Modular factors of regular graphs

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Abstract

For a graph G with $\delta(G) \geq 2s$, we apply the Combinatorial Nullstellensatz to prove that G contains a nontrivial subgraph H in which each vertex degree lies outside s specified congruence classes modulo a prime p. As a result, we prove the conjecture of Akbari and Kano that 5-regular graphs decompose into two factors where each vertex has degree 1 or 4. We obtain similar decompositions of other regular graphs, including another proof of the result of Akbari, Rahmati, and Zare that for $r \geq 3$, the edges of any r-regular graph can be labeled from $\{1, 2\}$ so that at each vertex the sum of the labels on incident edges is divisible by 3.

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Let G be a graph, and let v be a vertex of G. Let $d(v)$ and $\delta(G)$ denote the degree of v and the minimum degree of G, respectively. Let $\Gamma(v)$ be the set of all edges in G incident to vertex v. An $\{a, b\}$-factor of a graph G is a spanning subgraph G' of G in which every vertex degree is a or b. Similarly, for $A \subseteq \mathbb{Z}_p$, an A-modular factor of G is a spanning subgraph G' of G such that the degree of each vertex in G' is congruent to some element of A modulo p.

Conjecture 1 (Akbari and Kano [1]). *A 5-regular graph has a $\{1, 4\}$-factor.*

Conjecture 2 (Akbari and Kano [1]). *If r and k are odd integers such that $1 \leq k < r/2$, then every r-regular graph has a $\{k, r-k\}$-factor.*

Under the condition $1 \leq k < r/2$, Akbari and Kano [1] proved that every r-regular graph G has a $\{k, r-k\}$-factor when r is odd (with $r \geq 5$) and k is even. When r and k are both even, and $k \leq r$, Petersen's Theorem guarantees a $\{k\}$-factor. Also, Hanson, Loten, and

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Toft [6] proved that if r is odd and k is even and $r \geq 3k/2$, then every r-regular graph with at most $2r/k - 1$ cut-edges has a $\{k\}$-factor.

Our results imply Conjecture 1 and some of the cases of Conjecture 2. In addition, we obtain a simple proof of the result of Akbari, Rahmati, and Zare [2] that if $k \geq 3$, then the edges of any k-regular graph can be labeled 1 or 2 so that at each vertex the labels on the incident edges sum to a multiple of 3.

A fundamental tool for proving existence results in discrete mathematics is the Combinatorial Nullstellensatz (see Alon [3]).

Lemma 3 (Combinatorial Nullstellensatz)**.** *Given a polynomial f of degree t in m variables over a field $\mathbb{F}$. If there is a monomial $\prod {x_i}^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero (in $\mathbb{F}$), then f is nonzero at some point of $\prod S_i$, where each S_i is a set of $t_i + 1$ values in $\mathbb{F}$.*

Our main result, proved using the Combinatorial Nullstellensatz, guarantees subgraphs with vertex degrees in certain congruence classes for graphs with sufficiently large minimum degree. Alon, Friedland, and Kalai [4] proved a result with the same flavor, showing that when q is a power of a prime, every graph G with maximum degree $2q - 1$ and more than $(q-1)|V(G)|$ edges has a nontrivial subgraph whose vertex degrees are all divisible by q (a graph is *nontrivial* if it has at least one edge).

A *decomposition* of a graph G is a family of edge-disjoint subgraphs whose union is G.

Lemma 4. *Every n-vertex multigraph G decomposes into n subgraphs that are stars with distinct centers, and each having at least $\lfloor \delta(G)/2 \rfloor$ edges.*

Proof. Adding a new vertex adjacent to all vertices of odd degree in G yields a supergraph G' whose components are Eulerian. Orient each component of G' to follow an Eulerian circuit. For each vertex v in G assign to v all edges leaving v in the orientation of G'. □

The proof of our main lemma is very similar to the Alon-Tarsi [5] application of the Combinatorial Nullstellensatz to list coloring.

Lemma 5. *Let G be a graph with n vertices and sn edges. Let S be a set of s elements in a field $\mathbb{Z}_p$, where p is an odd prime and $s < p$. If the edges of G can be oriented so that the outdegree of each vertex is exactly s, then G has a $(\mathbb{Z}_p - S)$-modular factor.*

Proof. For each edge $e \in E(G)$ define a variable x_e, and let x be the vector of these variables. Define $f(x) = \prod_{v \in V(G)} \prod_{a \in S} (-a + \sum_{e \in \Gamma(v)} x_e)$ over the field $\mathbb{Z}_p$. We are given an orientation of G in which the outdegree of each vertex is s. Fix such an orientation D.

We claim that the coefficient of the monomial $\prod_{e \in E(G)} x_e$ is not zero. With each occurrence of $\prod_{e \in E(G)} x_e$ in the expansion of f (always contributing $+1$ to the coefficient), we associate an orientation of G. Orient the edge uv from u to v if uv is chosen from the factor $(-a + \sum_{e \in \Gamma(u)} x_e)$ in f. Let D' be the resulting orientation; by the definition of f, in D' the outdegree of each vertex is exactly s. Let D'' be the spanning subdigraph of D whose edges are those oriented oppositely in D and D'. Since each vertex has outdegree s in both D and D' the digraph D'' is a circulation. Conversely, reversing the edges of a circulation does not change any outdegree, so the map sending D' to D'' is a bijection. Hence the number of unit contributions to this coefficient is the number of circulations of D.

