

# Proper path-factors and interval edge-coloring of (3, 4)-biregular bigraphs

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## Abstract

An *interval coloring* of a graph  $G$  is a proper coloring of  $E(G)$  by positive integers such that the colors on the edges incident to any vertex are consecutive. A *(3, 4)-biregular bigraph* is a bipartite graph in which each vertex of one part has degree 3 and each vertex of the other has degree 4; it is unknown whether these all have interval colorings. We prove that  $G$  has an interval coloring using 6 colors when  $G$  is a (3, 4)-biregular bigraph having a spanning subgraph whose components are paths with endpoints at 3-valent vertices and lengths in  $\{2, 4, 6, 8\}$ . We provide sufficient conditions for the existence of such a subgraph.

Keywords: path factor, interval edge-coloring, biregular bipartite graph

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## 1 Introduction

An *interval coloring* or *consecutive coloring* of a graph  $G$  is a proper coloring of the edges of  $G$  by positive integers such that the colors on the edges incident to any vertex are consecutive. The notion was introduced by Asratian and Kamalian [2] (available in English as [3]), motivated by the problem of constructing timetables without “gaps” for teachers and classes. Hansen [9] suggested

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another scenario: a school wishes to schedule parent-teacher conferences in time slots so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for the people and edges for the required meetings has an interval coloring.

In the context of edge-colorings, and particularly edge-colorings of bipartite graphs, it is common to consider the general model in which multiple edges are allowed. In this paper, we adopt the convention that "graph" allows multiple edges, and we will explicitly exclude multiple edges when necessary (a *simple graph* is a graph without loops or multiple edges).

All regular bipartite graphs have interval colorings, since they have proper edge-colorings in which all color classes are perfect matchings. Not every graph has an interval coloring, since a graph  $G$  with an interval coloring must have a proper  $\Delta(G)$ -edge-coloring [3]. Furthermore, Sevastjanov [15] proved that determining whether a bipartite graph has an interval coloring is NP-complete. Nevertheless, trees [9, 12], complete bipartite graphs [9, 12], "doubly convex" bipartite graphs [12], grids [7], and simple outerplanar bipartite graphs [8, 4] all have interval colorings. Giaro [6] showed that one can decide in polynomial time whether bipartite graphs with maximum degree 4 have interval 4-colorings.

An  $(a, b)$ -biregular bigraph is a bipartite graph where the vertices in one part all have degree  $a$  and the vertices in the other part all have degree  $b$ . Hansen [9] proved that  $(2, b)$ -biregular bigraphs are interval colorable when  $b$  is even. This was extended to all  $b$  by Hanson, Loten, and Toft [11] and independently by Kostochka [13]. Kamalian [12] showed that the complete bipartite graph  $K_{b,a}$  has an interval coloring using  $t$  colors if and only if  $a + b - \gcd(a, b) \leq t \leq a + b - 1$ , where  $\gcd$  denotes the greatest common divisor. Asratian and Casselgren [1] showed that recognizing whether  $(3, 6)$ -biregular bigraphs have interval 6-colorings is NP-complete.

It is unknown whether all  $(3, 4)$ -biregular bigraphs have interval colorings. Hanson and Loten [10] proved that no  $(a, b)$ -biregular bigraph has an interval coloring with fewer than  $a + b - \gcd(a, b)$  colors; thus  $(3, 4)$ -biregular bigraphs need at least 6 colors. An  $X, Y$ -bigraph is a bipartite graph with partite sets  $X$  and  $Y$ . In our  $(3, 4)$ -biregular  $X, Y$ -bigraphs, the vertices of  $X$  will have degree 3. Pyatkin [14] proved that if a  $(3, 4)$ -biregular bigraph has a 3-regular subgraph covering the vertices of degree 4, then it has an interval 6-coloring.

