

Decomposition of sparse graphs, with application to game coloring number

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Abstract

Let k be a nonnegative integer, and let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$. We prove that every simple graph with maximum average degree less than m_k decomposes into a forest and a subgraph with maximum degree at most k . It follows that every simple graph with maximum average degree less than m_k has game coloring number at most $4 + k$.

1 Introduction

The *game coloring number* of a graph G is defined using a two-person game to produce an ordering of the vertices of G . In the *ordering game* on G , Alice and Bob take turns choosing vertices from the set of unchosen vertices of G . This places the vertices in a linear order L , with $x < y$ if x is chosen before y . The *back degree* of a vertex x with respect to L , written $b_L(x)$, is the number of neighbors of x that precede x in L . The *back degree* of L , written $b(L)$, is $\max_{x \in V(G)} b_L(x)$. Alice's goal is to minimize $b(L)$, and Bob's goal is to maximize it.

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The *game coloring number* $\text{col}_g(G)$ of G is defined to be $1+k$, where k is the least integer such that Alice can guarantee $b(L) \leq k$. Equivalently, k is the greatest integer such that Bob can guarantee $b(L) \geq k$. The game coloring number was first formally defined in [5] as a tool to prove upper bounds on the game chromatic number [1]. It is the game version of the *coloring number*, which is defined to be $1 + \min_L b(L)$ and received its somewhat unfortunate name because it is an upper bound on the chromatic number. Since the ordering game does not directly produce colorings, and the concept now has other applications, “game ordering number” might be more accurate, but we will use the traditional term and notation. Note that the definition of back degree makes multiple edges and loops irrelevant in the game, so we use the model of “graph” that forbids these.

Recently, Zhu [6] proved that $\text{col}_g(G) \leq 17$ when G is planar. Borodin et al. [2] and He et al. [4] improved this for planar graphs with large girth. Their results use structural properties of planar graphs with large girth. A *decomposition* of a graph G is a set of edge-disjoint subgraphs whose union is G .

Theorem 1 *Let G be a planar graph with girth at least g .*

1. [2] *If $g \geq 9$, then G decomposes into a forest and a matching.*
2. [4] *If $g \geq 7$, then G decomposes into a forest and a graph with maximum degree 2.*
3. [4] *If $g \geq 5$, then G decomposes into a forest and a graph with maximum degree 4.*

Two lemmas show the importance, for game coloring number, of decomposing a graph into a forest and a graph with small maximum degree.

Lemma 1 (Zhu [5]) *If a graph G decomposes into subgraphs G_1 and G_2 , then $\text{col}_g(G) \leq \text{col}_g(G_1) + \Delta(G_2)$.*

Lemma 2 (Faigle et al. [3]) *If T is a forest, then $\text{col}_g(T) \leq 4$.*

Combining these two lemmas with Theorem 1 yields

Corollary 1 ([2, 4]) *If G is a planar graph with girth at least 5, then $\text{col}_g(G) \leq 8$. The upper bound decreases to 6 for girth at least 7 and to 5 for girth at least 9.*

In this note, we derive bounds on the game coloring number of sparse graphs using this decomposition approach. We measure sparseness by avoidance of dense subgraphs. The *maximum average degree* of a graph G , written $\text{Mad}(G)$, is the largest average degree among the subgraphs of G . That is,

$$\text{Mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}.$$

We can now state our main result.

Theorem 2 *Every graph G satisfying $\text{Mad}(G) < \frac{4(k+1)(k+3)}{k^2+6k+6}$ decomposes into a forest and a subgraph with maximum degree at most k .*

The proof of Theorem 2 is inductive, using reducible configurations (discussed in Section 2) and a discharging procedure (discussed in Section 3). The key structures in the proof are “banks” and “cores” that allow the transfer of charge over unlimited distances.

In contrast to Theorem 1, Theorem 2 is not restricted to planar graphs. However, Theorem 1 does not follow from Theorem 2 merely by using the usual inequality $\text{Mad}(G) \leq 2g/(g-2)$ that holds for every planar graph G having girth at least g .

