Induced Turán problems: Largest P_m-free graphs with bounded degree

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(Joint work with Myung Chung and Tao Jiang)

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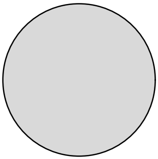
Defn. $G + H$ = disjoint union, tG = t copies of G; linear forest = disjoint union of paths, like $\sum_{i=1}^{t} P_{m_i}$.

Examples

P_2-free



P_3-free

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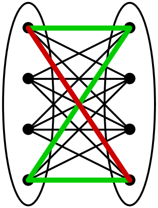
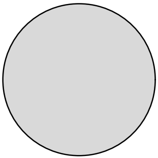


P_4-free



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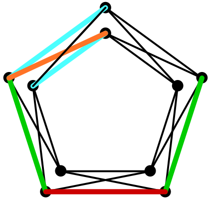

$2P_2$-free

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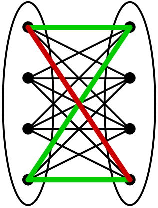

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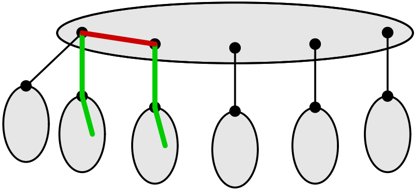
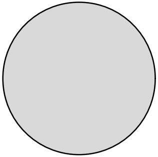


$2P_3$-free



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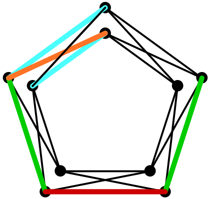

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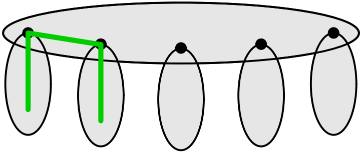

P_5-free

Results for Small or Special H

H	$\mathrm{ex}^*(D;H)$	graph	reference
P_2	0	K_1	trivial
P_3	$\binom{D+1}{2}$	K_{D+1}	trivial
P_4	D^2	$K_{D,D}$	M.Chung–West [1993]
$2P_2$	$\frac{5}{4}D^2$ (even D)	$C_5[\frac{D}{2}]$	F.Chung–Gyárfás–Tuza–Trotter [1990]
P_2+P_3	$< 2D^2$	$C_5[\frac{D}{2}]$??	M.Chung [1993]
$2P_3$	$\frac{1}{8}D^4+O(D^3)$	(later)	M.Chung–West [1996]
P_5	$\frac{2}{27}D^3+O(D^2)$	(later)	new (exact for $D \geq 187$)
P_m (even)	$\sim \frac{1}{2}D^{m/2}$	tree-like	new
P_m (odd)	$\sim \frac{1}{8}D^{(m+1)/2}$	tree-like	new (exact max #verts)
$2P_m$	$\Theta(D^2\mathrm{ex}^*(D;P_m))$	like $2P_3$	new

Importance of the Case $H = P_m$

Prop. Chung–West [1996] (for $H_2 = P_m$)

$\mathrm{ex}^*(D; H_1 + H_2) \le \max\{\mathrm{ex}^*(D; H_1), n(H_1)D^2(\mathrm{ex}^*(D; H_2) + 1)\}$.

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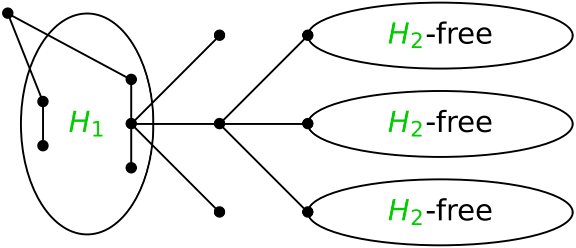

When G is not H_1-free, components of $G - N(V(H_1))$ must be H_2-free. ■

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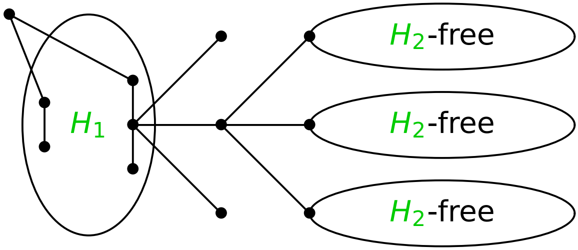