Here we obtain another sufficient condition for the existence of an interval 6-coloring of a  $(3, 4)$ -biregular  $X, Y$ -bigraph  $G$ : If  $G$  has a spanning subgraph whose components are paths with endpoints in  $X$  and lengths in  $\{2, 4, 6, 8\}$  (we call this a *proper path-factor* of  $G$ ), then  $G$  has an interval 6-coloring. A longer proof of this was found earlier by Casselgren [5].

We present infinitely many  $(3, 4)$ -biregular bigraphs that have proper path-factors but do not satisfy Pyatkin's condition. On the other hand,  $(3, 4)$ -biregular bigraphs with multiple edges need

not have proper path-factors, even if they satisfy Pyatkin's condition. For example, consider the graph formed from three triple-edges by adding a claw; that is, the pairs  $x_i y_i$  have multiplicity three for  $i \in \{1, 2, 3\}$ , and there is an additional vertex  $x_0$  with neighborhood  $\{y_1, y_2, y_3\}$ . A 3-regular subgraph covers  $\{y_1, y_2, y_3\}$ , but there is no proper path-factor. Therefore, neither our result nor Pyatkin's result implies the other.

Various difficulties disappear when multiple edges are forbidden. We have found no simple (3,4)-biregular bigraph that does not have a proper path-factor. We conjecture that every simple (3,4)-biregular bigraph has a proper path-factor. In Section 3 we present various sufficient conditions for the existence of a proper path-factor in such a graph.

## 2 Interval 6-Colorings from Proper Path-Factors

In general, an  $\mathcal{H}$ -factor of a graph is a spanning subgraph whose components lie in  $\mathcal{H}$ . We are interested in a particular family  $\mathcal{H}$ . Let  $d_H(v)$  denote the degree of a vertex  $v$  in a graph  $H$ .

**Definition 1.** A *proper path-factor* of a (3,4)-biregular  $X, Y$ -bigraph  $G$  is a spanning subgraph of  $G$  whose components are paths with endpoints in  $X$  and lengths in  $\{2, 4, 6, 8\}$ .

Henceforth let  $G$  be a (3,4)-biregular  $X, Y$ -bigraph. Given a proper path-factor  $P$  of  $G$ , let  $Q = G - E(P)$ . Observe that  $d_Q(y) = 2$  for all  $y \in Y$ . Furthermore,  $d_Q(x) = 2$  if  $x$  is an endpoint of a component of  $P$ , and  $d_Q(x) = 1$  if  $x \in X$  and  $x$  is an internal vertex of a component of  $P$ . Thus every component of  $Q$  is an even cycle or is a path with endpoints in  $X$ .

**Definition 2.** Given a proper path-factor  $P$  of  $G$ , the  *$P$ -graph of  $G$* , denoted  $G_P$ , is the graph with vertices  $\{x \in X : d_P(x) = 2\}$  having  $x_i$  and  $x_j$  adjacent when any condition below holds:

- (a)  $x_i$  and  $x_j$  are vertices of degree 2 in one component of  $P$  with length 6, or
- (b)  $x_i$  and  $x_j$  are vertices of degree 2 at distance 4 in one component of  $P$  with length 8, or
- (c)  $x_i$  and  $x_j$  are vertices of degree 1 in one component of  $Q$ .

**Lemma 3.** *If  $P$  is a proper path-factor of  $G$ , then  $G_P$  is bipartite.*

*Proof.* Every vertex of  $G_P$  has exactly one incident edge of type (c). Some vertices have one more neighbor, via an edge of type (a) or (b). Thus  $\Delta(G_P) \leq 2$ . Furthermore, the edges along any path or cycle in  $G_P$  alternate type (c) with type (a) or (b). Thus  $G_P$  has no odd cycle.  $\square$

We say that a color appears “at” a vertex if it appears on an edge incident to that vertex.

**Theorem 4.** *If  $G$  has a proper path-factor, then  $G$  has an interval 6-coloring.*

*Proof.* Let  $P$  be a proper path-factor of  $G$ . Let  $c$  be a proper 2-coloring of  $V(G_P)$  with colors  $A$  and  $B$ . We define a 6-coloring of  $E(G)$  that we will show is an interval coloring. Edges of  $P$  receive colors from  $\{1, 2, 5, 6\}$ ; edges of  $Q$  receive colors from  $\{3, 4\}$ .

