

# Vertex Degrees in Outerplanar Graphs

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Joint work with  
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# History

**Ques.** Erdős–Griggs [1991] How many vertices with degree  $\geq k$  can an  $n$ -vertex planar graph have?

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$\alpha_6(n) = \begin{cases} n - 4 & n \text{ even} \\ n - 5 & n \text{ odd} \end{cases}$  Grünbaum–Motzkin [1963]

$\alpha_k(n) = \lfloor \frac{3n-12}{k-3} \rfloor$  for  $7 \leq k \leq 10$  Griggs–Lin [1995]

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Also known: Maximum sum of the degrees of vertices with degree at least  $k$ .

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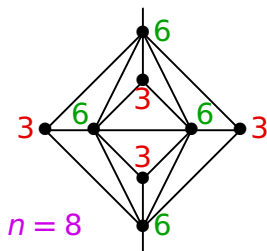
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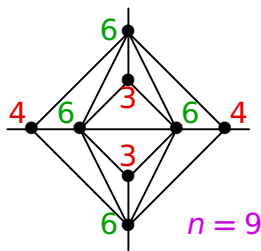
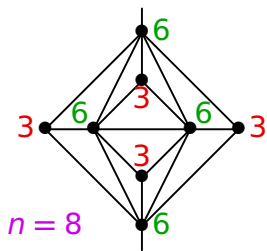
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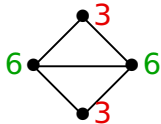
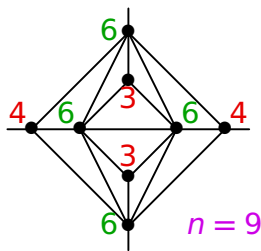
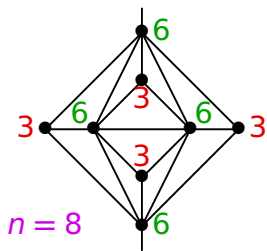
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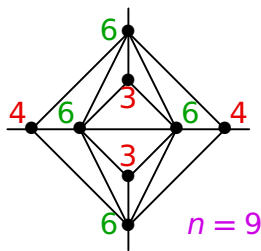
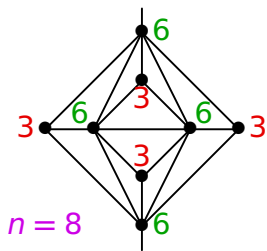
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We also determine:

Maximum sum of the  $s$  largest degrees.

Maximum degree-sum for the vertices with degree  $\geq k$ .

## Small $k$

- $\beta_k(n)$  attained by a maximal outerplanar graph (MOP).

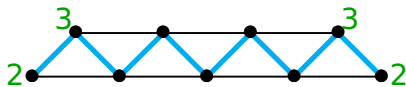
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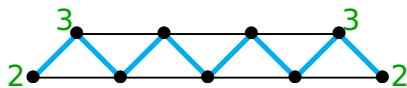
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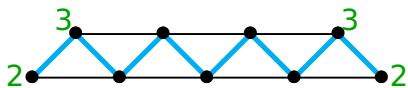


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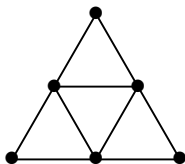
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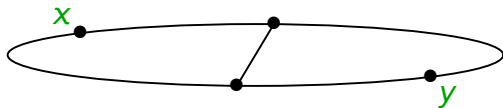
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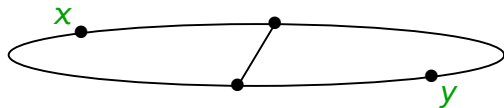
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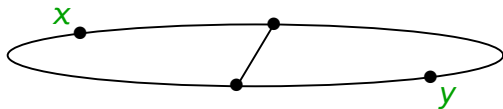
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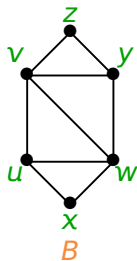
**Pf.** Now  $n_2 \leq n - \beta - 2$ . Put into  $(k - 3)\beta \leq n + n_2 - 6$ . ■

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**Lem.** If  $n \geq 4$ , then  $\beta_5(n) \geq \begin{cases} \lfloor \frac{2(n-5)}{3} \rfloor & \text{if } n \equiv 1 \pmod{6}, \\ \lfloor \frac{2(n-4)}{3} \rfloor & \text{otherwise.} \end{cases}$