Let A be a basis for the binary space of circulations of D. The number of circulations is $2^{|A|}$, which is nonzero in $\mathbb{Z}_p$ when p is an odd prime. Hence the coefficient in f of the monomial $\prod_{e \in E(G)} x_e$ is nonzero. By the Combinatorial Nullstellensatz, there exists $x^* \in \{0,1\}^{sn}$ such that $f(x^*) \neq 0$. Let H be the spanning subgraph of G with edge set $\{e \in E(G) \colon x^*_e = 1\}$. In H, each vertex has degree in $\mathbb{Z}_p - S$, since otherwise $f(x^*) = 0$. □

Theorem 6. *For an odd prime p and a proper subset S of $\mathbb{Z}_p$ with size s, every graph G with $\delta(G) \geq 2s$ that is not $2s$-regular has a nontrivial $(\mathbb{Z}_p - S)$-modular factor H.*

Proof. Let G be a graph with n vertices. Let $G_1, \ldots, G_n$ be the decomposition of G guaranteed by Lemma 4, with G_i being a star centered at vertex v_i. Choose s edges from each G_i to form a set E' of sn edges. Let D be an orientation of the spanning subgraph of G with edge set E' such that an edge v_iv_j is oriented away from v_i if $v_iv_j \in G_i$. Define a variable x_e for each edge e, and define a polynomial f over $\mathbb{Z}_p$ by

$$f(x) = \prod_{v \in V(G)} \prod_{a \in S} \left(-a + \sum_{e \in \Gamma(v) - E'} 1 + \sum_{e \in \Gamma(v) \cap E'} x_e \right) \prod_{e \in E(G) - E'} x_e$$

The degree of f is $|E(G)|$. As in the proof of Lemma 5, the coefficient of $\prod_{e \in E(G)} x_e$ in f is $2^{|A|}$, where A is a basis for the space of circulations of D. By the Combinatorial Nullstellensatz, $f(x^*) \neq 0$ for some $x^* \in \{0,1\}^{|E(G)|}$. Again let H be the spanning subgraph with $E(H) = \{e \in E(G) \colon x^*_e = 1\}$. Since G is not $2s$-regular, $x^*_e = 1$ for $e \in E(G) - E'$, and H is nontrivial. Since $f(x^*) \neq 0$, the degree in H of each vertex is not congruent to any element of S, as desired. □

The subgraph H generated by Theorem 6 contains all of $E(G) - E'$. The statement of the theorem excludes $2s$-regular graphs because in this case f would be nonzero when all variables are 0 and $0 \notin S$, and we would not obtain a nontrivial subgraph.

Corollary 7. *If p is an odd prime, $b \in \mathbb{Z}_p$, $\delta(G) \geq 2p - 2$, and G is not $(2p-2)$-regular, then G has a nontrivial $\{b\}$-modular factor.*

Proof. Let $S = \mathbb{Z}_p - \{b\}$ in Theorem 6. Note that $|S| = p - 1$. □

Corollary 8. *When p is an odd prime and G is an r-regular graph,*
(a) If $r > 2p - 2$ and $r \equiv 2j$ (mod p), then G decomposes into two $\{j\}$-modular factors.
(b) If $r > 2p - 4$, then G decomposes into two $\{a, b\}$-modular factors, where $a \not\equiv b$ (mod p) and $a + b \equiv r$ (mod p).

Proof. Corollary 7 (in case (a)) or Theorem 6 (in case (b)) yields a factor of the desired type, and deleting its edges leaves another such factor. □

Corollary 9. *For $r \geq 3$, the edges of any r-regular graph can be assigned labels in $\{1, 2\}$ so that at every vertex, the sum of the labels on incident edges is a multiple of 3.*

Proof. When $r = 3$, assign 1 to all edges. When $r = 4$, Petersen's Theorem yields a decomposition into two 2-factors; use label 1 on the first and label 2 on the second.

For $r > 4$, suppose that $r \equiv j \pmod 3$, where $j \in \{0, 1, 2\}$. Corollary 7 provides a spanning subgraph H in which every vertex degree is congruent to $-j$ mod 3. Each vertex has degree congruent to $2j$ among the remaining edges. Since $2j \equiv -j \mod 3$, it suffices to give every edge of H label 1 and every remaining edge label 2. □

Corollary 10. *If $\frac{r-4}{4} < k < \frac{r}{3}$ and $r - 2k$ is prime, then every r-regular graph decomposes into two $\{k, r-k\}$-factors.*

Proof. By the proof of Lemma 4, G has a spanning subgraph G' whose edges can be oriented such that the outdegree of each vertex is k. Now apply Lemma 5 on G' over the field $\mathbb{Z}_{r-2k}$ to obtain a $\{k, r-k\}$-factor of G. Its complement is also a $\{k, r-k\}$-factor, so we obtain a decomposition of G. □

Remark 11. Corollary 10 yields some special cases of Conjecture 2, including $(r, k) \in \{(5, 1), (7, 1), (11, 3), (13, 3), (17, 5)\}$, and so on. In addition, if there is a prime p such that $r - k < p < \frac{r+4}{2}$, then by Corollary 8(b) every r-regular graph G has a $\{k, r-k\}$-factor. These conditions apply only with r odd, $k = \lfloor \frac{r}{2} \rfloor$, and $1 + \lceil \frac{r}{2} \rceil$ prime; that is, $k = p - 2$ and $r = 2p - 3$ for p prime. This yields one case of Conjecture 2 for each odd prime; they are $(r, k) \in \{(5, 1), (7, 3), (11, 5), (19, 9), \ldots\}$.

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