First we color  $E(Q)$ . Properly color cycles arbitrarily using colors 3 and 4. A component of  $Q$  that is a path has both endpoints in  $G_P$ , and they are adjacent in  $G_P$ . Hence  $c(x) = A$  for one endpoint  $x$  of the path, and  $c(x') = B$  for the other endpoint  $x'$ . Alternate colors along the path, starting with color 3 on the edge at  $x$  and ending with color 4 on the edge at  $x'$ . Colors 3 and 4 both now appear at every vertex of  $G$  having degree 2 in  $Q$ .

The edges of every component of  $P$  are colored by alternating 2 and 1 (starting with 2) from one end, and alternating 5 and 6 (starting with 5) from the other end. We must specify which end is which and where to switch from using one pair of colors to using the other. The choice is based on the colors that  $c$  assigns to the internal vertices of the path that lie in  $X$ , as illustrated in Figure 1. Those vertices all have degree 1 in  $Q$ ; they appear in  $V(G_P)$  and have colors under  $c$ .

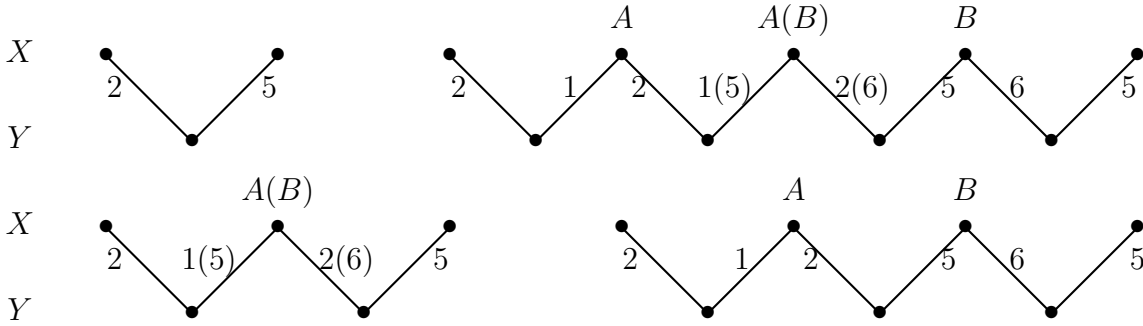


Figure 1: Coloring the edges of  $P$

Let  $H$  be a component of  $P$ . If  $H \cong P_3$ , then we assign 2 and 5 to the edges arbitrarily. If  $H \cong P_5$  with middle vertex  $x$ , then it does not matter which end edge gets color 2 and which gets color 5, but the middle edges get colors 1 and 2 if  $c(x) = A$ , 5 and 6 if  $c(x) = B$ . If  $H \cong P_7$ , then the internal vertices are adjacent in  $G_P$  and receive distinct colors under  $c$ ; use 2, 1, 2 from the end closest to the one colored  $A$  and 5, 6, 5 from the end closest to the one colored  $B$ . If  $H \cong P_9$ , then the internal vertices at distance 4 on the path again are adjacent in  $G_P$ , and the three edges from each end are colored in the same way as for  $P_7$ . The two central edges are colored like the middle edges of  $P_5$ , based on the color under  $c$  of the central vertex of the path.

We check that the resulting 6-edge-coloring is an interval coloring. Each vertex of  $Y$  has colors 3 and 4 on its incident edges in  $Q$  and receives  $\{2, 5\}$  or  $\{1, 2\}$  or  $\{5, 6\}$  on its incident edges in  $P$ ,

forming an interval in each case. Each endpoint of a component of  $P$  has colors 3 and 4 from  $Q$  and receives color 2 or 5 from  $P$ . Each internal vertex  $x$  of a component of  $P$  receives 3 from  $Q$  and  $\{1, 2\}$  from  $P$  if  $c(x) = A$ , while it receives 4 from  $Q$  and  $\{5, 6\}$  from  $P$  if  $c(x) = B$ .  $\square$

