Vertex Degrees in Outerplanar Graphs

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Abstract

For an outerplanar graph on n vertices, we determine the maximum number of vertices of degree at least k. For $k = 4$ (and $n \geq 7$) the answer is $n - 4$. For $k = 5$ (and $n \geq 4$), the answer is $\left\lfloor \frac{2(n-4)}{3} \right\rfloor$ (except one less when $n \equiv 1 \mod 6$). For $k \geq 6$ (and $n \geq k + 2$), the answer is $\left\lfloor \frac{n-6}{k-4} \right\rfloor$. We also determine the maximum sum of the degrees of s vertices in an n-vertex outerplanar graph and the maximum sum of the degrees of the vertices with degree at least k.

1 Introduction

For $n > k > 0$, Erdős and Griggs [1] asked for the minimum, over n-vertex planar graphs, of the number of vertices with degree less than k. For $k \leq 6$, the optimal values follow from results of Grünbaum and Motzkin [4]. West and Will [5] determined the optimal values for $k \geq 12$, obtained the best lower bounds for $7 \leq k \leq 11$, and provided constructions achieving those bounds for infinitely many n when $7 \leq k \leq 10$. Griggs and Lin [2] independently found the same lower bounds for $7 \leq k \leq 10$ and gave constructions achieving the lower bounds when $7 \leq k \leq 11$ for *all* sufficiently large n.

We study the analogous question for outterplanar graphs, expressed in terms of large-degree vertices. Let $\beta_k(n)$ be the maximum, over n-vertex outerplanar graphs, of the number of vertices having degree at least k. For $k \leq 2$, the problem is trivial; $\beta_k(n) = n$, achieved by a cycle (or by any maximal outerplanar graph).

When $k \in \{3, 4\}$, the square of a path shows that $\beta_3(n) \geq n - 2$ and $\beta_4(n) \geq n - 4$. Since every outerplanar graph with $n \geq 2$ has at least two vertices of degree at most 2, $\beta_3(n) = n - 2$. We will prove $\beta_4(n) = n - 4$ when $n \geq 7$ (Theorem 2.6). For $k = 5$ and

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$n \geq 4$, we prove $\beta_k(n) = \lfloor 2(n-4)/3 \rfloor$, except one less when $n \equiv 1 \mod 6$ (Theorem 2.5). For $k \geq 6$ and $n \geq k+2$, we prove $\beta_k(n) = \lfloor (n-6)/(k-4) \rfloor$ (Theorem 3.4).

We close this introduction with a general upper bound that is optimal for $k = 5$ when $n \not\equiv 1 \mod 6$. In Section 2 we improve the upper bound by 1 when $n \equiv 1 \mod 6$ and provide the general construction that meets the bound; these ideas also give the upper bound for $k = 4$. In Section 3 we solve the problem for $k \geq 6$. The bounds in [5] were obtained by first solving a related problem, which here corresponds to maximizing the sum of the degrees of the vertices with degree at least k. We use this approach in Section 3 to prove the upper bound on $\beta_k(n)$ when $k \geq 6$.

Adding edges does not decrease the number of vertices with degree at least k, so an n-vertex outerplanar graph with $\beta_k(n)$ vertices of degree at least k must be a maximal outerplanar graph, which we abbreviate to *MOP*. For a MOP with n vertices, let β be the number of vertices having degree at least k, and let n_2 be the number of vertices having degree 2. A MOP with n vertices has $2n-3$ edges, so summing the vertex degrees yields

$$2n_2 + 3(n - n_2 - \beta) + k\beta \leq 4n - 6. \tag{1}$$

This inequality simplifies to $(k-3)\beta \leq n + n_2 - 6$. Using $n_2 \leq n - \beta$ then yields $\beta_k(n) \leq \lfloor 2(n-3)/(k-3) \rfloor$. To improve the bound, we need a structural lemma.

