Acquisition Parameters of Graphs

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Results with or by
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Noah Prince, Chris Stocker, Paul S. Wenger,
Leslie Wiglesworth, Pratik Worah

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A person can give his or her votes to a friend
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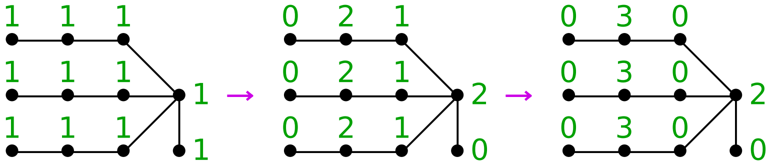
• We will study extremal problems for $\alpha(G)$.

An Example

Ex. $\alpha(T) = 4$, 11 vertices.



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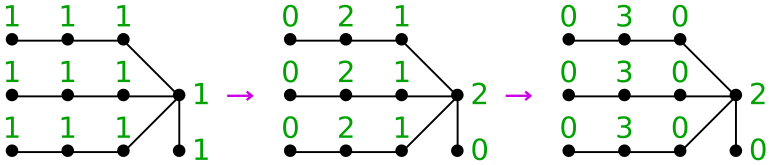

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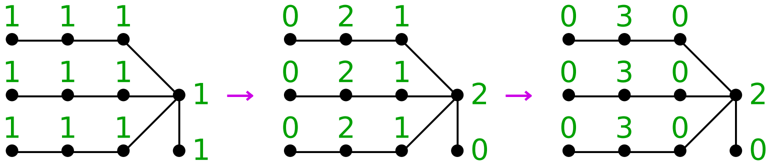

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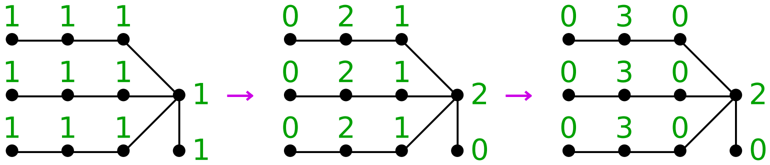

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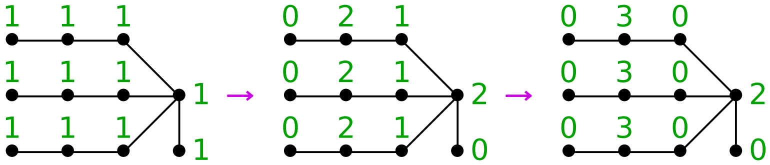

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game acquisition: move all weight, but two players Min and Max alternate moves — $a_g(G)$

Full Acquisition Number — Extremal Problem

All our graphs have n vertices.

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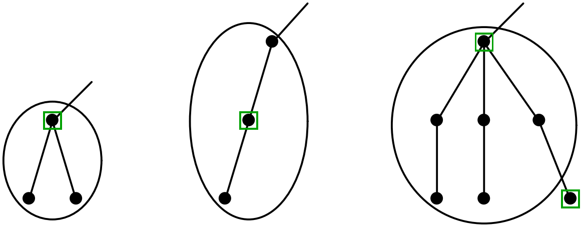
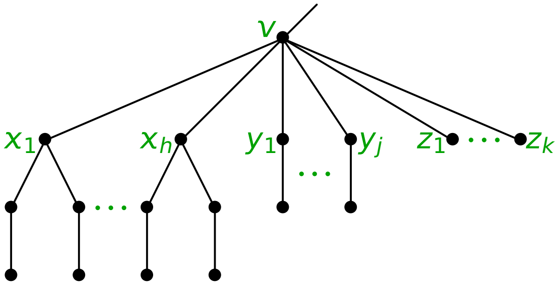


Hard Case:



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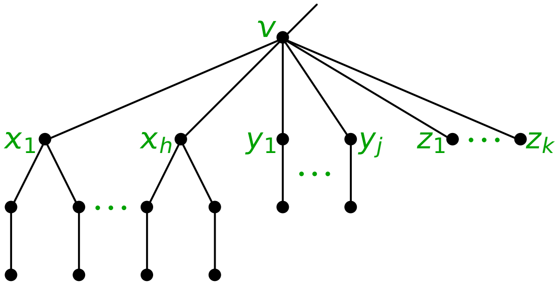
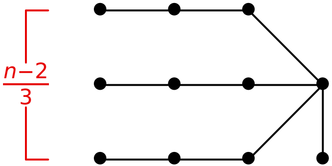


Sharpness:



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- View each initial unit of weight as a token or chip.

Lem. For full or partial acquisition, at most one chip can reach and leave a degree-2 vertex v, and it can only be the chip from a neighbor.