This technique does not extend to arbitrary path and cycle factors. We switch from 1, 2-alternation to 5, 6-alternation only once along a path in  $P$  and cannot switch back. Thus we need that along any path of  $P$ , the internal vertices with color  $A$  under  $c$  all precede those with color  $B$ . With longer paths, our technique offers no mechanism for achieving this; the graph  $G_P$  can only enforce that vertices receive different colors under  $c$ . Introducing more edges into  $G_P$  to prevent alternation of  $A$  and  $B$  along the path destroys the 2-colorability of  $G_P$ .

### 3 Constructions and Conditions for Proper Path-Factors

To apply the theorem, we seek proper path-factors of  $(3, 4)$ -biregular bigraphs. Here we will give some sufficient conditions for existence of proper path-factors and provide some examples related to Pyatkin's condition.

We call a 3-regular subgraph of a  $(3, 4)$ -biregular bigraph that covers the vertices of degree 4 a *full 3-regular subgraph*. Pyatkin proved that a  $(3, 4)$ -biregular bigraph with a full 3-regular subgraph has an interval 6-coloring. We begin with an example that satisfies our condition but not Pyatkin's condition. Let  $[n] = \{1, \dots, n\}$ .

**Example 1.** *The  $X, Y$ -bigraph  $G$  defined by letting  $X$  and  $Y$  be the 3-sets and 2-sets in [6], with adjacency defined by proper containment, has an interval 6-coloring.* By Theorem 4, it suffices to find a proper path-factor. In fact,  $G$  has a  $P_7$ -factor as shown below.

$$\begin{aligned}
 &124 \rightarrow 12 \rightarrow 123 \rightarrow 23 \rightarrow 235 \rightarrow 35 \rightarrow 345 \\
 &135 \rightarrow 13 \rightarrow 134 \rightarrow 34 \rightarrow 346 \rightarrow 46 \rightarrow 456 \\
 &146 \rightarrow 14 \rightarrow 145 \rightarrow 45 \rightarrow 245 \rightarrow 25 \rightarrow 256 \\
 &125 \rightarrow 15 \rightarrow 156 \rightarrow 56 \rightarrow 356 \rightarrow 36 \rightarrow 236 \\
 &136 \rightarrow 16 \rightarrow 126 \rightarrow 26 \rightarrow 246 \rightarrow 24 \rightarrow 234
 \end{aligned}$$

Here  $|X| = 20$  and  $|Y| = 15$ , with  $Y$  corresponding to the edge set of  $K_6$ . The neighborhood of a vertex in  $X$  corresponds to a triangle in  $K_6$ . Hence five vertices can be deleted from  $G$  to leave a full 3-regular subgraph if and only if  $K_6$  decomposes into five triangles. It does not, because the vertices of  $K_6$  have odd degree.  $\square$

We next construct infinitely many examples that satisfy our condition but not Pyatkin's, starting with a graph smaller than that of Example 1.

**Example 2.** The smallest simple  $(3, 4)$ -biregular bigraph is  $K_{3,4}$ ; it satisfies Pyatkin's condition. The next smallest such graphs have eight vertices of degree 3 and six of degree 4. For example, consider an  $X, Y$ -bigraph where  $Y = [6]$  and the neighborhoods of the vertices in  $X$  are eight triples from  $[6]$ , with each element used in four triples. The graph fails Pyatkin's condition if and only if the triple system does not have two disjoint triples.