Lemma 1.1. *Let G be an n-vertex MOP with external cycle C. If $n \geq 4$, then G has two vertices with degree in $\{3,4\}$ that are not consecutive along C.*

Proof. We use induction on n. Note that G contains $n-3$ chords of C. If every chord lies in a triangle with two external edges, then $n \leq 6$ and $\Delta(G) \leq 4$, and the two neighbors of a vertex of degree 2 are the desired vertices. This case includes the MOPs for $n \in \{4,5\}$.

Otherwise, a chord xy not in a triangle with two external edges splits G into two MOPs with at least four vertices, each with x and y consecutive along its external cycle. By the induction hypothesis, each has a vertex with degree 3 or 4 outside $\{x,y\}$. In G, those two vertices retain their degrees, and they are separated along C by x and y. □

Corollary 1.2. *If $k \geq 5$ and $n \geq 4$, then $\beta_k(n) \leq \lfloor 2(n-4)/(k-2) \rfloor$.*

Proof. Lemma 1.1 yields $n - n_2 - \beta \geq 2$, and hence $n_2 \leq n - \beta - 2$. Substituting this improved inequality into the inequality $(k-3)\beta \leq n + n_2 - 6$ that follows from (1) yields $\beta_k(n) \leq 2(n-4)/(k-2)$. □

Corollary 1.2 gives us a target to aim for in the construction for $k = 5$. For $k \geq 6$ we will need further improvement of the upper bound. The argument of Corollary 1.2 is not valid for $k = 4$, since vertices of degree 4 are counted by β. The upper bound $\beta_4(n) \leq n - 4$ (for $n \geq 7$) will come as a byproduct of ideas in the next section.

2 The solution for $k = 5$

We begin with the construction. Let $\langle v_1, \ldots, v_k \rangle$ and $[v_1, \ldots, v_k]$ denote a path and a cycle with vertices $v_1, \ldots, v_k$ in order, respectively. Let B be the graph formed from the cycle $[v, u, x, w, y, z]$ by adding the path $\langle u, w, v, y \rangle$. (see Fig. 1). The reason for naming the vertices in this way is that we will create copies of B in a large graph by adding the vertices in the order u, v, w, x, y, z.

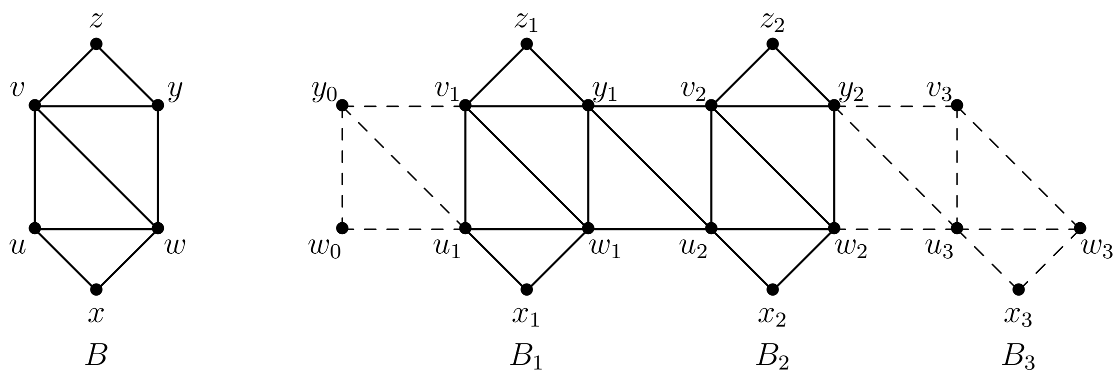

Figure 1: The graph B and its use in constructing F.

To facilitate discussion, define a *j-vertex* to be a vertex of degree j, and define a *j^+-vertex* to be a vertex of degree at least j.

