex order* on $[k]^d$ is a bijection $f\colon\ [k]^d\to\{1,\dots,k^d\}$ defined as follows. The numbering gives the smallest $2l^d+(d-1)l^{d-1}$ values to X and the largest $2l^d+(d-1)l^{d-1}$ values to X'. Within X, the sets $A,B_1,\dots,B_{d-1},\hat{A}$ appear in that order, and within each of these $d+1$ sets the elements are numbered in lexicographic order. Similarly, within X' the sets $\hat{A}',B'_1,\dots,B'_{d-1},A'$ appear in that order, and within each of these $d+1$ sets the elements are numbered in lexicographic order. The elements of $[k]^d-X-X'$ are assigned the remaining (intermediate) values in lexicographic order.

We will show that f defines a linear extension of $\mathbf{k}^d$ and compute its uncertainty to obtain an upper bound on $\mathrm{ld}(\mathbf{k}^d)$. When $d=2$, the sets $\hat{A}$ and $\hat{A}'$ are identical, and $B_1\cap B'_d=\{(l+1,l)\}\ne\varnothing$, so in this case the numbering must be defined differently. Since $\mathrm{ld}(\mathbf{k}^2)$ was already determined in [3], we omit this degenerate case.

Lemma 3.2. *When k is even and $d \geq 3$, the modifed lexicographic order f in Definition 3.1 specifies a linear extension of $\mathbf{k}^d$.*

Proof. Because the subsets we have ordered have been defined to be disjoint, and lexicographic order defines a bijection on any set of distinct d-tuples, the numbering f is a bijection. If $x \prec y$ in $\mathbf{k}^d$, then also $x < y$ in the lexicographic order, so lexicographic order on the elements of any subposet is a linear extension of that subposet.

A numbering specifies a linear extension if and only if each initial segment of the resulting ordering is an ideal. By symmetry, it suffices to show that no element in any of $A, B_1, \ldots, B_{d-1}, \hat{A}$ is above any element that comes in a later set in the partition used to define f. First, A is an ideal; no later element is less than an element of A.

Every element not in A has some coordinate greater than l. For each element of $\bigcup_{j=1}^{d-1} B_j$, there is one such coordinate, and it has the minimum possible value, $l+1$. In particular, for $x \in B_j$ all elements below x lie in $A \cup B_j$.

Finally, consider $x \in \hat{A}$. Every element below x has value at most l in each coordinate other than the last. Hence every such element is in $A \cup \hat{A}$. $\square$

Theorem 3.3. *If k is even and $d \geq 3$, then*

$$\mathrm{ld}(\mathbf{k}^d) \leq \left(1 - \frac{1}{2^{d-1}}\right) k^d - \left(\frac{d-1}{2^{d-1}}\right) k^{d-1} - 1.$$

Proof. With $l = k/2$, by Lemma 3.2 it suffices to show that

$$|f(x) - f(y)| \leq k^d - 1 - 2l^d - (d-1)l^{d-1}$$

for any two incomparable elements x and y, where f is the modified lex order of Definition 3.1. The desired bound equals $|Y| - 1$, where $Y = [k]^d - X - X'$. Hence we may ignore incomparable pairs contained in Y. Since all of A is below all of $A' \cup \bigcup_{j=2}^{d} B_j'$ and all of A' is above all of $A' \cup \bigcup_{j=2}^{d} B_j'$, it suffices to consider pairs (x, y) of the following types:

(1) $x \in A$ and $y \in \hat{A}'$, (2) $x \in A'$ and $y \in \hat{A}$, (3) $x \in B_i$ and $y \in B_j'$.

For type (3) we may assume that $i = j$, since otherwise $x \prec y$. All of B_j is incomparable to all of B_j', so it suffices to compare the first element of B_j with the last element of B_j', where $2 \leq j \leq d-1$. There are $l^d + (j-1)l^{d-1}$ elements before B_j and $l^d + (d-j)l^{d-1}$ after B_j'. Hence the reduction from $k^d - 1$ in $|f(x) - f(y)|$ is $2l^d + (d-1)l^{d-1}$, as desired.