Case analysis shows that it is not possible to avoid two disjoint triples without a repeated triple. However, it is possible using a repeated triple, as in  $\{123, 124, 235, 346, 346, 145, 156, 256\}$ . The resulting  $(3, 4)$ -biregular bigraph has a  $P_7$ -factor as shown in bold in Figure 2.  $\square$

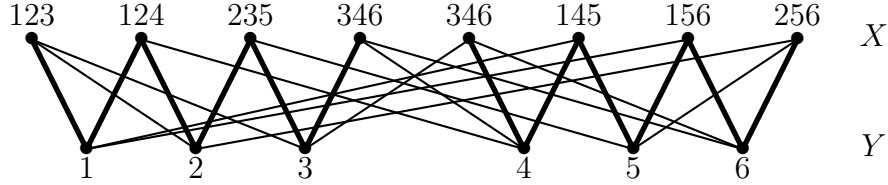


Figure 2:  $P_7$ -factor in a bigraph with no full 3-regular subgraph

Using the next lemma, we can generate infinitely many examples that have  $P_7$ -factors but have no full 3-regular subgraphs. The number of vertices can be any nontrivial multiple of 7. Here multiple edges are allowed.

**Lemma 5.** For  $i \in \{1, 2\}$ , let  $G_i$  be a 2-edge-connected  $(3, 4)$ -biregular bigraph having a  $P_7$ -factor  $F_i$ , and choose  $e_i \in E(G_i) - E(F_i)$ . Let  $G$  be the  $(3, 4)$ -biregular bigraph obtained from the disjoint union of  $G_1$  and  $G_2$  by deleting  $e_1$  and  $e_2$  and replacing them with two other edges  $e'_1$  and  $e'_2$  joining their endpoints. If  $G_1$  has no full 3-regular subgraph, then  $G$  is a larger 2-edge-connected  $(3, 4)$ -biregular bigraph having a  $P_7$ -factor but no full 3-regular subgraph.

*Proof.* Since  $e_i \notin E(F_i)$ , the subgraph  $F_1 \cup F_2$  is a  $P_7$ -factor of  $G$ . Since each  $G_i$  is 2-edge-connected,  $G$  is connected. Also, a cycle through  $e_i$  in  $G_i$  can detour through  $G_{3-i}$  using  $e'_1$  and  $e'_2$ . Thus  $G$  is 2-edge-connected.

Suppose that  $G$  has a full 3-regular subgraph  $H$ . By considering vertex degrees,  $H$  must have an even number of edges in  $\{e'_1, e'_2\}$ . If  $H$  uses neither, then  $H$  restricts to full 3-regular subgraphs of  $G_1$  and  $G_2$ . If  $H$  uses both, then replacing  $e'_1$  and  $e'_2$  with  $e_1$  and  $e_2$  yields full 3-regular subgraphs of  $G_1$  and  $G_2$ .  $\square$

If  $G_1$  and  $G_2$  in Lemma 5 have no multiple edges, then neither does the resulting graph  $G$ .

Our next theorem gives a sufficient condition for existence of a proper path-factor in a  $(3, 4)$ -biregular bigraph. First we note an easy lemma.

**Lemma 6.** *Every  $(2, 4)$ -biregular bigraph  $H$  has a  $(1, 2)$ -biregular factor with every component isomorphic to  $P_3$ . (Indeed,  $H$  decomposes into two such factors.)*

*Proof.* Each component of  $H$  is Eulerian and has an even number of edges. Taking the even-indexed edges from an Eulerian circuit in each component takes half the edges from each vertex. Thus it yields a spanning subgraph in which every vertex of one partite set has degree 1 and every vertex of the other has degree 2. Hence each component of the subgraph is isomorphic to  $P_3$ .  $\square$

**Theorem 7.** *A  $(3, 4)$ -biregular bigraph  $G$  has a  $P_7$ -factor (and hence an interval 6-coloring) if  $G$  has a  $(2, 4)$ -biregular subgraph covering the set of vertices of degree 3.*

*Proof.* Let  $G$  have bipartition  $X, Y$ , where  $|X| = 4k$  and  $|Y| = 3k$ . Let  $H$  be a  $(2, 4)$ -biregular subgraph of  $G$  covering  $X$ ; we obtain  $H$  from  $G$  by deleting vertices  $u_1, \dots, u_k$  of  $Y$  that have disjoint neighborhoods. Let  $\hat{Y} = \{u_1, \dots, u_k\}$  and  $Y' = Y - \hat{Y}$ , so  $H$  has bipartition  $X, Y'$ .