Lemma 2.1. *If $n \geq 4$, then* $\beta_5(n) \geq \begin{cases} \lfloor 2(n-5)/3 \rfloor & \text{if } n \equiv 1 \mod 6, \\ \lfloor 2(n-4)/3 \rfloor & \text{otherwise.} \end{cases}$

Proof. Begin at $n = 2$ with one edge having endpoints w_0 and y_0. Add vertices one by one, always adding a vertex adjacent to two earlier neighboring external vertices; the result is always a MOP. For $6q - 3 \leq n \leq 6q + 2$, add the vertices $u_q, v_q, w_q, x_q, y_q, z_q$ in order. The added vertex is the center of a 3-vertex path; the paths added are successively $\langle w_{q-1}, u_q, y_{q-1} \rangle$, $\langle y_{q-1}, v_q, u_q \rangle$, $\langle u_q, w_q, v_q \rangle$, $\langle u_q, x_q, w_q \rangle$, $\langle v_q, y_q, w_q \rangle$, and $\langle v_q, z_q, y_q \rangle$. After z_q is added, the subgraph induced by $\{u_q, v_q, w_q, x_q, y_q, z_q\}$ is isomorphic to B; call it B_q (see Fig. 1).

When $n = 6$, the addition of x_1 augments u_1 to degree 5 (the first such vertex). The second occurs at v_1 when z_1 is added to reach eight vertices. This agrees with the claimed values for n from 4 through 8. Subsequently, addition of u_q, v_q, x_q, z_q raises $w_{q-1}, y_{q-1}, u_q, v_q$, respectively, to degree 5. Addition of w_q and y_q does not introduce a 5-vertex, and x_q and z_q never exceed degree 2.

When $n = 6 \cdot 2 - 4$, we have two 5-vertices; note that $2 = \lfloor 2(8-4)/3 \rfloor$. For $6q - 3 \leq n \leq 6q + 2$, the values required by the stated formula for the number of 5^+-vertices are

$4q-5, 4q-4, 4q-4, 4q-3, 4q-3, 4q-2$, respectively. The induction hypothesis for an induction on q states that when $n = 6q-4 = 6(q-1)+2$, the graph has $4(q-1)-2$ vertices of degree 5. Starting from this point, we augmented one vertex to degree 5 when n is congruent to each of $\{-3,-2,0,2\}$ modulo 6, matching the formula. $\square$

The lower bound in Lemma 2.1 and upper bound in Corollary 1.2 are equal except when $n \equiv 1 \mod 6$. In this case we improve the upper bound by showing that there is no outerplanar graph having the vertex degrees required to achieve equality in the upper bound. The construction shows $\beta_5(n) \geq \lfloor 2(n-5)/3 \rfloor$. We used the existence of two vertices with degree 3 or 4 to improve the upper bound from $\lfloor 2(n-3)/3 \rfloor$ to $\lfloor 2(n-4)/3 \rfloor$, which differs from $\lfloor 2(n-5)/3 \rfloor$ by 1 when $n \equiv 1 \mod 6$. We begin by showing that slightly stronger hypotheses further reduce the bound.

Lemma 2.2. *If G is a MOP having a 6^+-vertex, or a 3-vertex and a 4-vertex, or two 4-vertices, or at least three 3-vertices, then G has at most $\lfloor (2n-9)/3 \rfloor$ vertices of degree at least 5.*

Proof. We have proved that $\beta \leq (2n-8)/3$. If we cannot improve the upper bound to $(2n-9)/3$, then equality must hold in all the inequalities that produced the upper bound $(2n-8)/3$. Thus $n_2 = n - \beta - 2$, which forbids a third vertex with degree 3 or 4. Also, the sum of the degrees must equal $2n_2 + 3(n - n_2 - \beta) + 5\beta$ (see (1)). This requires that the two vertices of degree 3 or 4 both have degree 3 and that no vertex has degree at least 6. $\square$

Let T be the subgraph of the dual graph of G induced by the vertices corresponding to bounded faces of G; we call T the *dual tree* of G. Since G is a MOP, T is a tree. A triangular face in G having j edges on the external cycle corresponds to a $(3-j)$-vertex in T. The next lemma will reduce the proof of the theorem to the case where T is a special type of tree.

Lemma 2.3. *If in the dual tree T the neighbor of a leaf t has degree 2, then G has a 3-vertex on the triangle corresponding to t. If two leaves in T have a common neighbor, then the common vertex of the corresponding triangles in G is a 4-vertex in G.*

Proof. Let x, y, z be the vertices of the triangle in G corresponding to t, with x having degree 2 in G. The neighboring triangle t' raises the degree of y and z to 3. If t' has degree 2 in T, then in G only one of $\{y, z\}$ can gain another incident edge.

