The Chinese Postman Problem in Regular Graphs of Odd Degree

Suil O*, Douglas B. West†

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Abstract

The Chinese Postman Problem in a graph is the problem of finding a shortest closed walk traversing all the edges. In a $(2r+1)$-regular graph, the problem is equivalent to finding a smallest spanning subgraph in which all vertices have odd degree. We establish a sharp upper bound for the solution in 3-regular graphs, characterize when equality holds, and conjecture the answer for general r.

1 Introduction

The Chinese Postman Problem was introduced in the early 1960s by the Chinese mathematician Guan Meigu. Roughly speaking, a postman wishes to traverse every road in a city to deliver the mail, using the least possible total distance. A *postman tour* in a connected graph G is a closed walk containing all the edges of G. An *optimal* postman tour in a connected graph G is a shortest closed walk traversing all edges in G. Since all edges of G must be used, we are interested only in the additional length needed. Let $p(G) = l - |E(G)|$, where l is the minimum length of a postman tour.

Since a postman tour is an Eulerian supergraph obtained by repeating some edges, $p(G)$ equals the minimum number of edges in a parity subgraph of G, where a *parity subgraph* is a spanning subgraph H of G such that $d_G(v) \equiv d_H(v) \mod 2$ for every vertex v in G. Thus we call $p(G)$ the *parity number* of G. Kostochka and Tulai [5] obtained upper bounds on

*Department of Mathematics, University of Illinois, Urbana, IL, 61801, suilo2@math.uiuc.edu. research partially supported by the Korean Research Foundation (MOEHRD, Basic Research Promotion Fund), grant KRF-2005-C00003.

†Department of Mathematics, University of Illinois, Urbana, IL, 61801, west@math.uiuc.edu; research is partially supported by the National Security Agency under Award No. H98230-10-1-0363.

the parity number of $(2r+1)$-regular $2t$-edge-connected multigraphs; their bounds are sharp among multigraphs and when t is near r, but they are not sharp among graphs.

We obtain a sharp upper bound on the parity number of a connected 3-regular graph with n vertices. We will show that for $n > 10$, equality holds only for graphs in a family $\mathcal{H}'_r$ we introduced in [6] and describe in Section 2; these graphs exist when n lies in an appropriate congruence class. We conjecture that the result extends for regular graphs with larger odd degree. Let $\mathcal{F}_{n,r}$ be the family of all connected $(2r+1)$-regular graphs with n vertices.

Conjecture 1.1. *If $G \in \mathcal{F}_{n,r}$, then $p(G) \leq \frac{(2r^2+3r-1)n-2(r+1)}{4r^2+4r-2} - 1$. Furthermore, for $n > 4r+6$, equality holds if and only if $n \equiv (4r+6) \mod (4r^2+4r-2)$ and $G \in \mathcal{H}'_r$.*

In Section 3, we prove the conjecture for $r = 1$.

2 The Construction

We construct a family $\mathcal{H}'_r$ of $(2r+1)$-regular graphs with large parity number. We introduced this family in [6], where we showed that these graphs have the most cut-edges among graphs in $\mathcal{F}_{n,r}$ when $n \equiv (4r+6) \mod (4r^2+4r-2)$. In addition, we showed that the graphs in $\mathcal{F}_{n,r}$ minimizing the matching number are a particular subfamily of $\mathcal{H}'_r$ (for such n); this is generalized in [7].

Construction 2.1. Let B_r be the graph obtained from the complete graph K_{2r+3} by deleting a matching of size $r+1$ and deleting one more edge incident to the remaining vertex. This is the smallest graph in which one vertex has degree $2r$ and the others have degree $(2r+1)$. Note that deleting the vertex of degree $2r$ from B_r leaves a subgraph having a perfect matching.

Let $\mathcal{T}'_r$ be the family of trees such that every non-leaf vertex has degree $2r+1$. Let $\mathcal{H}'_r$ be the family of $(2r+1)$-regular graphs obtained from trees in $\mathcal{T}'_r$ by identifying each leaf of such a tree with the vertex of degree $2r$ in a copy of B_r.

The smallest graph in $\mathcal{H}'_r$ has $4r+6$ vertices. Larger graphs in $\mathcal{H}'_r$ are grown from smaller ones by turning a copy of B_r back into a leaf x, attaching $2r$ new leaf neighbors to x, and attaching B_r at each new leaf. The new graph has $(2r-1)(2r+3)+1$ more vertices. Hence the number of vertices of each graph in $\mathcal{H}'_r$ is congruent to $4r+6$ modulo $4r^2+4r-2$. □

We next compute the parity number for n-vertex graphs in $\mathcal{H}'_r$; this proves sharpness of Conjecture 1.1. A *balloon* in a graph G is a maximal 2-edge-connected subgraph that is incident to exactly one cut-edge of G. The copies of B_r in a graph in $\mathcal{H}'_r$ are balloons, and they correspond to leaves in a tree in $\mathcal{T}'_r$. Note that B_r is the smallest possible balloon in a $(2r+1)$-regular graph (it plays a crucial role also in [2]). We have observed in Construction 2.1 that when growing graphs in $\mathcal{H}'_r$, each increase of $4r^2+4r-2$ in n increases the number of balloons (leaves of the tree) by $2r-1$ and increases the number of cut-edges (edges of the tree) by $2r$. This inductively proves the computation below.