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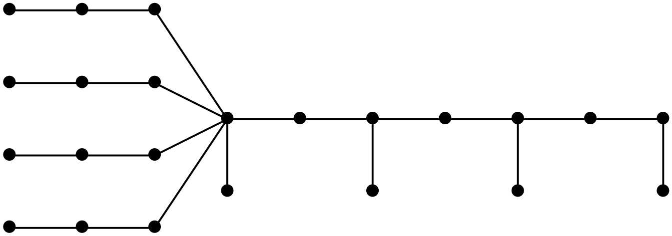
Thm. For $d \geq 3$ and $k \geq 6$, there is a tree T with $\Delta(T) = d$, $\text{diam}\, T \geq k$, and $a(T) = (n+1)/3$.

Trees with $\alpha(G)$ Large

Ex. The tree $T_{4,3}$.



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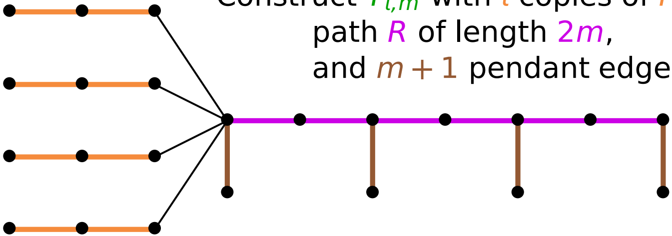

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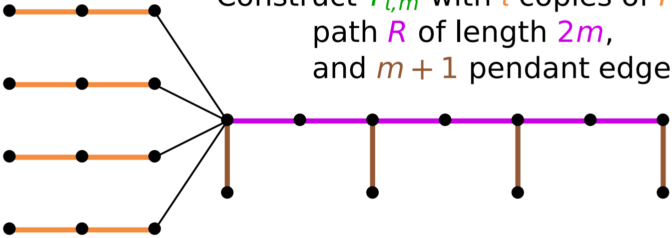

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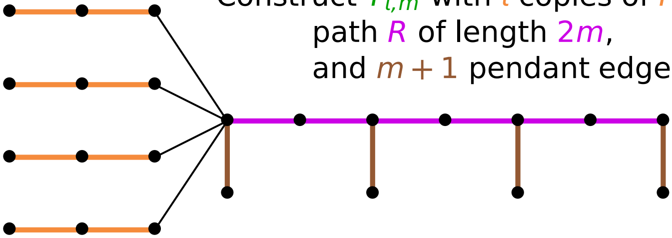

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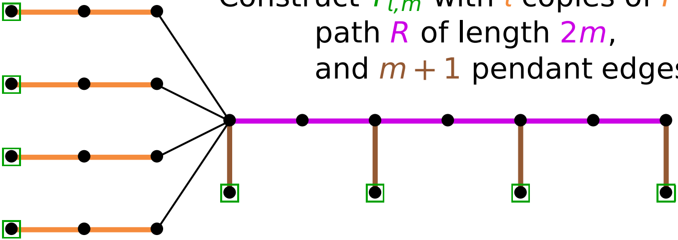

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Pf. Chips from marked vertices can never combine. ∎

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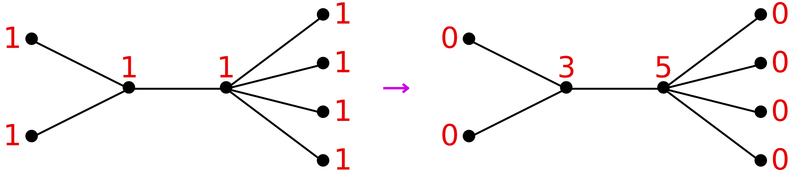

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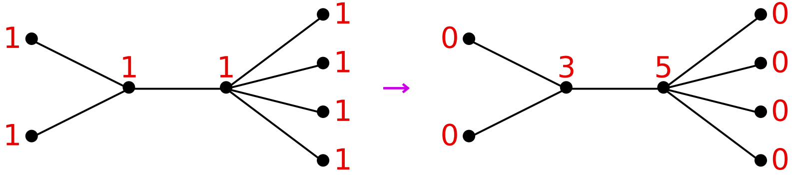

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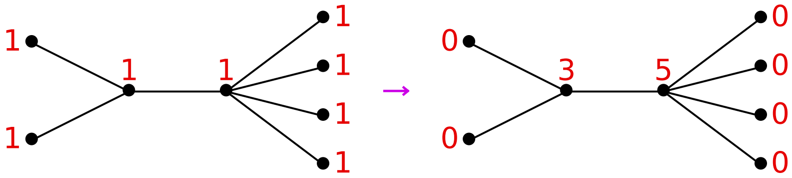

What about diameter 4 and diameter 5?

Thm. For diameter 4 or 5, $\max \alpha(T) = \Theta(\sqrt{n \lg n})$.