Two leaves in T having a common neighbor $\hat{t}$ correspond to two triangles in G having a common vertex z. The vertex $\hat{t}$ in T corresponds to a triangle in G that shares an edge with each of them. No further edges besides the four in these triangles are incident to z. $\square$

A triangle in a MOP is *internal* if none of its edges lie on the external cycle.

Lemma 2.4. *In a MOP with n vertices, let n_2 be the number of 2-vertices and t be the number of internal triangles. If $n \geq 4$, then $t = n_2 - 2$.*

Proof. Let T be the dual tree. Note that T has $n-2$ vertices, of which n_2 have degree 1, t have degree 3, and the rest have degree 2. By counting the edges in terms of degrees, $2(n-3) = n_2 + 3t + 2(n-2-n_2-t)$, which yields $n_2 - 2 = t$. $\square$

Theorem 2.5. *If $n \geq 4$, then* $\beta_5(n) = \begin{cases} \lfloor 2(n-5)/3 \rfloor & \text{if } n \equiv 1 \mod 6, \\ \lfloor 2(n-4)/3 \rfloor & \text{otherwise.} \end{cases}$

Proof. It suffices to prove the upper bound when $n \equiv 1 \mod 6$. Let $n = 6q+1$. Corollary 1.2 yields $\beta \leq 4q-2$, and we want to improve this to $4q-3$, which equals $\lfloor (2n-9)/3 \rfloor$. If the upper bound cannot be improved, then by the computation in the proof of Lemma 2.2 we may assume that G has exactly two 3-vertices, exactly $4q-2$ vertices of degree 5, and exactly $2q+1$ vertices of degree 2.

A MOP with n vertices has $n-2$ bounded faces, so T has $n-2$ vertices. Since G has $2q+1$ vertices of degree 2, there are $2q+1$ leaves in T; by Lemma 2.4, T has $2q-1$ vertices of degree 3. The remaining $2q-1$ vertices of T have degree 2.

A *caterpillar* is a tree such that deleting all the leaves yields a path, called its *spine*. A tree that is not a caterpillar contains as a subtree the graph Y obtained by subdividing each edge of the star $K_{1,3}$. If $Y \subseteq T$, then consider longest paths in T starting from the central vertex v of Y along each of the three incident edges. Each such path reaches a leaf. By Lemma 2.3, each such path generates a vertex of degree 3 or 4 in G. Since G has at most two such vertices, T is a caterpillar. Furthermore, since G has no 4-vertices, Lemma 2.3 implies that each endpoint of the spine of T has degree 2 in T.

Consider vertices $a, b, c \in V(T)$ such that $ab, bc \in E(T)$. The corresponding three triangles in G have a common vertex x. If a and c are 3-vertices in T, then x has degree 6 in G. Hence no two 3-vertices in T have a common neighbor. This implies that along the spine of T (which starts and ends with 2-vertices), there are at most two consecutive 3-vertices, and nonconsecutive 3-vertices are separated by at least two 2-vertices.

In particular, every run of 3-vertices has at most two vertices, every run of 2-vertices has at least two vertices (except possibly the runs at the ends), and the number of runs of 2-vertices is one more than the number of runs of 3-vertices. The only way this can produce the same number of 2-vertices and 3-vertices is $2, 3, 3, 2, 2, 3, 3, \ldots, 2, 2, 3, 3, 2$. However, in this configuration the number of vertices of each type is even and cannot equal $2q-1$.

We have proved that no outerplanar graph has the required vertex degrees. $\square$

Theorem 2.6. *If $n \geq 7$, then $\beta_4(n) = n-4$.*

Proof. When the dual tree T is a path, the graph G has two 2-vertices, two 3-vertices, and $n-4$ vertices of degree 4; this proves the lower bound. For the upper bound, since leaves of T correspond to triangles in G having 2-vertices, we may assume that T has at most three leaves. If T has only two leaves, then the neighbor of each has degree 2 in T, and Lemma 2.3 provides two 3-vertices in G, matching the construction.