Proposition 2.2. ([6]) *Let $q_r = 2r^2 + 2r - 1$, and let $b(G)$ and $c(G)$ denote the numbers of balloons and cut-edges in a graph G, respectively. For any n-vertex graph G in $\mathcal{H}'_r$,*

$$b(G) = \frac{(2r-1)n+2}{2q_r} \qquad \text{and} \qquad c(G) = \frac{r(n-2)-2}{q_r} - 1.$$

Lemma 2.3. *If G is regular of odd degree, then every cut-edge is in every parity subgraph.*

Proof. Let e be a cut-edge in G. By the Degree-Sum Formula, each component of $G-e$ has an odd number of vertices. Since a parity subgraph has odd degree at each vertex, the Degree-Sum Formula then implies that the parity subgraph must contain e. □

Next, we determine the parity number of graphs in $\mathcal{H}'_r$.

Proposition 2.4. *If G is a graph in $\mathcal{H}'_r$, then*

$$p(G) = \frac{(2r^2+3r-1)n - 2(r+1)}{4r^2+4r-2} - 1,$$

which reduces to $\frac{2n-5}{3}$ for 3-regular graphs.

Proof. Let T be the tree obtained by shrinking all the balloons in G. Each edge of T is a cut-edge in G and belongs to every parity subgraph of G, by Lemma 2.3. Each vertex of each balloon other than the one having degree $2r$ in the balloon must also be incident to an edge in the parity subgraph, so a parity subgraph must have at least $b(G)(2r+2)/2$ edges besides the cut-edges. Thus

$$\begin{aligned} p(G) \geq c(G) + (r+1)b(G) &= \frac{r(n-2)-2}{q_r} - 1 + (r+1)\frac{(2r-1)n+2}{2q_r} \\ &= \frac{(2r^2+3r-1)n - 2(r+1)}{4r^2+4r-2} - 1, \end{aligned}$$

by Proposition 2.2. By taking all edges of T plus a near-perfect matching in each copy of B_r, equality is achieved. □

3 The Upper Bound

Definition 3.1. *A k-graph*[1] *is a k-regular multigraph G with an even number of vertices such that for every odd-sized subset X of $V(G)$, the number of edges with exactly one endpoint in X is at least k.*

[1] There are at least three different meanings for this term in the literature. Seymour [8] used the definition above, conjecturing that if G is a k-graph, then $\chi'(G) \leq k+1$. In Berge's book [1], a k-graph is a directed multigraph with edge-multiplicity at most k. In [4] and elsewhere, a k-graph is a k-uniform hypergraph.

Remark 3.2. Every 2-edge-connected 3-regular multigraph is a 3-graph, since the Degree-Sum Formula requires three edges leaving every odd-sized subset S of $V(G)$. More generally, if G is a $(k-1)$-edge-connected k-regular multigraph with even order, then G is a k-graph for the same reason. Also, every k-edge-colorable k-regular graph is a k-graph.

We need a fundamental result about k-graphs due to Edmonds.

Theorem 3.3. (Edmonds [3]) *If G is a k-graph, then there is an integer p and a family $\mathcal{M}$ of perfect matchings such that each edge of G is contained in precisely p members of $\mathcal{M}$. (The members of $\mathcal{M}$ need not be distinct.)*

Lemma 3.4. *If G is a $2r$-edge-connected $(2r+1)$-regular multigraph, in which each edge e has weight $w(e)$, then there exists a perfect matching with weight at most $\frac{1}{2r+1}W$, where $W = \sum_{e\in E(G)} w(e)$. For 3-regular graphs, the bound reduces to $\frac{1}{3}W$.*

Proof. Let $\mathcal{M}$ be a family of perfect matchings as guaranteed by Theorem 3.3. By counting two ways, $|\mathcal{M}|\frac{n}{2} = \frac{(2r+1)n}{2}p$, which yields $|\mathcal{M}| = p(2r+1)$. Let $\mathcal{M} = \{M_1, \cdots, M_{p(2r+1)}\}$, and let $w(M_i)$ be the total weight of all edges in M_i. Since $\sum w(M_i) = p\sum_{e\in E(G)} w(e) = pW$, a lightest matching in the family has weight at most $\frac{1}{2r+1}W$, by the pigeonhole principle. $\square$

For the proof of the main result, we need the concept of "threads". A *thread* in a graph G is a maximal path in G whose internal vertices have degree 2 in G. We also need a bound on the number of balloons in a graph in $\mathcal{F}_{n,r}$. When $r = 1$, the bound reduces to $(n+2)/6$.