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Cor. $\mathrm{ex}^*(D; tP_m) \le (1 + o(1))(m\sqrt{t}D^2)^{\lceil \lg t \rceil}\mathrm{ex}^*(D; P_m)$

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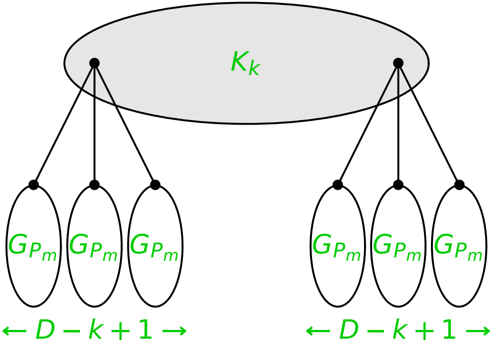


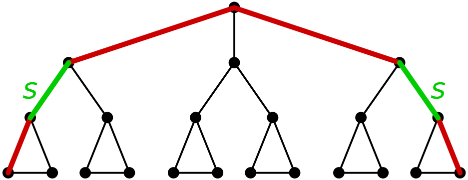
Let G_{P_m} = P_m-free graph with maximum degree D.

Each induced P_m must contain a vertex of K_k.

Optimize by setting k = $\lceil (D+1)/2 \rceil$. ■

Constructions for $H = P_m$ with $m \geq 6$

length $m-2$ allowed



$m = 8,\ D = 3,\ s = \frac{m-6}{2}$

$e(G_{P_m}) \sim D(D-1)^{(m-6)/2}\binom{D}{2} \sim \frac{1}{2}D^{m/2}$

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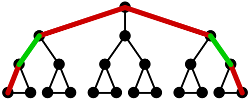

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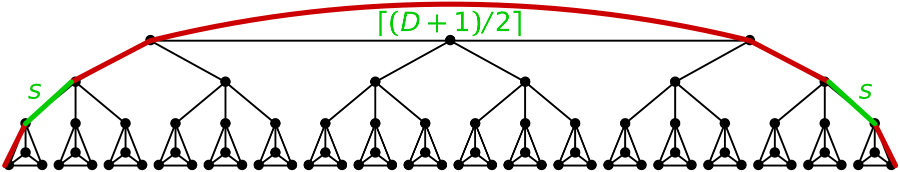

$m = 9,\ D = 4,\ s = \frac{m-7}{2}$

$$e(G_{P_m}) \sim \left\lfloor \frac{(D+1)^2}{4} \right\rfloor (D-1)^{(m-7)/2} \binom{D}{2} \sim \frac{1}{8}D^{(m+1)/2}$$

Preliminaries for $m > 5$

- $e(G) \leq \frac{D}{2} n(G)$; bound $e(G)$ by bounding $n(G)$.
- A tree of depth s has $O(D^s)$ vertices.
- P_m-free $\Rightarrow$ $\text{diam}\, G \leq m-2$ $\Rightarrow$ $n(G) \leq O(D^{m-2})$.

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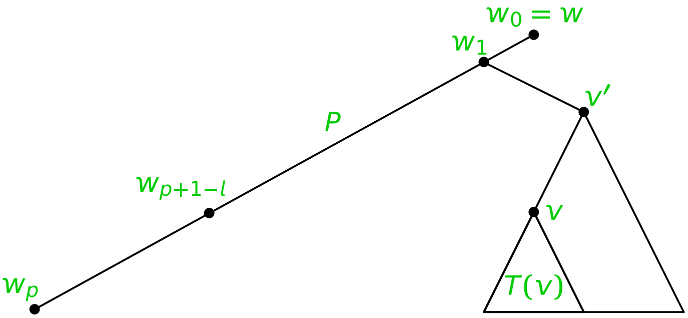
- Idea: Capture the vertices in trees only half as deep.