If T has exactly three leaves, then T has one 3-vertex and at least four other vertices, since T has $n-2$ vertices and $n \geq 7$. Hence at least one leaf in T has a neighbor of degree 2, and Lemma 2.3 provides one 3-vertex in addition to the three 2-vertices in G. (In fact, this case yields another construction having exactly $n-4$ vertices with degree at least 4.) □

3 The solution for $k \geq 6$

Trivially, $\beta_k(n) = 1$ when $k = n-1$, which does not satisfy the general formula. We restrict our attention to $n \geq k+2$ and begin with the construction. Fix k with $k \geq 6$.

Form a graph B' from B in Section 2 by respectively replacing edges yz and ux with paths P and Q, each having $k-6$ internal vertices. Make v adjacent to all of $V(P)$ and w adjacent to all of $V(Q)$ (see Fig. 2). Since P and Q have $k-4$ vertices each, B' has $2k-6$ vertices; also, B' is a MOP. Its vertices v and w of maximum degree have degree $k-2$.

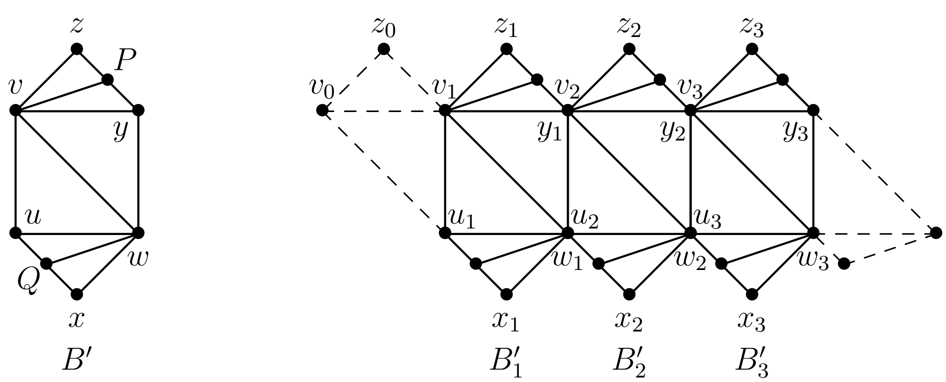

Figure 2: The graph B' and its use in constructing F'.

Lemma 3.1. *If $n \geq k+2$, then* $\beta_k(n) \geq \left\lfloor \dfrac{n-6}{k-4} \right\rfloor$.

Proof. Letting $n = 2(k-4)q + 6 + r$ with $0 \leq r < 2(k-4)$, the claim is equivalent to $\beta_k(n) \geq 2q$ when $0 \leq r < k-4$ and $\beta_k(n) \geq 2q+1$ when $k-4 \leq r < 2(k-4)$.

Let F' be the union of q copies $B'_1, \ldots, B'_q$ of B', modifying the names of vertices in B'_i by adding the subscript i and taking $y_i = v_{i+1}$ and $w_i = u_{i+1}$ for $i \geq 1$ (see the solid graph in Fig. 2). Note that F' is a MOP with $2+2q(k-4)$ vertices, and the $2q-2$ vertices that lie in two copies of B' have degree k. To obtain the lower bound, we will add vertices one by one as in Lemma 2.1, but we will start with the appended vertices z_0 and v_0 in order to raise v_1 to degree k quickly.

Each vertex when added will be a 2-vertex appended to an edge of the external cycle, so we always have a MOP. Begin with the triangle on $\{v_1, z_0, v_0\}$. Add u_1, w_1, y_1 and all of P in B'_1 in order. We now have $k+1$ vertices, and v_1 has degree k. There remains only one k-vertex as we add the rest of Q to complete B'_1, at which point w_1 has degree $k-2$.

Having completed B'_q, we have $2q(k-4)+4$ vertices (including v_0 and z_0), of which $2q-1$ vertices have degree k (including v_1). We show that adding $V(B'_{q+1})$ in the right order produces two more k-vertices at the right times.