By Lemma 6,  $H$  has a spanning subgraph  $F$  whose components are copies of  $P_3$  with endpoints in  $X$ . Let  $T_1, \dots, T_{2k}$  be these paths. Index  $X$  so that  $V(T_i) = \{x_{2i-1}, y_i, x_{2i}\}$  (we maintain the flexibility to decide later which end is  $x_{2i-1}$  and which is  $x_{2i}$ ).

Next we obtain from  $G - Y'$  a graph  $H'$  by combining the endpoints of each path  $T_i$  into a single vertex  $x'_i$ . Since  $G - Y'$  is a  $(1, 4)$ -biregular  $X, \hat{Y}$ -bigraph,  $H'$  is a  $(2, 4)$ -biregular  $X', \hat{Y}$ -bigraph, where  $X' = \{x'_1, \dots, x'_{2k}\}$ . Note that multiple edges may arise in  $H'$ .

For each of the  $k$  vertices of  $\hat{Y}$ , we construct a path of length 6 in  $G$  with endpoints in  $X$ . By Lemma 6,  $H'$  has a spanning subgraph  $F'$  whose components are copies of  $P_3$  with endpoints in  $X'$ . For  $u \in \hat{Y}$ , let  $x'_i$  and  $x'_j$  be the neighbors of  $u$  in  $F'$ . Thus in  $G$  the vertex  $u$  is adjacent to one endpoint of  $T_i$  and one endpoint of  $T_j$ . We may complete the indexing of  $X$  so that these neighbors of  $u$  are  $x_{2i} \in V(T_i)$  and  $x_{2j-1} \in V(T_j)$ . The path we associate with  $u$  is then  $\langle x_{2j-1}, y_i, x_{2i}, u, x_{2j-1}, y_j, x_{2j} \rangle$ , isomorphic to  $P_7$ .

We check that these paths are pairwise disjoint. Each uses exactly one vertex of  $\hat{Y}$ . Since  $F'$  has exactly one edge incident to each vertex of  $X'$ , for each  $i$  the vertices of  $T_i$  occur in exactly one of the paths. Hence these paths form a  $P_7$ -factor, and Theorem 4 applies.  $\square$

We now return to simple  $(3, 4)$ -biregular bigraphs. Although the examples constructed so far in this section all have  $P_7$ -factors, Casselgren [5] found a simple  $(3, 4)$ -biregular bigraph with no  $P_7$ -factor. We conjecture that every simple  $(3, 4)$ -biregular bigraph has the weaker property of having

a proper path-factor. It should also hold that Pyatkin's condition guarantees the existence of a proper path-factor in a simple  $(3, 4)$ -biregular bigraph, but this also seems difficult. We present a condition that guarantees a proper path-factor when combined with Pyatkin's condition.

Let  $G$  be a simple  $(3, 4)$ -biregular  $X, Y$ -bigraph having a full 3-regular subgraph  $H$ . Since  $|X| = 4k$  and  $|Y| = 3k$  for some  $k$ , we may let  $X' = X \cap V(H)$  and  $X_0 = X - X'$ , where  $X_0 = \{x_1^0, \dots, x_k^0\}$ . Since  $H$  is 3-regular,  $H$  has a proper 3-edge coloring. Fix such a coloring  $c$ , and let  $H'$  be the spanning subgraph of  $H$  whose edges are those with color 1 or 2 under  $c$ . Define an auxiliary graph  $F$  with vertex set  $Y$  by putting  $y_i y_j \in E(F)$  if  $H'$  has a  $y_i, y_j$ -path of length 2. Note that  $F$  may have multiple edges and is 2-regular, since each vertex of  $Y$  has one incident edge with each color under  $c$ . Since  $G$  is simple, the components of  $F$  are cycles of length at least 2.