Theorem 2 combines with Lemmas 1 and 2 to yield:

Corollary 2 *If a graph G satisfies $\text{Mad}(G) < \frac{4(k+1)(k+3)}{k^2+6k+6}$, then $\text{col}_g(G) \leq 4+k$.*

Let $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$. The proof of Theorem 2 makes it clear that m_k is the largest value for which our approach works. However, we do not know whether the result is sharp. A construction gives an upper bound: the complete bipartite graph $K_{2,2k+2}$, with maximum average degree $\frac{4k+4}{k+2}$, has no such decomposition. Letting $f(k)$ be the infimum of $\text{Mad}(H)$ over graphs H that do not decompose into a forest and a graph with maximum degree at most k , our results yield

$$4 - \frac{8k+12}{k^2+6k+6} \leq f(k) \leq 4 - \frac{4}{k+2}.$$

Answering the following question would solve the problem completely.

Question 1 *For every k , what are the graphs with smallest maximum average degree that do not decompose into a forest and a subgraph with maximum degree at most k ?*

2 Reducible Configurations and Special Subgraphs

Let $d(x)$ denote the degree of a vertex x in a graph G . A k -vertex is a vertex of degree k . A $\geq k$ -vertex or $\leq k$ -vertex is a vertex of degree at least k or at most k , respectively. An (a, b) -alternating cycle is an even cycle that alternates between a -vertices and b -vertices. A k_l -vertex is a k -vertex adjacent to exactly l vertices of degree 2.

Let $\sigma(G) = |V(G)| + |E(G)|$. We prove our theorem by considering a counterexample with smallest $\sigma(G)$. Since $\text{Mad}(H) \leq \text{Mad}(G)$ when H is an induced subgraph of G , every proper subgraph of G decomposes into a forest and a graph with maximum degree at most k , but G has no such decomposition. We use this to exclude various configurations from G . Since $m_0 = 2$, and $\text{Mad}(G) < 2$ implies that G is a forest, we may assume that $k \geq 1$.

Lemma 3 *A minimal counterexample G contains (a) no 1-vertices, (b) no adjacent $\leq(k+1)$ -vertices, and (c) no $(k+2, 2)$ -alternating cycles.*

PROOF. When G contains such a configuration, we decompose an appropriate subgraph of G into a forest F and a subgraph D with maximum degree at most k .

(a) Let v be a 1-vertex in G , with u the neighbor of v . The decomposition of $G - u$ into F and D extends to G by adding the edge uv to F , a contradiction.

(b) Let u and v be adjacent $\leq(k+1)$ -vertices in G . Consider the decomposition of $G - uv$ into F and D . If u or v has k neighbors in D , then add uv to F ; otherwise, add uv to D . This extends the decomposition to G , a contradiction.

(c) Let C be a $(k+2, 2)$ -alternating cycle in G . Consider the decomposition of $G - E(C)$ into F and D . We may assume that each $(k+2)$ -vertex on C has an incident edge in F , since otherwise we can move an incident edge from D to F . Now adding one perfect matching in C to D and the other to F extends the decomposition to G , a contradiction. \square

To apply the discharging method, we first give each vertex in a supposed minimal counterexample G a “charge” equal to its degree. We then use “discharging rules” to redistribute the charge (without changing the total charge) so that the resulting charge on each vertex will be at least m_k . To facilitate the discharging argument, we will also move some charge to special subgraphs. Since they start with charge 0 and end with nonnegative charge, the final configuration contradicts the hypothesis on $\text{Mad}(G)$ by having more charge than the initial configuration. This completes the proof.

By Lemma 3, we may assume that our minimal counterexample has no 1-vertex, no adjacent $\leq(k+1)$ -vertices, and no $(k+2, 2)$ -alternating cycle. We use discharging to show that $\text{Mad}(G) \geq m_k$ when G is such a graph. This prohibits counterexamples to Theorem 2.