Consider type (1): $x \in A$ and $y \in \hat{A}'$; by symmetry, the argument also handles pairs of type (2). Since $x_i \leq l < l+1 \leq y_i$ for $2 \leq i \leq d$, incomparability requires $x_1 > y_1$. Note that f respects lexicographic order within A and within $\hat{A}'$, with the first coordinate most important. The largest difference occurs for x_1 earliest and y_1 latest, so we may assume that $x_1 = y_1 + 1$. Similarly, we may assume that $x_i = 1$ for $i > 1$ and $y_i = k$ for $i > 1$.

Both A and $\hat{A}'$ have l^{d-1} elements with each given value of the first coordinate. Hence $y_1 l^{d-1}$ elements come before x under f, and $|X'| - y_1 l^{d-1}$ elements come after y. Hence again the savings from $k^d - 1$ in $|f(x) - f(y)|$ is $|X'|$, which is $2l^d + (d-1)l^{d-1}$, as desired. □

Corollary 3.4. *For all k (when d is fixed and $d \geq 3$), $\mathrm{ld}(\mathbf{k}^d) \leq (1 - 2^{-(d-1)})k^d + O(k^{d-1})$.*

Proof. The linear discrepancy of a poset is at least the linear discrepancy of any subposet. Hence when k is odd it suffices to apply Theorem 3.3 to $k+1$ and observe that $(k+1)^d \in k^d + O(k^{d-1})$. □

4 Lower Bounds on $\mathrm{ld}(\mathbf{k}^d)$ for $d \in \{3, 4\}$

Our approach to computing $\mathrm{ld}(\mathbf{k}^d)$ is to show that optimal extensions have a special form. Lower bounds for $d \in \{3, 4\}$ result from studying extensions of this form more closely.

The first part of the argument is valid for general chain products. Consider the poset $\prod_{j=1}^d \mathbf{k}_j$, consisting of $[k_1] \times \cdots \times [k_d]$ under the coordinate-wise ordering. We focus on special elements of $\prod_{j=1}^d \mathbf{k}_j$. The element $z_j^{(i)}$ has value 1 in every coordinate except for i in coordinate j. The element $\hat{z}_j^{(i)}$ has the maximum value in every coordinate except for i in coordinate j. Note that $z_j^{(i)} < \hat{z}_{j'}^{(i')}$ if $j \neq j'$, and $z_j^{(i)} \| \hat{z}_{j'}^{(i')}$ if $j = j'$ and $i > i'$.

We write x_j for the value in coordinate j of a k-tuple x, while $z_j^{(i)}$ denotes a particular element of $\prod [k_i]$. To avoid confusion, we will not use z as a generic element. Let $A = \{z_j^{(i)} \colon 1 < i \leq k_j \text{ and } 1 \leq j \leq d\}$, and let $\hat{A} = \{\hat{z}_j^{(i)} \colon 1 \leq i < k_j \text{ and } 1 \leq j \leq d\}$. We call $A \cup \hat{A}$ the *special elements* of $\prod_{j=1}^d \mathbf{k}_j$. Here we omitted $z_j^{(1)}$ and $\hat{z}_j^{(k_j)}$, which are the bottom and top elements of the poset regardless of j. Disjointness of A and $\hat{A}$ requires $d \geq 3$.

For a linear extension L, the *associated numbering* f is the injection from $\mathbf{k}^d$ to $[k^d]$ that gives the position of an element on L. Without reference to L, the term for such a function in [4] is *linear labeling*; it is an order-preserving injection from a poset P to $[|P|]$. Without further remark, we henceforth always use f and f' to denote the associated numberings for linear extensions named L and L', respectively.

Lemma 4.1. *If L is a linear extension of $\prod_{j=1}^d \mathbf{k}_j$, then* $\mathrm{uncert}(L)$ *equals $f(\hat{z}_j^{(i-1)}) - f(z_j^{(i)})$ for some choice of i and j.*

Proof. Consider any elements x and y such that $x \| y$ and $f(x) < f(y)$. Since $x \| y$, we have $y_j < x_j = i$ for some i and j. Now $\hat{z}_j^{(i-1)} \succeq y$ and $z_j^{(i)} \preceq x$. Hence $f(\hat{z}_j^{(i-1)}) - f(z_j^{(i)}) \geq f(y) - f(x)$, and $\hat{z}_j^{(i-1)} \| z_j^{(i)}$. □

Let x and y be incomparable elements of $A \cup \hat{A}$. A linear extension L' is an *x, y-switch* of an extension L if (1) no element of $A \cup \hat{A}$ lies between x and y on L, and (2) L' agrees with L except for reordering the the interval from x to y so that x and y switch order.