Since  $G$  is  $(3, 4)$ -biregular, the neighborhoods of the vertices of  $X_0$  partition  $Y$  into triples; let  $T_i = N_G(x_i^0) = \{y_i^1, y_i^2, y_i^3\}$ . Let  $\mathbf{T}$  denote the family  $T_1, \dots, T_k$ .

**Definition 8.** For families of disjoint triples, we define a *transversal* to be a set  $S$  having exactly one element from each triple. For a family  $\mathbf{T}$  defined on the vertices of a 2-regular graph  $F$ , an *independent transversal* is a transversal  $S$  that is an independent set in  $F$ . A *spread transversal* is a transversal  $S$  such that, given directions on the cycles of  $F$ , for every vertex  $v$  of  $F$  that does not belong to  $S$ , there is a vertex of  $S$  among the next three vertices after  $v$  along the forward direction of its cycle in  $F$ . Let  $F^*$  be the 4-regular graph obtained from  $F$  by adding triangles whose vertex sets are the triples of  $\mathbf{T}$ . A *mixed transversal* is a transversal that restricts on each component of  $F^*$  to an independent transversal or a spread transversal.

Note that a spread transversal intersects each cycle of  $F$ .

**Theorem 9.** *Let  $G$  be a simple  $(3, 4)$ -biregular  $X, Y$ -bigraph having a full 3-regular subgraph  $H$ , and let  $F$  and  $\mathbf{T}$  be the 2-regular graph and triple system defined as above. If  $\mathbf{T}$  has a mixed transversal, then  $G$  has a proper path-factor.*

*Proof.* Let  $c$  be a proper 3-edge-coloring of  $H$ , and let  $M$  be the perfect matching in color 1. The  $i$ th triple in  $\mathbf{T}$  is  $\{y_i^1, y_i^2, y_i^3\}$ ; we may let  $y_i^1$  be the vertex of  $T_i$  in the mixed transversal  $Y_1$ . Let  $Y_2 = \{y_1^2, \dots, y_k^2\}$  and  $Y_3 = \{y_1^3, \dots, y_k^3\}$ . For  $x \in X' = X - X_0$ , put  $x \in X_j$  if the other endpoint of the edge of  $M$  at  $x$  lies in  $Y_j$ , and write  $x$  as  $x_i^j$  if that neighboring vertex is  $y_i^j$ . Since each vertex of  $Y$  has one neighbor via  $M$ , we have labeled  $X$  so that  $X' = \{x_i^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq 3\}$ .

We construct a proper path-factor of  $G$ , dealing separately with each component  $C$  of  $F^*$ . From  $C$  we generate paths in  $G$  that together cover  $V(C)$ , the neighbors of  $V(C)$  via  $M$ , and the vertices of  $X_0$  whose neighborhoods lie in  $V(C)$ . The construction depends on whether the restriction of

$\mathbf{T}$  to  $C$  has an independent transversal or a spread transversal. For simplicity of notation, we describe the construction in the case that  $F^*$  is connected. In the general case,  $V(C)$  is the union of  $T_i$  for  $i$  in some subset of  $\{1, \dots, k\}$ , and the construction in Case 1 or Case 2 covers all vertices in  $T_i \cup \{x_i^0, x_i^1, x_i^2, x_i^3\}$  for each such index  $i$ .

**Case 1:**  $Y_1$  is an independent transversal. We specify  $k$  paths of lengths 4, 6, or 8, each containing one vertex of  $X_0$ . Consider the paths  $\langle x_i^2, y_i^2, x_i^0, y_i^3, x_i^3 \rangle$  for  $1 \leq i \leq k$ . These paths are disjoint and cover  $V(G) - (X_1 \cup Y_1)$ . The  $2k$  endpoints of these paths form  $X_2 \cup X_3$ . Each vertex  $y_i^1$  of  $Y_1$  has one incident edge  $y_i^1 x$  having color 2 under  $c$ . Since  $Y_1$  is an independent transversal, this neighbor  $x$  lies in  $X_2 \cup X_3$ , not in  $X_1$ . Extend the original path of length 4 ending at  $x$  by adding  $xy_i^1$  and  $y_i^1 x_i^1$ . Altogether there are  $k$  such extensions to absorb  $Y_1 \cup X_1$ . Each of the original paths extends by zero or two edges at each end, so we have the factor using paths of the desired lengths.