Construction of Tree T'

- Let T = BFS tree (of depth p) from a vertex w.

Let $P = w_0, \ldots, w_p$ and $T(v)$ = subtree of T rooted at v.



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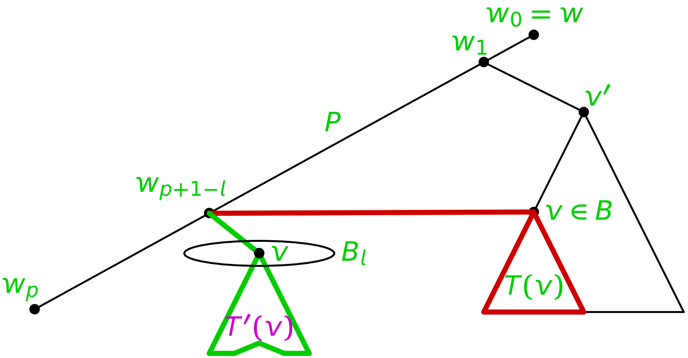
Form T' by moving $T(v)$ for $v \in B$ under last nbr on P.
Let B_l = vertices of B under w_{p+1-l} in T'.



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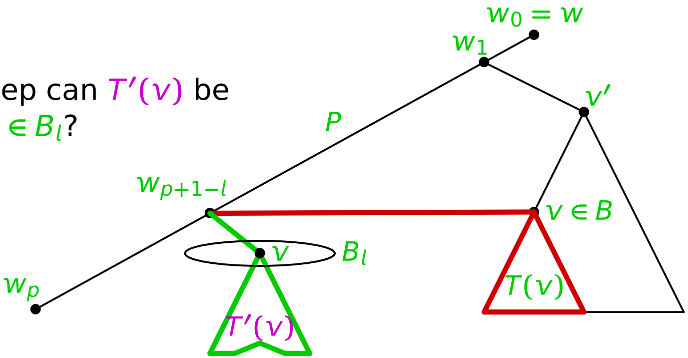
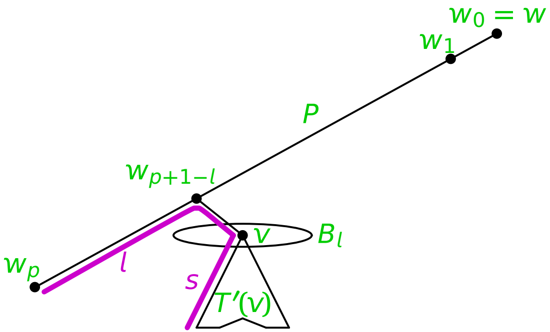
How deep can $T'(v)$ be when $v \in B_l$?



Depth of Subtrees

Lem. If $v \in B_l$ and s = depth of $T'(v)$, then $s \leq \min\{l, (m-2)-l\}$.

Pf. w_p, v-path in T' plus path to leaf of $T'(v)$ is induced path of length $l+s$, so $s \leq (m-2)-l$.



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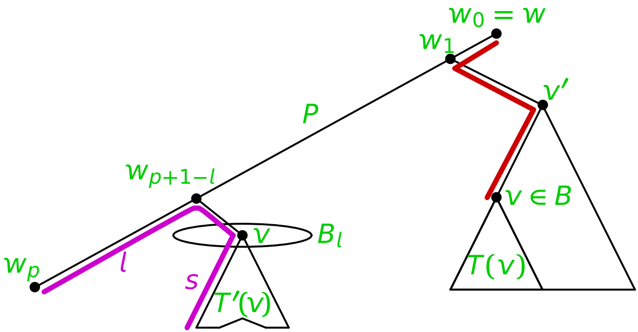

Level of v in T is $d_G(w, v)$; max level p is $d_G(w, w_p)$.
So, $s \le d_G(w, w_p) - d_G(w, v) \le d_G(v, w_p) \le l$. ∎

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Lem. $\mathrm{ex}^*(D; P_m) \le D^{\lceil m/2 \rceil + 1} + O(D^{\lceil m/2 \rceil})$.