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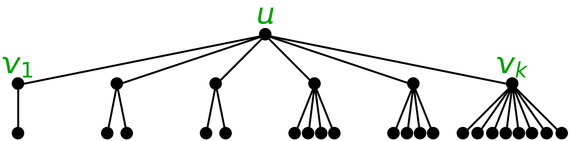

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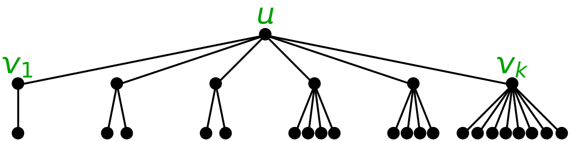

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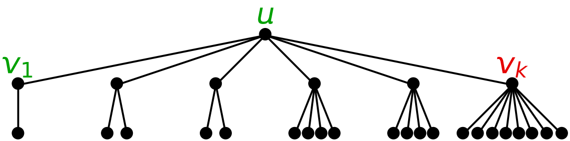

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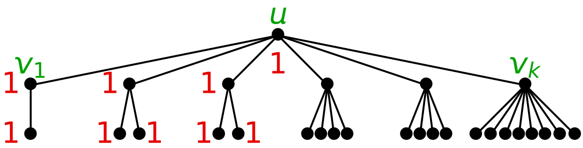

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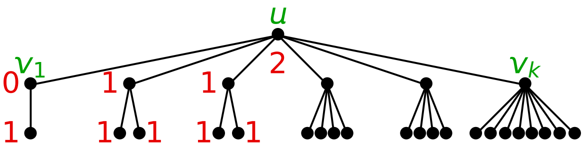

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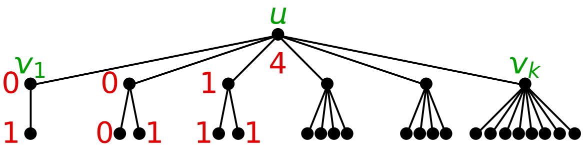

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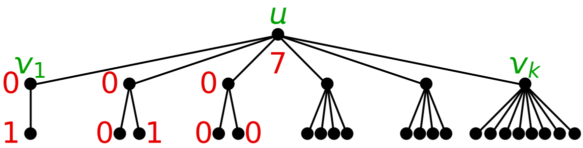

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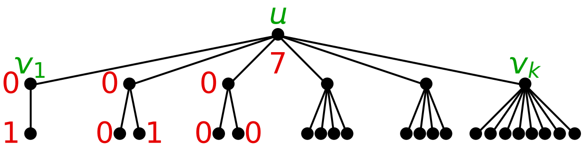

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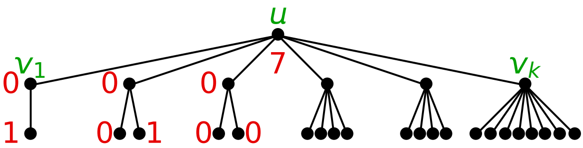

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$$m \le w_m < d(v_m) \le d(v_k) < 2\sqrt{n} < k/2.$$

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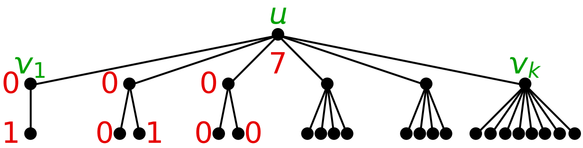

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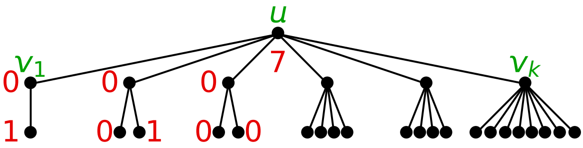

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Hence $a(T) < 1 + |S|(2n/k) < 2\sqrt{n \lg(2n)}$. ■

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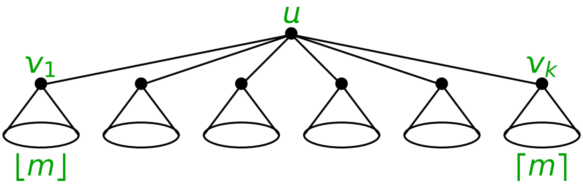

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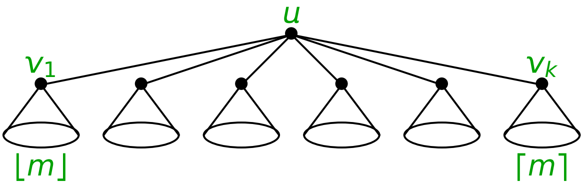

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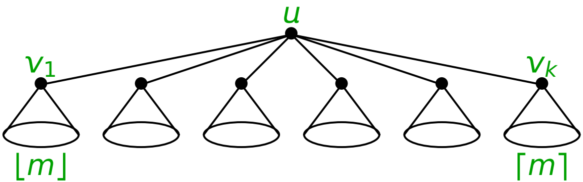

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Pf. Let $k = \sqrt{n \lg n}$ and $m = (n - k - 1)/k$.