With two more vertices, $n = 2q(k-4)+6$ and $r=0$, and the number of k-vertices should be $2q$. Add w_{q+1} and then the first vertex of Q from B'_{q+1}, raising the degree of w_q to k.

The next k-vertex should arrive when $n = 2q(k-4)+6+(k-4)$, with $r = k-4$. Adding the $k-4$ vertices from y_{q+1} to z_{q+1} along P in B'_{q+1} raises the degree of y_q from 4 to k. Finally, we add the rest of Q to complete B'_{q+1}. No k-vertex appears, but the degree of w_{q+1} rises to $k-2$ to be ready for the next iteration. □

In order to prove the upper bound, we consider a related problem. Let $D(n,s)$ be the maximum sum of the degrees of s vertices in an n-vertex outerplanar graph. Of course, these will be s vertices of largest degrees, and the maximum will be achieved by a MOP. We will obtain an upper bound on $D(n,s)$ by considering the structure of the subgraph induced by the vertices with largest degrees. Subsequently, we will apply the bound on $D(n,s)$ to prove that $\beta_k(n) \le \left\lfloor \frac{n-6}{k-4} \right\rfloor$ when $k \ge 6$.

Lemma 3.2. *Fix s with $1 \le s \le n$. If G is a MOP in which the sum of the degrees of some s vertices is $D(n,s)$, then each set of s vertices with largest degrees in G induces a MOP.*

Proof. Let C be the external cycle in an outerplanar embedding of G, with vertices $v_1, \ldots, v_n$ in order. Let S be a set of s vertices with largest degrees.

We show first that the outer boundary of the subgraph induced by some such set S is a cycle. For $x \in S$, let y be the next vertex of S along C. Let P be the path from x to y along C. If x is not adjacent to y, then let u be the last vertex of P adjacent to x. Since G is a MOP, xu lies in two triangles, and by the choice of u there is a triangle containing xu whose third vertex is not on P; let z be this third vertex. Let v be the next vertex of P after u. Let U be the set of neighbors of u not on the x,u-subpath of P; note that $z \in U$. Replace the edges from u to U with edges from x to $U \cup \{v\}$. Each edge moved increases the degree of x, and hence the sum of the s largest degrees does not decrease. (If we removed an edge from a neighbor of x that is in S and its degree is no longer among the s largest, then it was replaced by a vertex of the same degree; thus we have increased the sum of the s largest degrees, which contradicts the choice of G.)

The last neighbor of x is now farther along P. When it reaches y the sum of the m largest degrees increases. Since we started with a MOP maximizing this sum, the edge xy must have been present initially.

We have shown that the outer boundary of $G[S]$ is a cycle. Every bounded face is a face of G, since there are no vertices of G inside it. Hence $G[S]$ is a MOP. □

In fact, when $s < n-1$ there is always a unique set of s vertices with largest degrees in a graph maximizing the sum of those degrees.

Theorem 3.3. *The maximum value $D(n,s)$ of the sum of s vertex degrees in an n-vertex outerplanar graph is given by*

$$D(n,s) = \begin{cases} n-1 & \text{if } s = 1, \\ n-6+4s & \text{if } s < n/2, \\ 2n-6+2s & \text{if } s \geq n/2. \end{cases}$$

Proof. Let G be a MOP in which some set S of s vertices has degree-sum $D(n,s)$. If $s=1$, then $n-1$ is clearly an upper bound, achieved by a star. For $s \geq 2$, let G be a MOP achieving the maximum; we know that $G[S]$ is also a MOP and hence has $2s-3$ edges. The question then becomes how the remaining $n-s$ vertices can be added to produce the maximum sum of the degrees in S.

Consider an outerplanar embedding of G. The subgraph induced by S is also an outerplanar embedding of S. Since $G[S]$ has $2s-3$ edges, the outer boundary of the subgraph is a cycle. In the embedding of G, no vertex of $V(G)-S$ appears inside this cycle. Also, vertices outside S can be adjacent to only two vertices of S, and they can be adjacent to two only if those two are consecutive on the outer boundary of $G[S]$. This implies that at most s vertices of $V(G)-S$ can have two neighbors in S, and the rest have at most one neighbor in S. Furthermore, the vertices outside S can be added to achieve this bound.