Pf. Let $V_l = \bigcup_{v \in B_l} T'(v)$.

If $v \in B_l$ or $v \in B_{m-2-l}$ with $l \le m/2 - 1$, then $s \le l$.

Also $|B_l| < D$, so $|V_l| \le D^{s+1} + O(D^s)$.

Max only at $l \in \{\lfloor m/2 \rfloor - 1, \lceil m/2 \rceil - 1\}$, so
$n(G) \le 2D^{\lceil m/2 \rceil} + O(D^{\lceil m/2 \rceil - 1})$. (times $D/2$) ■

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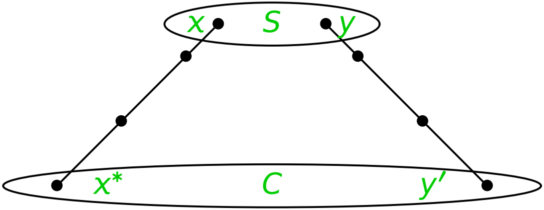
- Idea: To save a factor of D, note that #vertices outside $\bigcup_{l=\lceil m/2 \rceil - 2}^{\lfloor m/2 \rfloor} V_l$ is $O(D^{\lceil m/2 \rceil - 2})$.

Capture the remainder using fewer full-depth trees.

Key Lemma

Defn. S is k-distant from C if $d(v, S) = k$ for $v \in C$.

For $x, y \in S$ and $y' \in C$, if $d(y', y) = k$ and $d(y', x) > k$, then y' requires y (relative to x).



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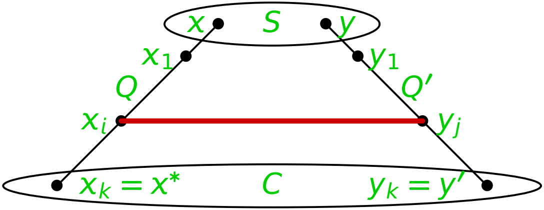

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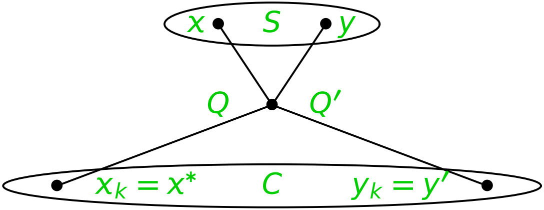

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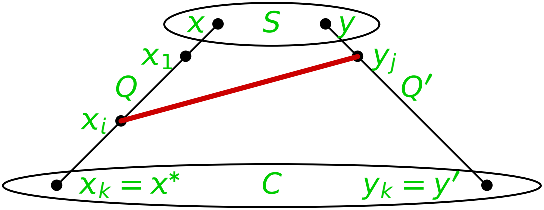

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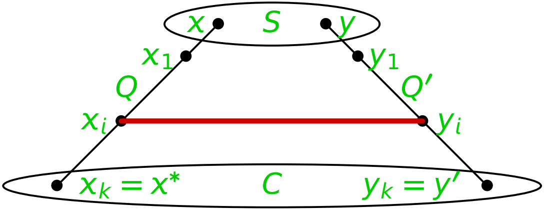

3) $x_i \leftrightarrow y_i \Rightarrow d(y',x_1) \le k$; equal since $d(y',x)>k$. ∎

Asymptotic Answer for Odd m

Thm. m odd and $m > 5 \Rightarrow \mathrm{ex}^*(D; P_m) \sim \frac{1}{8}D^{(m+1)/2}$.

Pf. It suffices to show $|A| \leq \frac{1}{4}D^{\lfloor m/2 \rfloor} + O(D^{\lfloor m/2 \rfloor - 1})$, where $A = V_{\lfloor m/2 \rfloor - 1} \cup V_{\lceil m/2 \rceil - 1}$.