Given such a graph G , let X be the set of all $(k+2)$ -vertices in G that are adjacent to at least $k+1$ vertices of degree 2, and let Y be the set of all 2-vertices adjacent to at least one vertex of X . Define the *bank* of G to be the maximal bipartite subgraph of G with partite sets X and Y . When $k \geq 4$, we modify this slightly by restricting X to use only the $(k+2)_{k+2}$ -vertices, omitting the $(k+2)_{k+1}$ -vertices.

A cycle in the bank would be a $(k+2, 2)$ -alternating cycle in G , which is forbidden. Hence the bank is a forest. We call each component of the bank a *core*. By construction, each vertex of X has at least $k+1$ neighbors in the bank ($k+2$ when $k \geq 4$). Hence each leaf in the bank belongs to Y .

The initial charge at each vertex of G is its degree, and also each core has initial charge 0. We use three discharging rules (plus a special rule when $k \geq 4$) to redistribute charges. In most discharging arguments, movement of charge is local. Assigning charge to cores permits charge to move long distances within a core via a simple rule.

3 The Discharging Argument

A vertex that belongs to no core is said to be *adjacent to a core* C if it is adjacent in G to a 2-vertex belonging to C .

For the computations, recall that $m_k = \frac{4(k+1)(k+3)}{k^2+6k+6}$. Each discharging rule R_i specifies moving a constant amount r_i of charge. These constants r_1, r_2, r_3, r_4 are defined in terms of m_k by

$$r_1 = \frac{m_k - 2}{2}, \quad r_2 = 1 - r_1 - \frac{m_k}{k+3}, \quad r_3 = m_k - (k+2)(1 - r_1), \quad r_4 = \frac{m_k - 3}{3}.$$

Note that r_4 is negative for $k \leq 3$ but positive for $k \geq 4$.

The discharging rules are as follows, with R4 used only when $k \geq 4$. We add R4 because $m_k > 3$ if and only if $k \geq 4$, so when $k \geq 4$ the 3-vertices need to gain charge.

- R1.** Every $\geq(k+2)$ -vertex gives r_1 to each neighbor that is a 2-vertex.
- R2.** If C is a core, v is a $\geq(k+2)$ -vertex that belongs to no core, and v is adjacent to l 2-vertices of C , then v gives lr_2 to C .
- R3.** Every core gives r_3 to each of its own $(k+2)_{k+2}$ -vertices.
- R4.** (For $k \geq 4$ only.) Every $\geq(k+2)$ -vertex gives r_4 to each neighboring 3-vertex.

The proof of Theorem 2 is now completed by proving the following lemma.

Lemma 4 *If a graph G satisfies (a), (b), and (c) of Lemma 3, then $\text{Mad}(G) \geq m_k$.*

PROOF. As described above, we give initial charge $d(v)$ to each vertex v and initial charge 0 to each core C . After applying the discharging rules, let $\omega(v)$ and $\omega(C)$ denote the final charges. We prove that $\omega(v) \geq m_k$ for each vertex v and $\omega(C) \geq 0$ for each core C .

By (b), the neighbors of 2-vertices are $\geq(k+2)$ -vertices. Using R1, the final charge of each 2-vertex is $2 + 2r_1$, which equals m_k , as desired.

If $3 \leq d(v) \leq k+1$, then v does not give or receive charge, unless $d(v) = 3 < k$. Thus $\omega(v) = d(v) > m_k$ except in that case. If $d(v) = 3 < k$, then v is a $\leq(k+1)$ -vertex, so by (b) its neighbors are all $\geq(k+2)$ -vertices. Via R4 it receives $3r_4$, and hence $\omega(v) = 3 + 3r_4 = m_k$.