Suppose u receives weight from q nbrs, say $v_1, \ldots, v_q$.

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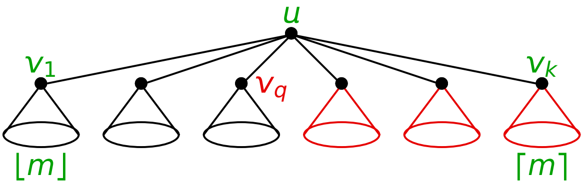

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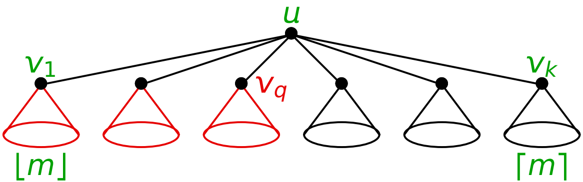

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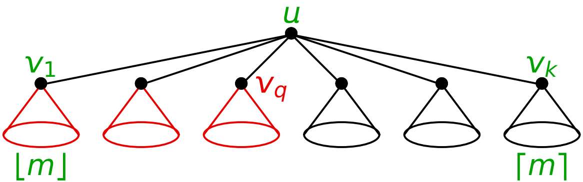

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#leaves stranded $\geq \sum_{i=1}^{q} \max\{d(v_i) - 2^{i-1}, 0\}$

$\geq m \lg n - 2m = \frac{n-k-1}{k}(\lg n - 2) \geq \frac{1}{20}\sqrt{n \lg n}$. ■

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Thm. (Lampert–Slater [1995]) Testing $\alpha(G) = 1$ on general graphs is NP-complete.

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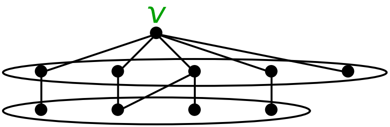

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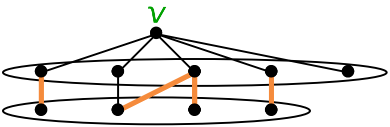

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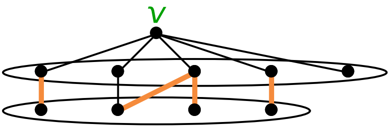

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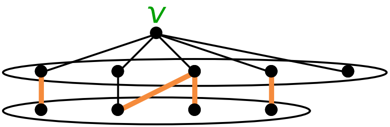

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Hence v can absorb the weight of neighbors with weight 1 and then absorb the remaining weight. ■

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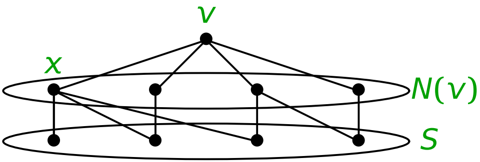
Lem. If G is $\frac{n-1}{2}$-regular and not C_5, then $\alpha(G) = 1$.



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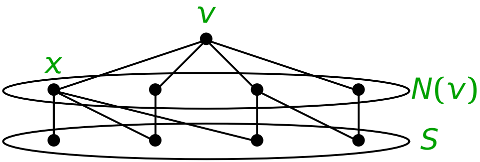

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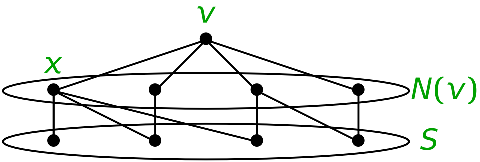

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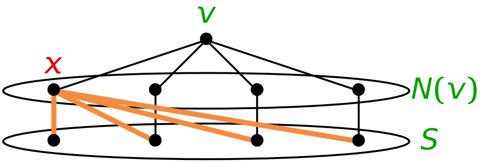

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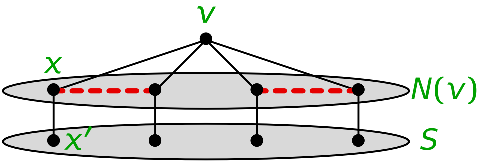

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Pf. The hypothesis gives a common nbr to nonadjacent vertices, so every neighborhood dominates.

Either $\Delta(G) \geq n/2$, or G is $(n-1)/2$-regular. ■

Results Not Presented

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Thm. If $p_n \geq \sqrt{3 \ln n / n}$, then almost always $\alpha(G) = 1$.

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Diameter 2

Thm. If $\operatorname{diam} G = 2$, then $\alpha(G) \leq 250 \lg n \lg \lg n$.
If $\operatorname{diam} G = 2$ and $C_4 \not\subseteq G$ and $\Delta(G) \geq 8$, then $\alpha(G) = 1$.

Open Problems (for Full Acquisition)

Conj. There is a constant c such that, if $\mathrm{diam} G = 2$, then $a(G) \leq c$.