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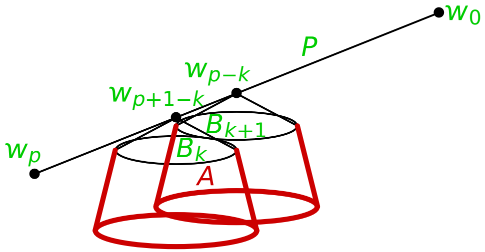

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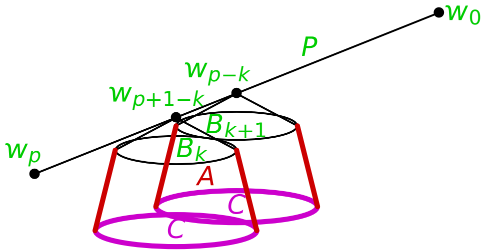
$C = \{v \in A\colon\ d(k, B') = k\}$.

Note $|B'| < 2D$, so $A - C$ is covered by $2D$ trees, each with depth $\le k - 1$, so $|A - C| < 2D^k + O(D^{k-1})$.



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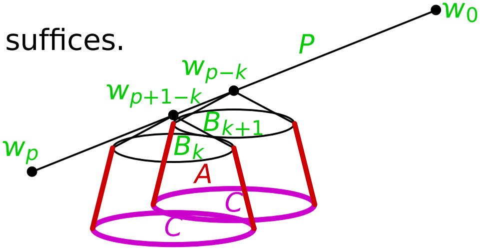
$\therefore$ $|C| \le \frac{1}{4} D^{k+1} + O(D^k)$ suffices.



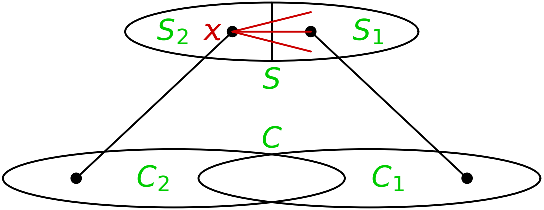
Proving $|C| \leq \frac{1}{4}D^{k+1} + O(D^k)$

Let S = a minimal subset of B' that is k-distant from C.

Choose $x \in S$ with fewest neighbors in S.

Let S_1 = $N(x) \cap S$; $S_2 = S - S_1$.

Let C_i = $\{v \in C :\ d(v, S_i) = k\}$;   note that $C = C_1 \cup C_2$.



Proving $|C| \leq \frac{1}{4}D^{k+1} + O(D^k)$

Let S = a minimal subset of B' that is k-distant from C.

Choose $x \in S$ with fewest neighbors in S.

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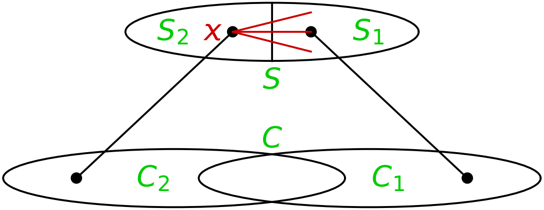


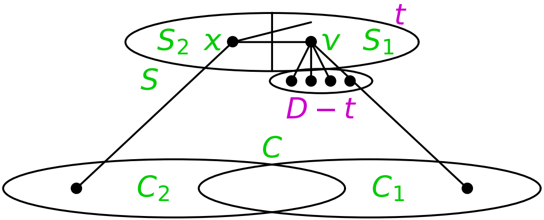
Claim 1: $|C_1| < \frac{1}{4}D^{k+1}$.

Claim 2: $|C_2| < 2D^k$.

Proof of Claim 1: $|C_1| < \frac{1}{4}D^{k+1}$

Let $t = |S_1| = |N(x) \cap S|$.

By the choice of x, each $v \in S_1$ has $\geq t$ nbrs in S and $\leq D - t$ nbrs outside S.