If $s \geq n/2$, then we add $2(n-s)$ to the degree-sum within $G[S]$, obtaining $D(n,s) = 2n-6+2s$. If $s \leq n/2$, then we add $2s+1(n-2s)$, obtaining $D(n,s) = n-6+4s$. □

Theorem 3.4. *If $k \geq 6$, then $\beta_k(n) \leq \left\lfloor \dfrac{n-6}{k-4} \right\rfloor$.*

Proof. In an extremal graph, the $\beta_k(n)$ vertices with degree at least k have the largest degrees. With $s = \beta_k(n)$, we have $sk \leq D(n,s)$. Using the bound obtained in Theorem 3.3, we have

$$sk \le \begin{cases} n - 6 + 4s & \text{if } s < n/2, \\ 2n - 6 + 2s & \text{if } s \ge n/2. \end{cases}$$

If $k \ge 6$ and $s \ge n/2$, then

$$6s \le ks \le 2n - 6 + 2s \le 6s - 6.$$

Hence $k \ge 6$ implies $s < n/2$, and therefore $ks \le n - 6 + 4s$, which simplies to $s \le \frac{n-6}{k-4}$. □

Finally, we consider the maximum sum of the degrees of the vertices with degree at least k. Essentially, the point is that we cannot increase this sum by using fewer than $\beta_k(n)$ vertices with degrees larger than k.

Corollary 3.5. *For $k \ge 6$, the maximum sum of the degrees of the vertices with degree at least k in an n-vertex outerplanar graph is $n - 6 + 4\left\lfloor \frac{n-6}{k-4} \right\rfloor$.*

Proof. Let G be an n-vertex outerplanar graph, and let $S = \{v \in V(G) \colon d(v) \ge k\}$. Let $s = |S|$. Since these are the vertices of largest degree, $\sum_{v \in S} d(v) \le D(n, s)$.

For $k \ge 6$, since $D(n, s)$ is monotone increasing in s, we obtain a bound on the sum by using the bound on $\beta_k(n)$ obtained in Theorem 3.4. Since $\beta_k(n) < n/2$, Theorem 3.3 yields $\sum_{v \in S} d(v) \le D(n, s) \le D(n, \beta_k(n)) = n - 6 + 4\left\lfloor \frac{n-6}{k-4} \right\rfloor$.

We show that a modification of the construction in Lemma 3.1 achieves equality in the bound. When $n = (k - 4)q' + 6$ for some integer q', let G be the graph constructed in Lemma 3.1, having $\beta_k(n)$ vertices of degree at least k; here $\beta_k(n) = q'$. All q' of these vertices have degree exactly k, so the sum of their degrees is $q'k$, equaling the upper bound here. For each increase in n over the next $k - 5$ vertices, adding one vertex of degree 2 can increase the degree-sum of these vertices by 1, again equaling the upper bound here. When the $(k - 4)$th addition is reached, start over with the construction from Lemma 3.1 for the new value of q'. □

Remark 3.6. Similar analysis solves the problem of maximizing the sum of the degrees of the vertices with degree at least 5. We remark that the maximum value when $n \not\equiv 1 \mod 6$ is $2n - 8 + 4\beta_k(n)$, which is smaller by 2 than the former in terms of $\beta_k(n)$ when $k \ge 6$. Writing the expression as $2(n - \beta_k(n)) - 2 + (4\beta_k(n) - 6)$, we begin with degree-sum $4\beta_k - 6$ within the MOP H induced by the vertices of degree at least 5. Since $\beta_5(n)$ is roughly $2n/3$, we should be able to augment the sum of the degrees by 2 for each of the remaining $n - \beta_k(n)$ vertices. However, H has at least two vertices of degree 2. For each such vertex, raising its degree to 5 requires one of the added vertices to contribute only 1 instead of 2. With this adjustment, the improved upper bound meets the construction. When $n \equiv 1 \mod 6$, there are additional technicalities we leave to the reader.

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