**Case 2:**  $Y_1$  is a spread transversal. Again each path contains one vertex of  $X_0$ , but now we may also use length 2. Specify an orientation of each cycle in  $F$ , and delete the incoming edge to each vertex of  $Y_1$ . Since  $Y_1$  is a spread transversal, each cycle is cut, and what remains of  $F$  consists of  $k$  disjoint paths  $P_1, \dots, P_k$ , starting at  $y_1^1, \dots, y_k^1$ , respectively, each with length at most 3. By the definition of  $G$ , each edge in  $P_i$  expands to a path of length 2 in  $G$  having edges of colors 1 and 2 under  $c$ , yielding paths of even length (at most 6) ending in  $Y_2 \cup Y_3$ . If some  $P_i$  ends at  $y \in Y_2 \cup Y_3$ , then the neighbor of  $y$  in  $M$  is not covered by any of these paths. Thus extending each path  $P_i$  by adding  $x_i^0 y_i^1$  at the beginning and the edge  $yx$  of  $M$  at the end yields  $k$  disjoint paths of lengths in  $\{2, 4, 6, 8\}$  that cover  $V(G)$ .  $\square$

This method for finding proper factors is robust, since any proper 3-edge-coloring of  $H$  and any indexing of its colors can be used. Care is needed, since there exist 2-regular graphs  $F$  and triple systems  $\mathbf{T}$  where no mixed transversal exists, as shown in our final example.

**Example 3.** First we construct  $F_1$  with no independent transversal. Let  $k_1$  be a multiple of 6, and let  $F_1$  consist of  $k_1/2$  cycles of length 4 and  $k_1/3$  cycles of length 3. Name the 4-cycles as  $[y_{2i-1}^1, y_{2i}^1, y_{2i-1}^2, y_{2i}^2]$  for  $1 \leq i \leq k_1/2$ . Name the 3-cycles as  $[y_{6i-3}^3, y_{6i-1}^3, y_{6i+1}^3]$  and  $[y_{6i-4}^3, y_{6i-2}^3, y_{6i}^3]$  for  $1 \leq i \leq k_1/6$  (with  $y_{k_1+1} = y_1$ ). An independent partial transversal has at most one vertex in each cycle, and hence the largest independent partial transversal has at most  $k_1/2 + k_1/3$  elements.

Next we construct  $F_2$  with no spread transversal; for clarity, we use vertices  $z_i^j$  instead of  $y_i^j$ . Let  $k_2$  be a multiple of 2, and let  $F_2$  consist of  $3k_2/2$  cycles of length 2. Name the 2-cycles as  $[z_i^1, z_{i+1}^2]$  for  $1 \leq i \leq k_2$  (where  $z_{k_2+1} = z_1$ ) and  $[z_{2i-1}^3, z_{2i}^3]$  for  $1 \leq i \leq k_2/2$ . A transversal has only  $k_2$  elements, but a spread transversal must have an element in each of the  $3k_2/2$  cycles.

Both  $F_1^*$  and  $F_2^*$  are connected graphs. To construct an example with no mixed transversal,

we start with disjoint copies of  $F_1$  and  $F_2$ , with  $k_1 = 12$  and  $k_2 = 8$ . Exchange vertex  $y_1^1$  in  $F_1$  for  $z_1^1$  in  $F_2$ . This creates a new graph  $F$  such that  $F^*$  is connected; hence a mixed transversal must be an independent transversal or a spread transversal. A transversal can cover at most 11 of the 2-cycles from  $F_2$  and thus cannot be spread. On the other hand, at most 11 vertices of  $F_1$  can be chosen for an independent transversal, since at most one vertex from each 3-cycle and 4-cycle can be selected. We conclude that there is no mixed transversal.  $\square$

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