Lemma 4.2. *If a linear extension L of $\prod_{j=1}^{k} \mathbf{k}_j$ has no elements of $A \cup \hat{A}$ between incomparable elements x and y of $A \cup \hat{A}$, then there is an x, y-switch of L.*

Proof. Let x and y be named so that $f(x) < f(y)$. Let $D[y] = \{w\colon\ w \preceq y\}$ and $U[x] = \{w\colon\ w \succeq x\}$. Let $S = \{w\colon\ f(x) \le f(w) \le f(y)\}$.

We construct an extension L' from L. Let L' agree with L outside S. Within S, let L' put the set $D[y] \cap S$ first, then $S - D[y] - U[x]$, then $U[x] \cap S$. Within each of these sets, the order on L' agrees with the order on L.

No element of $D[y]$ lies above any element outside $D[y]$, and no element of $U[y]$ lies below any element outside $U[y]$. Hence L' is a linear extension. As required, x and y have switched order, and the elements not originally between them have the same positions, so L' is an x, y-switch of L. □

By Lemma 4.1, we need only compare positions of incomparable elements of the form $\hat{z}_j^{(i-1)}$ and $z_j^{(i)}$ to determine the uncertainty of a linear extension. To obtain lower bounds, it helps to restrict attention to extensions with more constraints on the ordering of $A \cup \hat{A}$. For $a \in A$ with $a = z_j^i$, let $\hat{a} = \hat{z}_j^{(i-1)}$, and say that a and $\hat{a}$ are in *axis j*. To motivate this terminology, note that translating the multidimensional grid of elements to have its lowest point at the origin would indeed leave the elements of A lying along the coordinate axes.

Lemma 4.3. *Among linear extensions of $\prod_{j=1}^{d} \mathbf{k}_j$, uncertainty is minimized by an extension putting $\hat{a}$ later than a for all $a \in A$ and having the elements of $\hat{A}$ in the same order as the corresponding elements of A, except possibly for one fixed axis having some elements after the other elements of A and before the other elements of $\hat{A}$.*

Proof. Let L be an extension with associated numbering f. If $f(\hat{a}) < f(a)$, then no element of $A \cup \hat{A}$ lies between $\hat{a}$ and a on L, since elements of A in other axes are below $\hat{a}$ (in the poset), elements of $\hat{A}$ in other axes are above a, and elements of $A \cup \hat{A}$ in the same axis as a and $\hat{a}$ are below both or above both.

The argument of Lemma 4.1 produces $b \in A$ such that $f(\hat{b}) - f(b) > f(a) - f(\hat{a})$. By Lemma 4.2, there is an $a, \hat{a}$-switch of L. Since $f(\hat{b}) - f(b) > f(a) - f(\hat{a})$, Lemma 4.1 implies that the uncertainty of the resulting extension is no larger than that of L. Hence we can "correct" all reversed pairs without increasing the uncertainty, and the first claim holds.

Next we want to put the elements of $\hat{A}$ in the same order as the corresponding elements of A, without increasing the uncertainty. When two elements of $\hat{A}$ are "out-of-order", we have $x, y, \hat{x}, \hat{y}$ with $f(x) < f(y) < f(\hat{y}) < f(\hat{x})$; call this an *inversion* in $\hat{A}$. If no special elements lie between $\hat{y}$ and $\hat{x}$ on L, then Lemma 4.2 permits a $\hat{y}, \hat{x}$-switch. Let L' be the resulting extension. For $a \in A - \{x, y\}$, we have $f'(\hat{a}) - f'(a) = f(\hat{a}) - f(a)$. Also, $f'(\hat{x}) - f'(x) < f(\hat{x}) - f(x)$ and $f'(\hat{y}) - f'(y) < f(\hat{x}) - f(y) < f(\hat{x}) - f(x)$. By Lemma 4.1, $\mathrm{uncert}(L') \le \mathrm{uncert}(L)$.