Perhaps $c = 2$. This suffices for Moore graphs, polarity graphs, and graphs without 4-cycles.

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For $k = 2$, there exist G with $\alpha(G) > (\frac{1}{4} + \frac{1}{1024})n$ (binary trees with triangles at the leaves).

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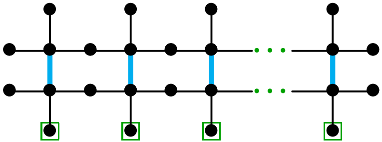
Ex. $a_p(G) \le a(G) \le M$, where M is the least number of double-stars that cover $V(G)$, and this may be sharp.



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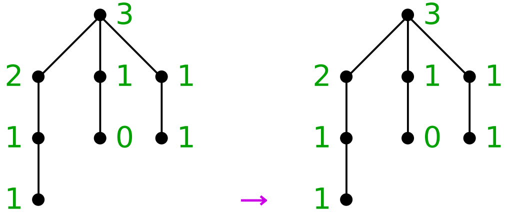

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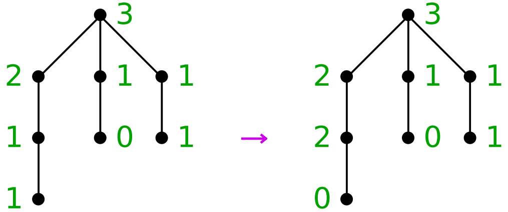

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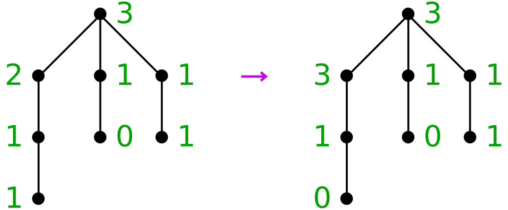

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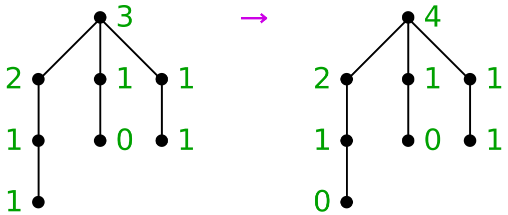

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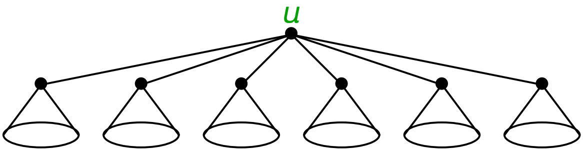

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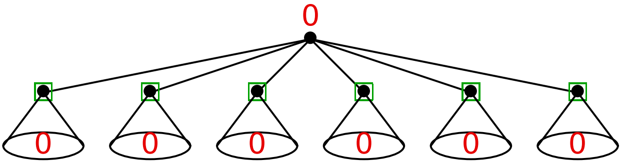

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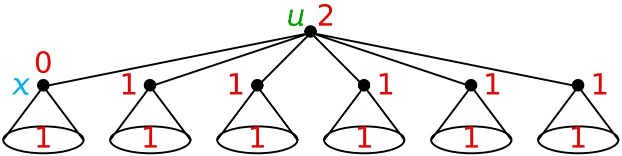

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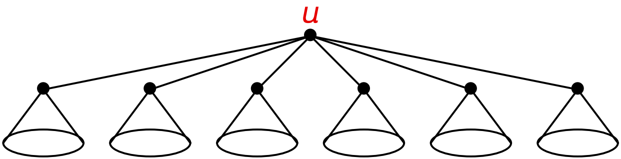

Construction: Make tree with $d(u) = m$ and $d(x) = m$ for $x \in N(u)$, so $n = m^2 + 1$. The first move involving u creates m components with positive weight.

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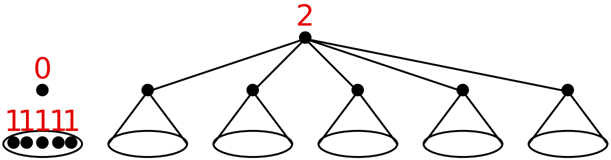

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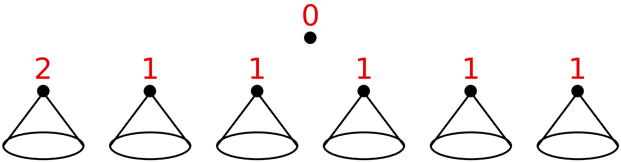

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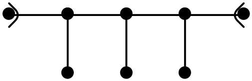

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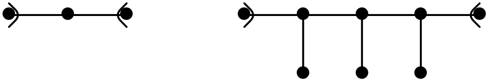

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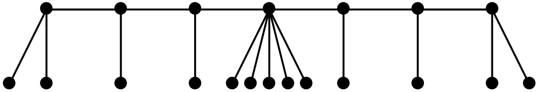

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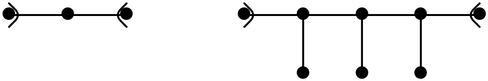

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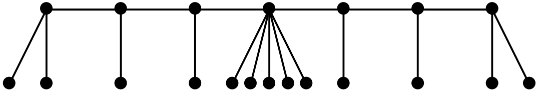

- Getting $a_p(T) = 1$ may need an edge both directions!