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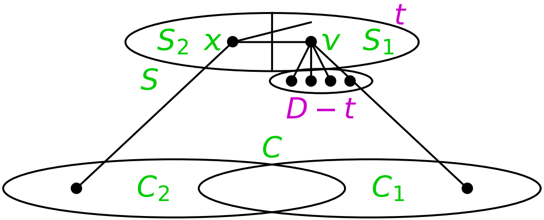
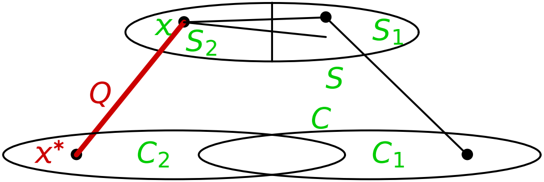


A path of length k from S_1 to C_1 leaves S immediately.

At most $t(D-t)$ vertices reached in one step;
$k-1$ more steps $\Rightarrow$ $|C_1| < \frac{1}{4}D^2D^{k-1}$.

Proof of Claim 2: $|C_2| < 2D^k$

Choose x^* that requires x, and Q a shortest x, x^*-path.



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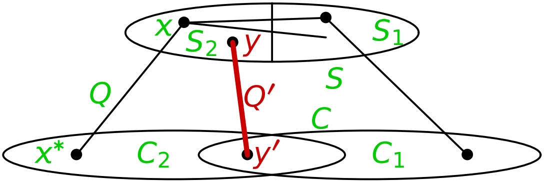

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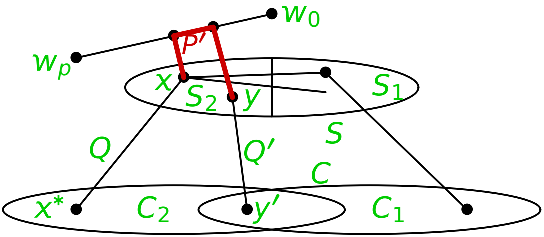

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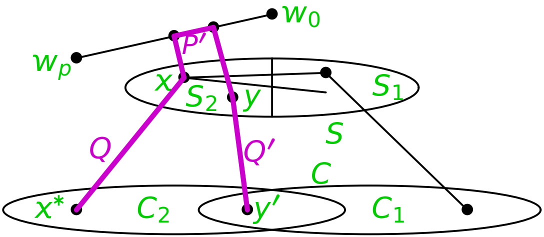

If $E(Q, Q') = \varnothing$, then $Q \cup P' \cup Q'$ is induced path;
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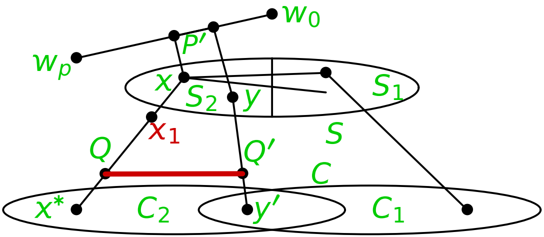

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$\therefore\; E(Q, Q') \neq \varnothing$, so Key Lemma $\Rightarrow\; d(y', x_1) = k$.

Now C_2 is covered by trees from x and x_1. ■

Further Results

Thm. [2 more pages] For m odd and $D \geq m + 20 > 25$, the general construction has the most vertices among connected P_m-free graphs with maximum degree D.

Thm. [7.5 more pages] When m is even and $m > 5$, $\mathrm{ex}^*(D; P_m) = \frac{1}{2}D^{m/2} + O(D^{m/2-1})$.

What about $m = 5$?

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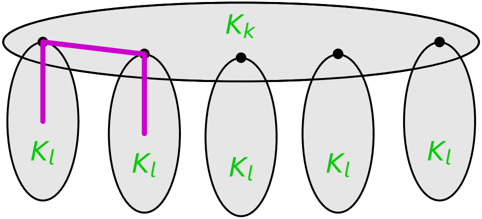
Construction



$l = D + 2 - k$

$e(G) = \binom{k}{2} + k\binom{D+2-k}{2}$

At $k = \left\lceil \frac{D+1}{3} \right\rceil$ equals $\frac{2}{27}D^3 + \frac{7}{18}D^2 + O(D)$.