If no element of A appears after any element of $\hat{A}$, then switches as described above eliminate all inversions. Otherwise, since special elements from A and $\hat{A}$ in different axes are comparable, all special elements appearing in such pairs lie in a single axis. They form a merging of some initial portion of $\hat{z}_j^{(1)}, \ldots, \hat{z}_j^{(k_j-1)}$ with some terminal portion of $z_j^{(2)}, \ldots, z_j^{(k_j)}$, such that always $z_j^{(i)}$ comes before $\hat{z}_j^{(i)}$.

No element of A in another axis follows $\hat{z}_j^{(1)}$. No element of $\hat{A}$ in another axis precedes $z_j^{(k_j)}$. Hence switches as described in the first part of the proof ensure that pairs in $\hat{A}$ not involving elements in axis j appear in the same order as the corresponding elements in A. $\square$

The next definition and lemma about the computation of uncertainty do not require the restricted form obtained in Lemma 4.3. We consider any linear extension L of $\prod_{j=1}^{d} \mathbf{k}_j$.

Definition 4.4. For $x \in A$, let $r(x) = \prod_{j=1}^{d} k_j - 1 - [f(\hat{x}) - f(x)]$. Also, let $t_j(x)$ be the maximum i such that $z_j^{(i)}$ precedes x in L, and let $\hat{t}_j(x)$ be the maximum i such that $\hat{z}_j^{(k_j+1-i)}$ follows $\hat{x}$ in L. Call $\{t_j(x)\}_{j=1}^{d}$ and $\{\hat{t}_j(x)\}_{j=1}^{d}$ the *thresholds* for x, and let $T(x) = \prod_{j=1}^{d} t_j(x) + \prod_{j=1}^{d} \hat{t}_j(x)$. For $x, y \in A$, say that y *captures* x if $f(y) < f(x) < f(\hat{x}) < f(\hat{y})$.

Note that each threshold for x is at least 1, and the number of elements outside the interval $[x, \hat{x}]$ in L is $r(x)$. By Lemma 4.1, $\mathrm{ld}(\prod_{j=1}^{d} \mathbf{k}^d) = \prod_{j=1}^{d} k_j - 1 - r(x)$ for some extension L and $x \in A$. Hence we seek upper bounds on $r(x)$ for appropriately chosen x.

Lemma 4.5. *If $x \in A$, then $r(x) \le T(x)$. Furthermore, if x is in axis j, then $t_j(x) + \hat{t}_j(x) = k_j$. Also $t_{j'}(x) + \hat{t}_{j'}(x) \le k_{j'} + 1$ for $j' \ne j$ if x is not captured by an element in axis j'.*

Proof. The quantity $T(x)$ is the sum of the sizes of copies of the posets $\prod \mathbf{t}_j(x)$ and $\prod \hat{\mathbf{t}}_j(x)$ at the bottom and top of $\prod_{j=i}^{d} \mathbf{k}_j$, respectively. The lower product contains all u such that $u_j \le t_j(x)$ for all j. Any other element v has $v_j > t_j(x)$ for some j. Changing the coordinates other than v_j to 1 yields an element of A that comes after x and is less than v; hence v is not before x. The symmetric argument holds for the top of L.

If $x = z_j^{(i)}$, then $t_j(x) = i - 1$ and $\hat{t}_j(x) = k_j - (i - 1)$. If $t_{j'}(x) = i'$, then let $y = z_{j'}^{(i')}$. If $\hat{x}$ is not captured by y, then $f(\hat{y}) < f(\hat{x})$, and hence $\hat{t}_{j'}(x) \le k_{j'} - (i' - 1)$. $\square$

After Lemma 4.5, we have a bound on $T(x)$ of the form $\prod_{j=1}^{d} t_j + (k_r - t_r) \prod_{j \ne r} (k_j + 1 - t_j)$ when x is in axis r and is not captured. If we can choose x so that each t_j is near $k_j/2$, then the resulting upper bound on $T(x)$ will be near $2^{-(d-1)} \prod_{j=1}^{d} k_j$, which would give the desired lower bound on $\mathrm{ld}(\prod_{j=1}^{d} k_j)$ asymptotically.