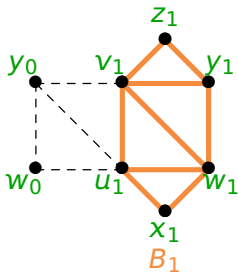
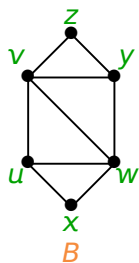
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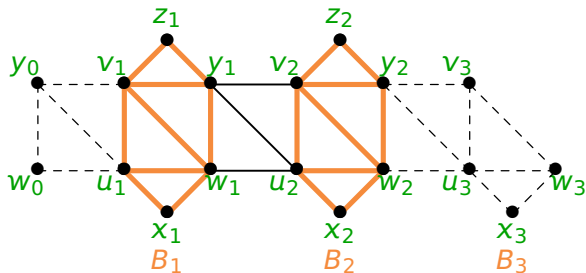
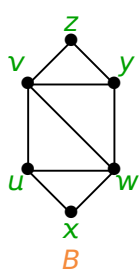
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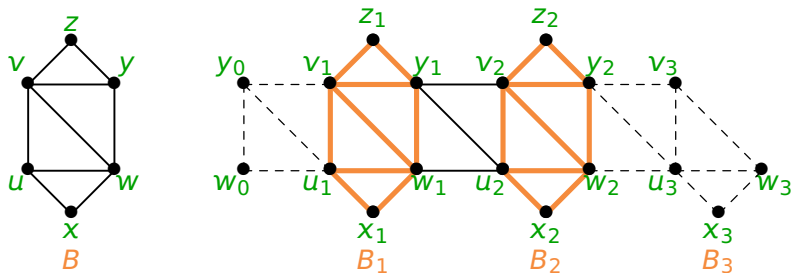


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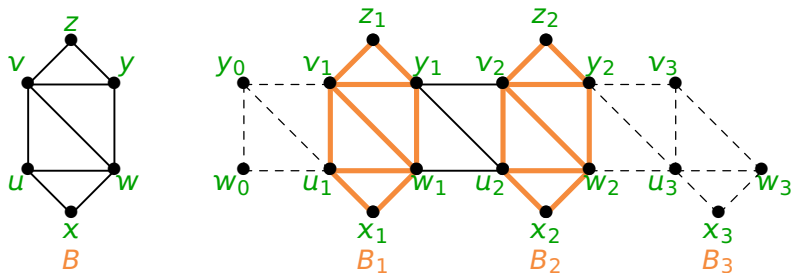
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Now  $k = 5$  done except upper bound for  $n \equiv 1 \pmod{6}$ .

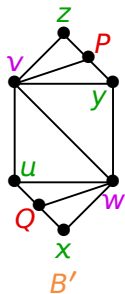
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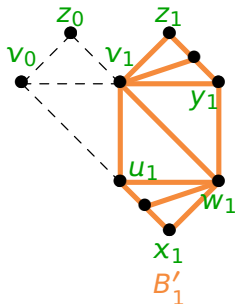
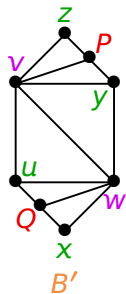
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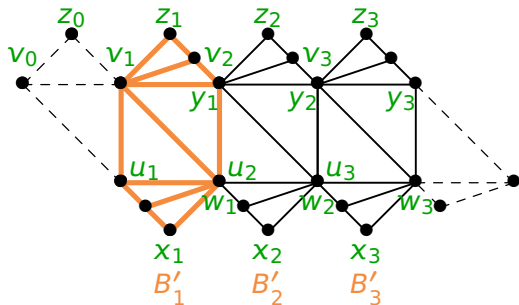
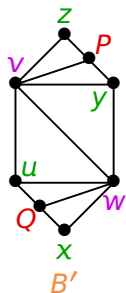


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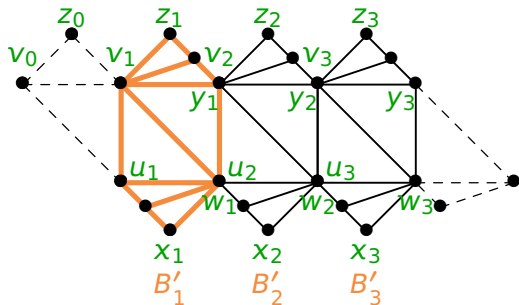
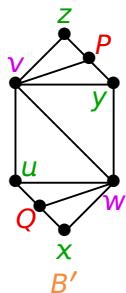


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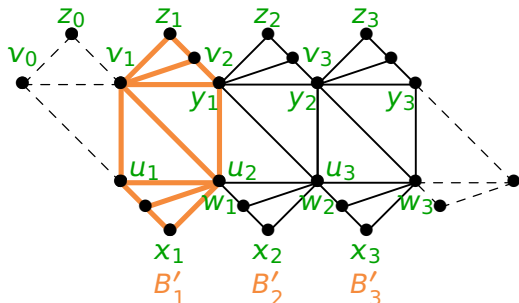
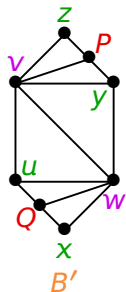
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**Pf.** Otherwise,  $G$  can be altered to get  $s$  vertices with larger degree-sum. ■

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Hence  $ks \leq n - 6 + 4s$ , which yields  $s \leq \frac{n-6}{k-4}$ . ■