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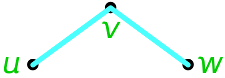

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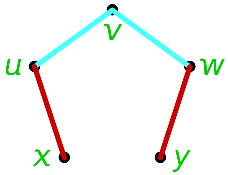

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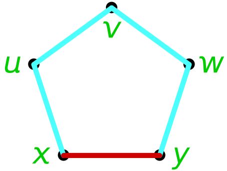
P_5-free $\Rightarrow x \leftrightarrow y$, creating induced 5-cycle C.



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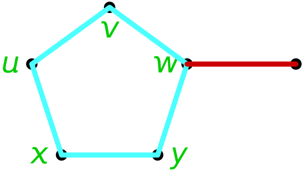

If some vertex has $=1$ nbr on C, then $\exists$ induced P_5.
If each has ≥ 2 nbrs, then $n(G) \le 5 + \frac{1}{2}5(D-2) = \frac{5}{2}D$. ■

Optimality for $H = P_5$

Thm. If $D \geq 187$, then $\mathrm{ex}^*(D; P_5) =$

$$\frac{2}{27}D^3 + \frac{7}{18}D^2 + \begin{cases} D/2 & D \equiv 0 \mod 3 \\ D/2 + 1/27 & D \equiv 1 \mod 3 \\ 7D/18 + 2/27 & D \equiv 2 \mod 3 \end{cases}$$

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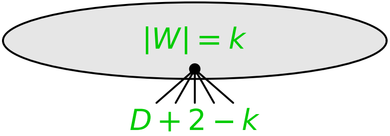

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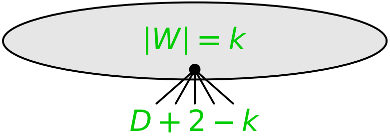


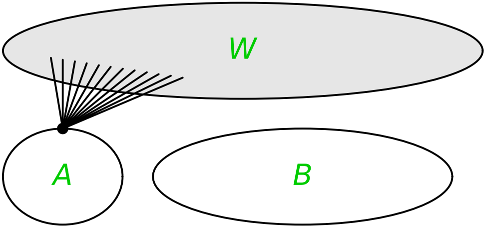
$k > \frac{5D}{6}$ or $k < \frac{D}{6}$ $\Rightarrow$ $n(G) \leq \frac{5}{36}D(D+2)$.

$\frac{5D(D+2)}{36} < \frac{4}{27}D^2$ for $D > 30$ $\Rightarrow$ may assume $\frac{D}{6} \leq k \leq \frac{5D}{6}$.

A Vertex Partition

Let $A = \{v \notin W : \ |N(v) \cap W| \geq 12\}$; $\quad B = V(G) - W - A$.

Let $a = |A|$ and $b = |B|$, so $\quad n(G) = k + a + b$.



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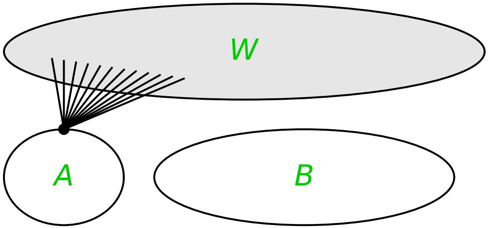

At most $k(D-k+1)$ edges leave W, so $a \leq \frac{k(D-k+1)}{12}$.

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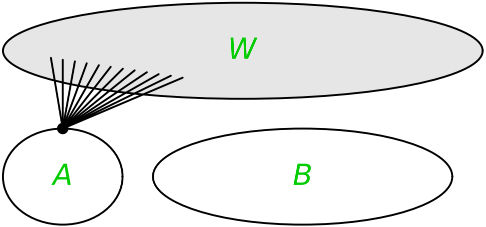

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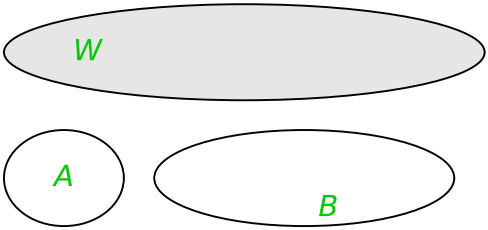
which is bounded by $\frac{4}{27}D^2 + \frac{14}{18}D$ when $D \geq 187$.