Unfortunately, optimizing the choice of x for a particular extension and computing the resulting bound on $T(x)$ are both very messy when the chain sizes are distinct. Therefore, we henceforth assume that all chain sizes equal k. With this assumption, we can simplify subsequent numerical arguments by proving that there is an optimal linear extension in which no special element is captured. When captured elements are allowed, the numerical arguments for $d \leq 4$ can be handled by extra cases, but this restriction on the optimal extensions may be useful in subsequent attacks on $\mathrm{ld}(\mathbf{k}^d)$ for general d.

Theorem 4.6. *Some linear extension of $\mathbf{k}^d$ with minimum uncertainty puts corresponding elements of A and $\hat{A}$ in the same order; equivalently, no element of A is captured.*

Proof. Consider an extension L with minimum uncertainty. By the argument in Lemma 4.3, it suffices to obtain an extension in which all of $\hat{A}$ appears after all of A, without increasing the uncertainty. If L does not have this form, then by Lemma 4.3 there is one axis j and integers r and s with $1 \leq r \leq k-1$ and $2 \leq s \leq k$ such that $\hat{z}_j^{(1)}, \ldots, \hat{z}_j^{(r)}$ is mixed with $z_j^{(s)}, \ldots, z_j^{(k)}$, following the other elements of A and preceding the other elements of $\hat{A}$. The first of these elements is $\hat{z}_j^{(1)}$, and the last is $z_j^{(k)}$.

Let S be this set of "in-between" elements. It suffices to obtain an extension L' such that (1) L' agrees with L before the first element of S and after the last element of S, (2) L' puts all elements of $S \cap A$ before all elements of $S \cap \hat{A}$, and (3) $\mathrm{uncert}(L') = \mathrm{uncert}(L)$.

It is easy to ensure (1) and (2). Since no element of A is above any element of $\hat{A}$, the subposet on S has an extension with $S \cap A$ before $S \cap \hat{A}$, and the remaining elements between the first and last elements of S on L can be inserted iteratively between the elements below them and the elements above them.

The problem is to ensure (3) for the resulting extension L'. Possibly $f'(\hat{a}) - f'(a) > f(\hat{a}) - f(a)$ for $a = z_j^{(i+1)}$ with $1 \leq i \leq \min\{r, s-1\}$. It suffices to produce an element $b \in A$ such that $f'(\hat{a}) - f'(a) < f'(\hat{b}) - f'(b) = f(\hat{b}) - f(b)$.

Let b be the first element of A on L not in axis j; let j' be its axis. Note that $\hat{b}$ is the first element of $\hat{A}$ not in axis j, and S is between b and $\hat{b}$. Since $f'(\hat{b}) > f'(\hat{a})$, the desired inequality holds unless $f(a) = f'(a) < f'(b) = f(b)$. It suffices to prove that L' has more elements between $\hat{a}$ and $\hat{b}$ than between a and b.

Elements between a and b on L and L' cannot be above any element of A outside axis j and hence exceed 1 only in position j. Thus they can only be $z_j^{(i+2)}, \ldots, z_j^{(s-1)}$, and there are at most $k-3$ of them. On the other hand, all elements of $\mathbf{k}^d$ between $z_j^{(k)}$ and $\hat{b}$ lie between $\hat{a}$ and $\hat{b}$ on both L and L'. Note that $\hat{b}$ and $z_j^{(k)}$ agree in positions j and j' (with values k and 1). In all other positions (at least one), $z_j^{(k)}$ has 1, while $\hat{b}$ has k. The elements we are counting have any value between 1 and k there, so there are at least $k-2$ of them.

The argument for elements of $S \cap \hat{A}$ is symmetric to this. □

In this proof, we used the assumption of equal chain sizes in comparing $k-2$ with $k-3$. Even with $d>3$, the argument could fail if the troublesome axis corresponded to a chain much longer than the others.