Lemma 3.5. ([6]) *If $G \in \mathcal{F}_{n,r}$, then $b(G) \le \frac{(2r-1)n+2}{4r^2+4r-2}$, with equality if and only if $G \in \mathcal{H}'_r$.*

Theorem 3.6. *If $G \in \mathcal{F}_{n,1}$ and $n \ge 10$, then $p(G) \le \frac{2n-5}{3}$. Equality holds when $G \in \mathcal{H}'_1$.*

Proof. Consider $G \in \mathcal{F}_{n,1}$. If G has no balloons or if $n = 10$, then G has a perfect matching and $p(G) = \frac{n}{2} \le \frac{2n-5}{3}$. Otherwise, G has a balloon and $n > 10$. For $n > 10$, we proceed by induction. Let e be a cut-edge. Let G_1 and G_2 be the components of $G-e$. Since a cut-edge must appear in every parity subgraph (Lemma 2.3), $p(G) = p(G_1) + p(G_2) + 1$.

Let G'_1 and G'_2 be the graphs obtained from G by replacing G_2 and G_1, respectively, with B_1. Every parity subgraph of G'_i contains e and a parity subgraph of G_i, and it uses at least two edges in its copy of B_1. Thus $p(G'_i) = p(G_i) + 3$, so $p(G) = p(G'_1) + p(G'_2) - 5$.

If neither G_1 nor G_2 is B_1, then G'_1 and G'_2 are smaller than G. Letting $n_i = |V(G'_i)|$, we have $n = n_1 + n_2 - 10$. By applying the induction hypothesis to both G'_1 and G'_2,

$$p(G) = p(G'_1) + p(G'_2) - 5 \le \frac{2n_1-5}{3} + \frac{2n_2-5}{3} - 5 = \frac{2n-5}{3}. \tag{1}$$

In the remaining case, every cut-edge is incident to a copy of B_1. Let each edge have weight 1. Form G' by deleting all the vertices of all the balloons (for each balloon, we lose eight edges). If G' is a cycle, then G has a perfect matching (each balloon plus its pendant edge has a perfect matching), and

$$p(G) = \frac{n}{2} < \frac{2n-5}{3}. \tag{2}$$

Otherwise, replace each thread of G' through vertices of degree 2 with a single edge whose weight is the length of the thread. Since the vertices of degree 2 have been suppressed and G' is 2-edge-connected, the resulting weighted graph G'' is a 3-graph, by Remark 3.2. Applying Lemma 3.4, G'' has a perfect matching with at most 1/3 of its total weight. The total weight is $\frac{m-8b}{3}$, where m is the number of edges in G and b is the number of balloons in G. Using Lemma 3.5 with $r = 1$, we have

$$p(G) \leq \ p(G')+3b \leq \frac{m-8b}{3}+3b = \frac{3n-16b}{6}+3b = \frac{n}{2}+\frac{b}{3} \leq \frac{n}{2}+\frac{1}{3}\left(\frac{n+2}{6}\right) \leq \frac{2n-5}{3}. \tag{3}$$

We have proved that $p(G) \leq \frac{2n-5}{3}$ for a connected 3-regular graph G.

By Proposition 2.4, equality holds for graphs in $\mathcal{H}_1$. □

Since $\frac{10}{2} = \frac{2*10-5}{3}$, 10-vertex connected 3-regular graphs without cut-edges also achieve equality even though they are not in $\mathcal{H}_1'$. However, a graph G with more than 10 vertices satisfying $p(G) = \frac{2n-5}{3}$ must be in $\mathcal{H}_1'$.

Theorem 3.7. *If $G \in \mathcal{F}_{n,1}$, then $p(G) = \frac{2n-5}{3}$ if and only if $n = 10$ or $G \in \mathcal{H}_1'$.*

Proof. It suffices to show that if $G \in \mathcal{F}_{n,1}$ and $p(G) = \frac{2n-5}{3}$, then $n = 10$ or $G \in \mathcal{H}_1'$. If $n < 10$, then G has a perfect matching, and $p(G) = \frac{n}{2} > \frac{2n-5}{3}$.

For $n > 10$, we use induction on n as in the proof of Theorem 3.6. To achieve equality in (1), we must have $p(G_i) = \frac{2n_i-5}{3}$ for $i \in \{1, 2\}$. Since neither component obtained by deleting the cut-edge is B_1, we have $|V(G_i)| > 10$. Thus, the induction hypothesis applies, and G_i' is in $\mathcal{H}_1'$. Since shrinking the balloons in G_1' or G_2' yields a tree whose internal vertices have degree 3, the same holds also for G, so also G is in $\mathcal{H}_1'$.

In the case where all cut-edges are incident to balloons, we have three subcases. If deleting the balloons leaves a cycle, then $p(G) = \frac{n}{2} < \frac{2n-5}{3}$. If it leaves a single vertex, then $n = 16$, $b = 3$ and $G \in \mathcal{H}_1'$. If it leaves a graph with maximum degree 3, then the last part of (3) states $p(G) \leq \frac{5n+1}{9} \leq \frac{2n-5}{3}$, with equality only when $n = 16$. □

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