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Moving one chip from x_i to v_i turns T_i into an ascending tree plus $d_{T_i}(x_i) - 1$ isolated vertices. Hence

$$a_p(G) \le \sum_{i=1}^{m} \frac{n_i - 1}{k} \le \frac{n-1}{k}. \quad \blacksquare$$

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Pf. **(Idea)** Inductive construction of a tree. ■

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Open Problems:

Find general lower bounds for $a_p(G)$ in terms of other parameters of G.

Characterize the trees with $a_p(T) = 1$.

What is the complexity of recognizing $a_p(G) = 1$?

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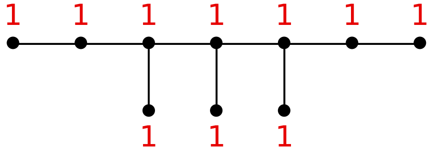

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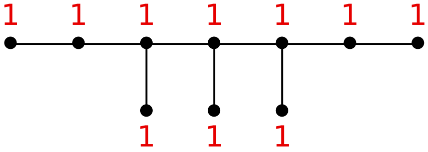
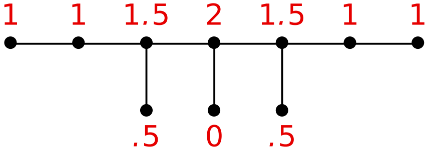
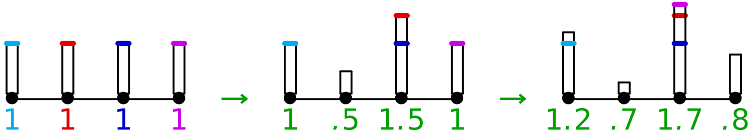


Move .5 twice to create an increasing tree with min difference .5: $a_f(G) = 1$.



Tool for Lower Bounds

Def. Represent the weight at a vertex by a tower of that height. The tops of the initial intervals are tips. When weight α is moved from u to v, the amount α is removed from the top of the tower at u and placed on top of the tower at v, along with any tips it contains.



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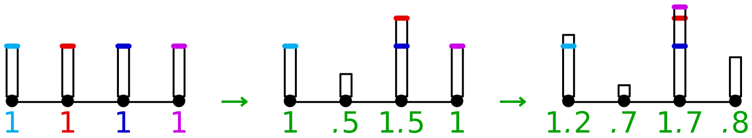

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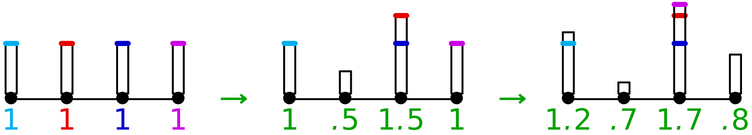

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To make $\alpha_f(G)$ small, tips must reach common towers.

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- Similarly, $\alpha_f(C_n) = \lceil n/4 \rceil$.

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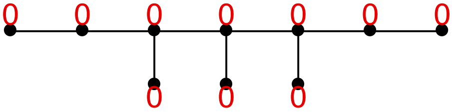

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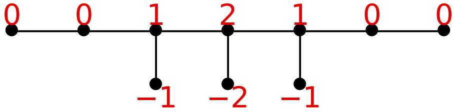

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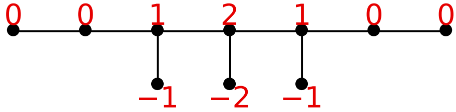

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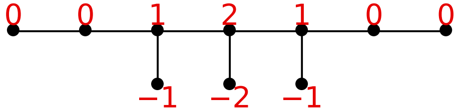

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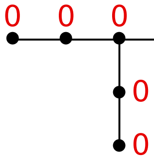
3) Inductively produce an ascending tree in this model.