With optimal extensions restricted as in Lemma 4.6, the lower bounds for $\mathrm{ld}(\mathbf{k}^3)$ and $\mathrm{ld}(\mathbf{k}^4)$ are easy to prove. Unfortunately, these simple arguments do not work for $d>4$, because the resulting bound on $T(x)$ for a single element x is not tight enough. Proving sharp bounds for larger d seems to require considering more elements on the extensions.

Theorem 4.7. $\mathrm{ld}(\mathbf{k}^3) \geq \frac{3}{4}k^3 - \frac{1}{2}k^2 - 1$ *when k is even.*

Proof. Let $m = k/2$ (suggesting "middle"). By Lemma 4.6, it suffices to consider a linear extension with corresponding elements of A and $\hat{A}$ in the same order and find an element $x \in A$ such that $r(x) \leq \frac{1}{4}(k^3+2k^2) = 2m^3+2m^2$.

Consider the following elements:

$$a=(m+1,1,1),\ b=(1,m+1,1),\ c=(1,1,m+1)$$

$$\hat{a}=(m,k,k),\ \hat{b}=(k,m,k),\ \hat{c}=(k,k,m).$$

By symmetry, we may assume that $f(a)<f(b)<f(c)$. By Lemma 4.5, it suffices to show that $T(b) \leq 2m^3+2m^2$.

The definition of b yields $t_2(b)=\hat{t}_2(b)=m$. Since b is uncaptured, $t_1(b)+\hat{t}_1(b) \leq k+1$ and $t_3(b)+\hat{t}_3(b) \leq k+1$, by Lemma 4.5. Let $t_1(b)=m+1+i$ and $t_3(b)=m-j$. Since $f(a)<f(b)<f(c)$, we have $t_1(b) \geq m+1$ and $t_3(b) \leq m$, so i and j are nonnegative. Now

$$\begin{aligned} T(b) &\leq (m+1+i)m(m-j)+(m-i)m(m+1+j) \\ &= 2m^3+2m^2-2ij-i-j \leq 2m^3+2m^2. \end{aligned}$$

□

The idea for $d=4$ is similar, but the increase in the number of coordinates requires a more careful choice of the key element x before bounding $T(x)$.

Theorem 4.8. $\mathrm{ld}(\mathbf{k}^4) \geq \frac{7}{8}k^4 - \frac{3}{8}k^3 - \frac{1}{4}k^2 - 1$ *when k is even.*

Proof. Again let $m=k/2$, and let L be an extension without captured elements. By Lemma 4.5, it suffices to find $x \in A$ such that $T(x) \leq \frac{1}{8}k^4 + \frac{3}{8}k^3 + \frac{1}{4}k^2 = 2m^4+3m^3+m^2$.

Consider the following elements:

$$a=(m+1,1,1,1),\ b=(1,m+1,1,1),\ c=(1,1,m+1,1),\ d=(1,1,1,m+1)$$

$$\hat{a}=(m,k,k,k),\ \hat{b}=(k,m,k,k),\ \hat{c}=(k,k,m,k),\ \hat{d}=(k,k,k,m).$$

By symmetry, we may assume that $f(a)<f(b)<f(c)<f(d)$.

We choose a special element $x \in A$ such that $f(b) \le f(x) \le f(c)$. For all such x, let $\alpha(x) = t_2(x) + t_3(x) - k$; we seek x such that $\alpha(x) = 0$. Let p and q be the numbers of elements of A in axes 2 and 3, respectively, in the part of L strictly between b and c. Since $t_2(b) = t_3(c) = m$, we have $\alpha(b) = -q$ and $\alpha(c) = p + 1 > 0$. Also, $t_2 + t_3$ increases by at most 1 with each step, increasing only when leaving an element in axis 2 or 3. Let x be the last element is in axis 2 or 3 such that $\alpha(x) = 0$. (For example, $x = b$ when $q = 0$, and x is the last element in axis 2 or 3 before c when $p = 0$ and $q > 0$).