Now suppose that $d(v) \geq k+2$. Vertex v may lose charge to each neighbor, and v may lose additional charge when v is not in a core and its neighbors are. Since always $r_1 > r_4$, we may assume that each neighbor getting charge from v is a 2-vertex. Hence the maximum charge lost from v , via $\{R1, R2, R4\}$, is $d(v)(r_1 + r_2)$. Hence $\omega(v) \geq d(v)(1 - r_1 - r_2) = \frac{d(v)m_k}{k+3}$. If $d(v) \geq k+3$, then $\omega(v) \geq m_k$.

The case $d(v) = k + 2$ is more delicate. If v is not in a core, then the definition of the bank limits the number of 2-neighbors of v (to k if $k \leq 3$, to $k + 1$ if $k \geq 4$). If $k \leq 3$, then $\omega(v) \geq 2 + k(1 - r_1 - r_2) = m_k + 2 - \frac{3m_k}{k+3}$. The formula for m_k yields $2 - \frac{3m_k}{k+3} = \frac{2k^2}{k^2+6k+6} > 0$, and hence $\omega(v) > m_k$. If $k \geq 4$, then v may have one 3-neighbor in addition to the maximum number of 2-neighbors. Hence $\omega(v) \geq 1 + (k + 1)(1 - r_1 - r_2) - r_4 = m_k + 2 - \frac{2m_k}{k+3} - \frac{m_k}{3}$. The formula for m_k converts the last expression to $m_k + \frac{2k(k-2)}{3(k^2+6k+6)}$, and hence $\omega(v) > m_k$.

Suppose now that $d(v) = k + 2$ and v is in a core. If v is a $(k + 2)_{k+2}$ -vertex, then it loses r_1 exactly $k + 2$ times, but it loses nothing by R2 and gains r_3 by R3. Hence $\omega(v) = (k + 2)(1 - r_1) + r_3 = m_k$. If v is a $(k + 2)_{k+1}$ -vertex, then $k \leq 3$. Now v loses r_1 exactly $k + 1$ times by R1 and is unaffected by $\{R2, R3, R4\}$. Hence $\omega(v) = (k + 2) - (k + 1)r_1$. Using $r_1 = m_k/2 - 1$ and the formula for m_k , we compute $(k + 2) - (k + 1)r_1 - m_k = \frac{k^2}{k^2+6k+2} > 0$, and hence $\omega(v) > m_k$.

Finally, we check that $\omega(C) \geq 0$ when C is a core. We have observed (using (c)) that C is a tree whose leaves are 2-vertices in G and whose non-leaves are $(k + 2)$ -vertices in G . Within C the non-leaves have degree $k + 1$ or $k + 2$; let there be n_1 of the first type and n_2 of the second, and let n_0 be the number of leaves. Since C is a tree, its vertex degrees must sum to $2(n_0 + n_1 + n_2) - 2$, so we obtain $n_0 = (k - 1)n_1 + kn_2 + 2$. Since (b) guarantees that the neighbor of a leaf of C outside C is a $\geq (k + 2)$ -vertex, and it is not in a core, C receives $n_0 r_2$ via R2. Via R3, C distributes $n_2 r_3$. Since $n_0 > kn_2$, it suffices to have $kr_2 \geq r_3$. Using the definitions of r_3 , r_2 , and then r_1 in terms of m_k , we compute

$$kr_2 - r_3 = (2k + 2)(1 - r_1) - \frac{k}{k + 3}m_k = 4(k + 1) - \frac{k^2 + 6k + 6}{k + 3}m_k = 0.$$

We have shown that all vertices and cores have sufficient final charge. □

Note that r_1 is defined in terms of m_k so that 2-vertices have final charge m_k , and then r_2 is defined in terms of m_k and r_1 to give $(k + 3)$ -vertices final charge m_k . Next r_3 is defined in terms of these so that $(k + 2)_{k+2}$ -vertices in cores have final charge m_k , and r_4 is defined so that 3-vertices have final charge m_k when $k \geq 4$. Given all this, and the fact that n_1 may equal 0 in a core, m_k has been chosen as the largest value allowing us to guarantee nonnegative final charge for cores. In this sense the theorem cannot be improved using the present argument.

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