An Example



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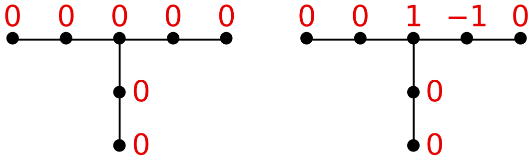

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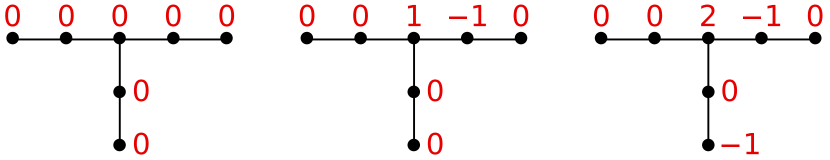

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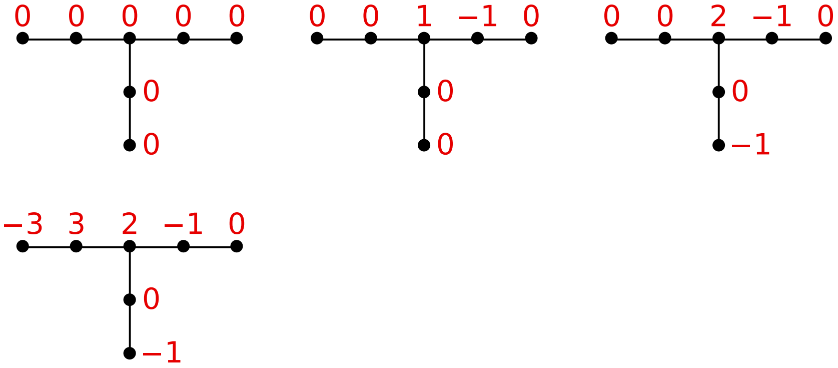

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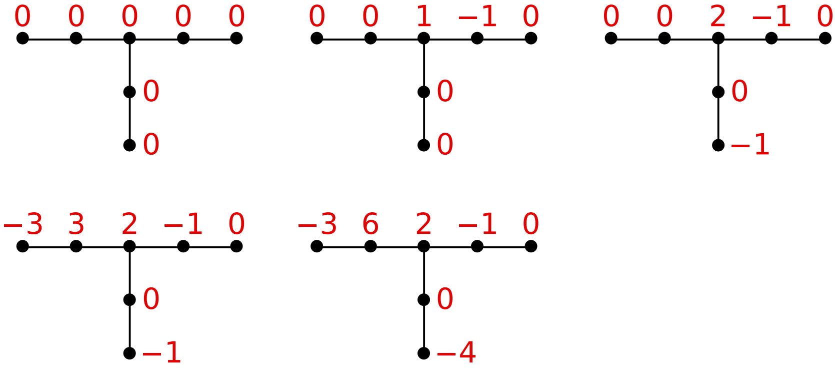

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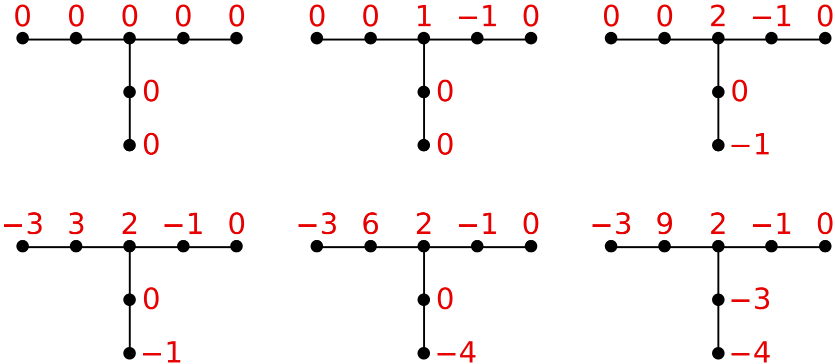

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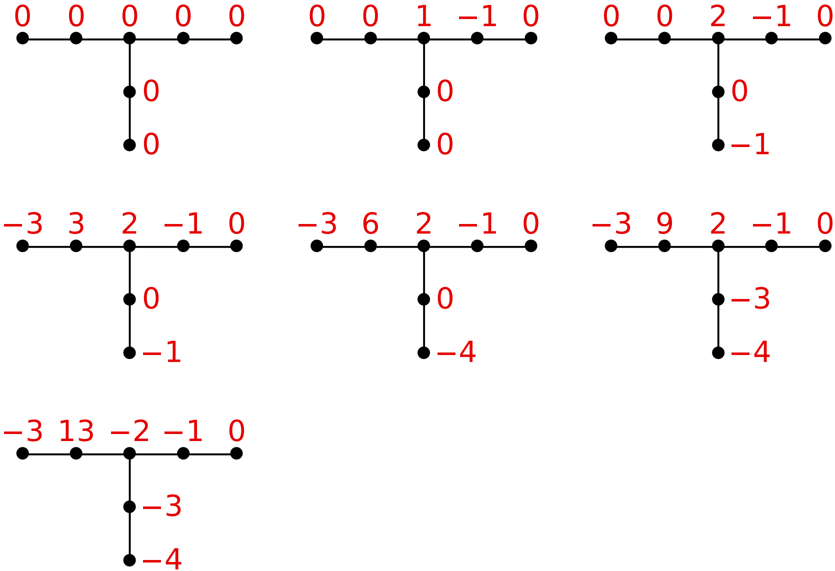