Since $\alpha(x) = 0$, we have $t_2(x) = m + h$ and $t_3(x) = m - h$ for some nonnegative h. This simple expression for $t_2(x)t_3(x)$ is the reason for choosing this x. We bound $T(x)$. Since $t_2(x) = m+h$ and $t_3(x) = m-h$ and x is uncaptured, Lemma 4.5 implies that $\hat{t}_2(x) = m-h$ and $\hat{t}_3(x) \le m + 1 + h$ if x is in axis 2, but $\hat{t}_3(x) = m + h$ and $\hat{t}_2(x) \le m + 1 - h$ if x is in axis 3. We keep the two cases together by writing $\hat{t}_2(x)\hat{t}_3(x) = (m - h)(m + 1 + h)$, where $0 \le h \le m$ if x is in axis 2, but $0 \ge h \ge -m$ if x is in axis 3.

Also $t_1(x) + \hat{t}_1(x) \le k + 1$ and $t_4(x) + \hat{t}_4(x) \le k + 1$. Since $f(a) < f(x)$, we have $t_1 = m+1+i$ with $0 \le i < m$, and $f(x) < f(d)$ implies $t_4(x) = m - j$ with $0 \le j \le m$. Now

$$T(x) \le (m + 1 + i)(m^2 - h^2)(m - j) + (m - i)(m - h)(m + 1 + h)(m + 1 + j).$$

To complete the proof, we maximize the upper bound over all choices of h, i, j. For fixed h and j, the derivative with respect to i is $(m^2 - h^2)(-2j - 1) - (m - h)(m + 1 + j)$, which is negative regardless of the sign of h. Hence we set $i = 0$. Now the derivative of $(m + 1)(m^2 - h^2)(m - j) + m(m - h)(m + 1 + h)(m + 1 + j)$ with respect to j is $h^2 - mh$, which has opposite sign to h. We set $j = 0$ when x is in axis 2 and $j = m$ when x is in axis 3. Now the upper bound is $m(m + 1)(m + h)2(m + .5 - h)$ with $0 \le h \le m$ or $m(m - h)(m + 1 + h)(2m + 1)$ with $0 \ge h \ge -m$. Each is maximized (for integers) when $h = 0$, where they both equal $2m^4 + 3m^3 + m^2$. □

Finally, we combine the upper and lower bounds. When k is even we have $\mathrm{ld}(\mathbf{k}^d) \le \frac{2^{d-1}-1}{2^{d-1}}k^3 - \frac{d-1}{2^{d-1}}k^2 - 1$ from Theorem 3.3. Combining this with Theorem 4.7 and Theorem 4.8 yields the following statement.

Corollary 4.9. *When k is even,* $\mathrm{ld}(\mathbf{k}^3) = \frac{3}{4}k^3 - \frac{1}{2}k^2 - 1$ *and* $\mathrm{ld}(\mathbf{k}^4) = \frac{7}{8}k^4 - \frac{3}{8}k^2 - O(k)$. □

Incorporating the possibility of odd k yields only an asymptotic solution.

Corollary 4.10. *For all k,* $\mathrm{ld}(\mathbf{k}^3) = \frac{3}{4}k^3 + O(k^2)$ *and* $\mathrm{ld}(\mathbf{k}^4) = \frac{7}{8}k^4 + O(k^3)$.

Proof. Let $\overline{\mathbf{k}}$ and $\hat{\mathbf{k}}$ denote the chains of sizes $k-1$ and $k+1$. When k is odd, $\overline{\mathbf{k}}^d \subseteq \mathbf{k}^d \subseteq \hat{\mathbf{k}}^d$. Hence

$$\tfrac{3}{4}(k-1)^3 - \tfrac{1}{2}(k-1)^2 - 1 = \mathrm{ld}(\overline{\mathbf{k}}^3) \le \mathrm{ld}(\mathbf{k}^3) \le \mathrm{ld}(\hat{\mathbf{k}}^3) = \tfrac{3}{4}(k+1)^3 - \tfrac{1}{2}(k+1)^2 - 1$$

and

$$\tfrac{7}{8}(k-1)^4 - \tfrac{3}{8}(k-1)^3 - \tfrac{1}{4}(k-1)^2 - 1 \leq \mathrm{ld}(\overline{\mathbf{k}}^4) \leq \mathrm{ld}(\mathbf{k}^4) \leq \mathrm{ld}(\hat{\mathbf{k}}^4) \leq \tfrac{7}{8}(k+1)^4 - \tfrac{3}{8}(k+1)^3 - 1.$$

□

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