Bounding $\Delta(G[B])$

$\dots$ We may therefore assume $b > 24D$.

Claim: $\Delta(G[B]) \leq D - k$ (otherwise find induced P_5).



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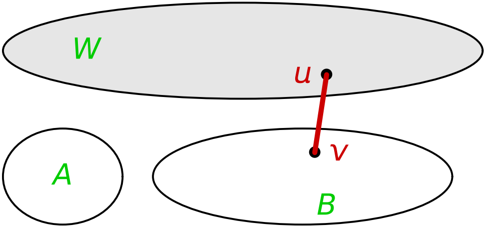

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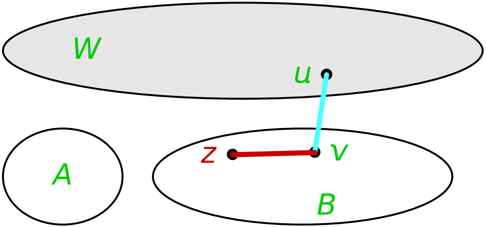

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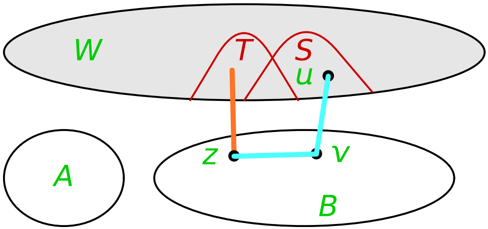

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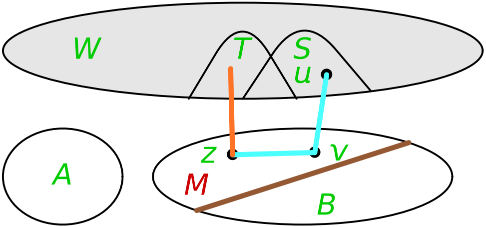

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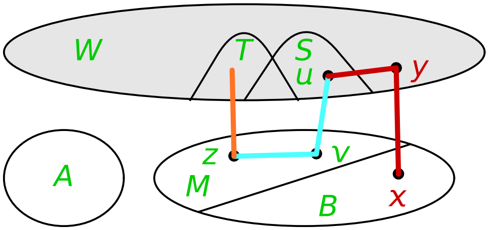

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Thus $\exists\ x \in B - M$. Choose $y \in N(x) \cap W$. Now $\exists\ P_5$!

End of proof

Count the edges of G.

Within W:	$= \binom{k}{2}$
From W to B:	$< k(D-k+1)$
Within $G[B]$:	$\leq b(D-k)/2$
Incident to A:	$\leq aD$

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- $k \leq 5D/6 \iff aD \leq 12a(D-k)/2$.

$$e(G) \leq \binom{k}{2} + k(D-k+1) + (12a+b)\frac{D-k}{2}$$

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$$
\begin{aligned}
e(G) &\leq \binom{k}{2} + k(D-k+1) + (12a+b)\frac{D-k}{2} \\
&\leq \binom{k}{2} + k\binom{D-k+2}{2}
\end{aligned}
$$

- Count edges from W to get $12a + b \leq k(D-k+1)$.

Open Problems

- What is $\mathrm{ex}^*(D; P_2 + P_3)$?

M. Chung proved $\frac{5}{4}D^2 \leq \mathrm{ex}^*(D; P_2 + P_3) < 2D^2$.
She proved $= \frac{5}{4}D^2$ for triangle-free graphs.

- Let $f(D) = \mathrm{ex}^*(D; 2P_m)/(D^2 \mathrm{ex}^*(D; P_m))$.

We know $\frac{1}{4} \leq f(D) \leq m$; is there a limit?

- What is $\mathrm{ex}^*(D; tP_m)$? For $m = 2$,

we know $(3t-1)D^2/4 \leq ex*(D; tP_2) < (2\sqrt{t}D)^{\lceil \lg t \rceil}$.

- What happens for D-regular P_m-free graphs?

A construction achieves roughly $\mathrm{ex}^*(D; P_m)/2^{m/2}$ edges.