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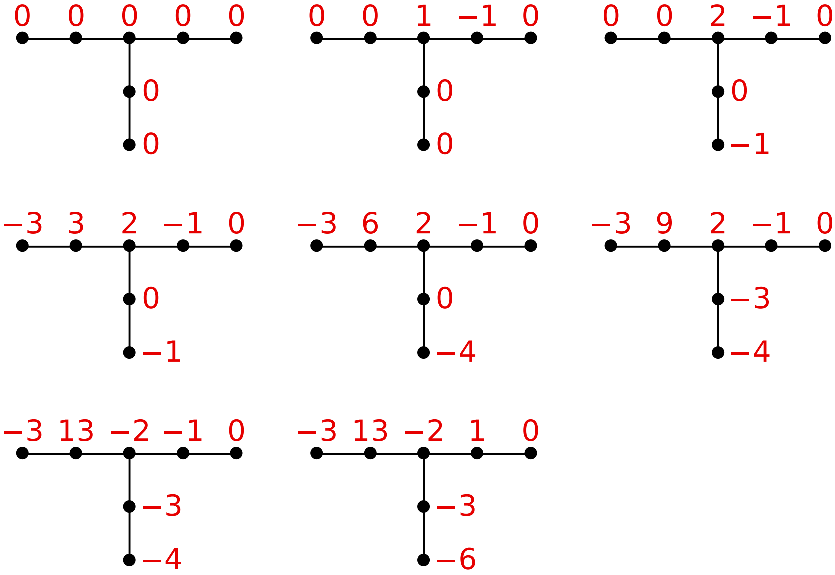
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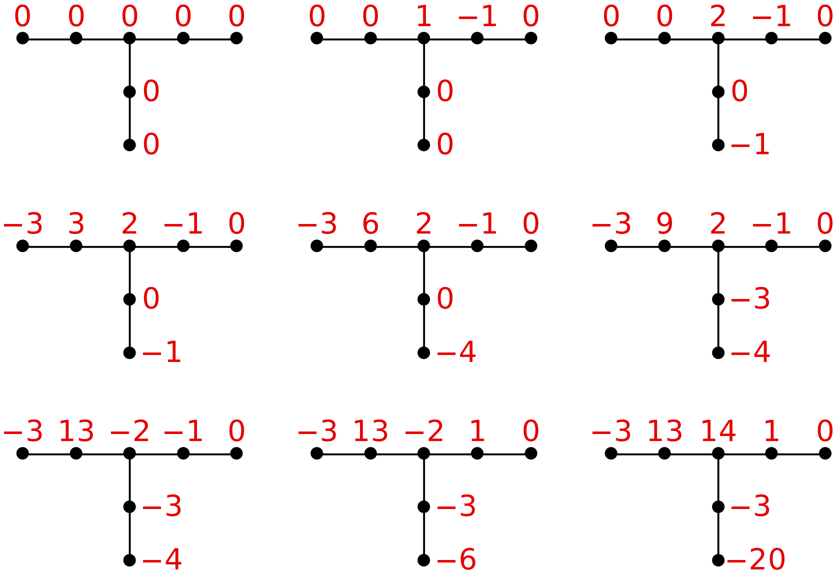
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Ex. Who moves first can make a big difference:
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Upper bounds = strategies for Min.
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Ques. Note that $a_g(K_{n,n}) = 2$ but $\hat{a}_g(K_{n,n}) = 1$.
What are the graphs with $a_g(G) > \hat{a}_g(G)$?
Can the excess be more than 1?

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Def. Live vertices are serfs (wt $= 1$) or kings (wt > 1).

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If Min makes a new X-king, Max eats it with a Y-king. Otherwise, Max eats an X-serf with a Y-king (if X has no serf, the game is over). Since X has no king, Y has no other way to disturb the conditions. ■

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If Max eats a serf with a king, Min eats a serf with the other king; excess unchanged. If Max makes a new king, Min eats it with the opposite king; both sides lose one serf. If Max eats a king, then Min replaces it (the game is over if this can't be done). ■

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If Max makes a king, then Min eats it, unless Max left two kings in X and none in Y; Min then makes a king in Y (if impossible, then the two kings in X eat the serfs in Y to end with 2). If Max eats a king, then Min restores it. Otherwise, Max eats a serf with a king; Min maintains the imbalance by eating a serf on the other side unless no king is available. In that case, Max ate a serf from Y and reduced the imbalance; Min makes a king in Y. ■

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The conjecture includes the statement that this bound holds for all n and is optimal when $n > 5m/2$.

Conj. $\alpha_g(K_{m,n}) = 2 + \min\{\left\lfloor \frac{n-m}{3} \right\rfloor, \left\lfloor \frac{m}{2} \right\rfloor\}$.