The Chinese Postman Problem in Regular Graphs of Odd Degree

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Joint work with
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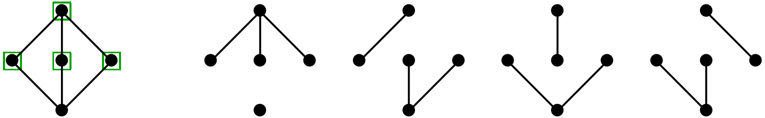

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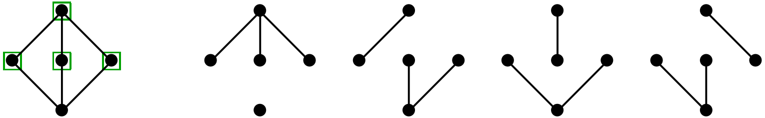

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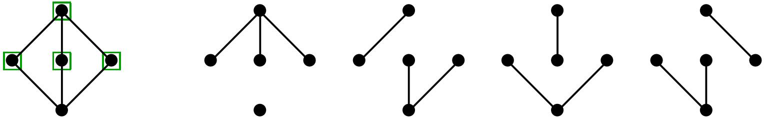

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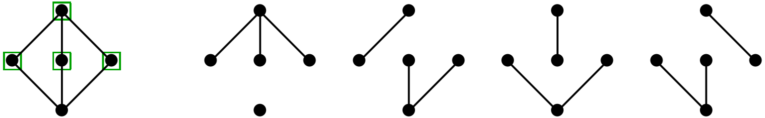

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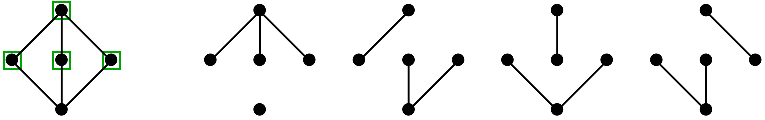

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- Applied by Kostochka–Tulai [1996] to upper bounds on $p(G)$ for $(2r+1)$-regular n-vertex G (by edge-conn.).

Results

Thm. (special case of Kostochka–Tulai [1996]) If G is a $(2r+1)$-regular n-vertex graph ($n \geq 4r+6$), then

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Equal denominators for $r = 1$, but not for larger r. ???

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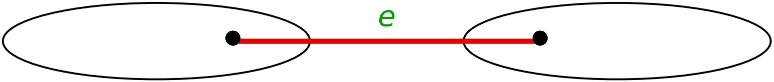

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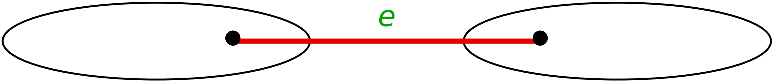

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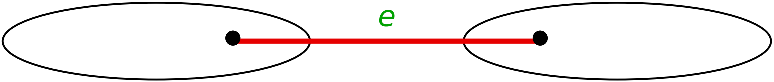

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Seek regular graphs with many cut-edges.

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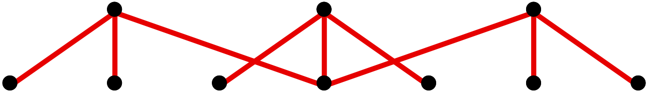

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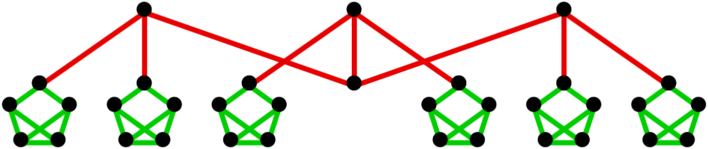

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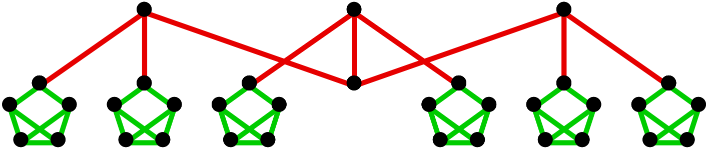

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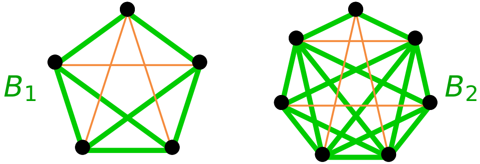
$B_r = \overline{rK_2 + P_3}$ (from K_{2r+3}, delete a maximum matching plus one edge at the last vertex).

B_r is nearly $(2r+1)$-regular (one vertex of degree $2r$).



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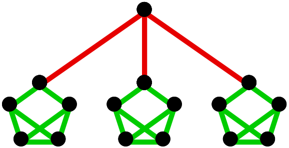

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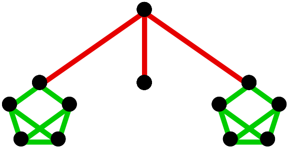

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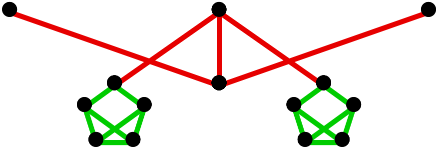

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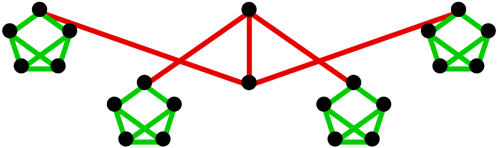

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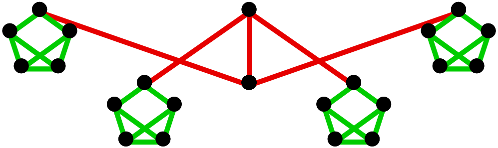

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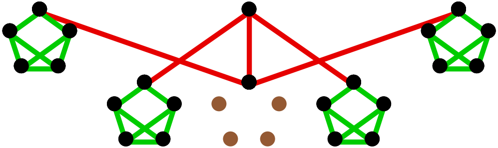

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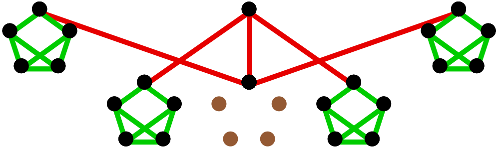

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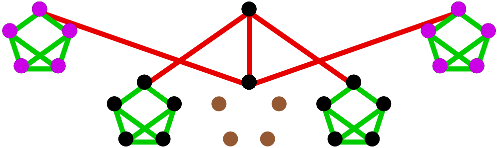

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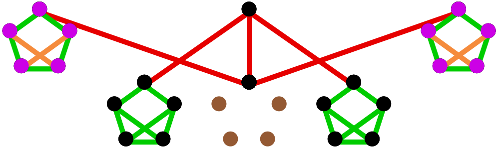

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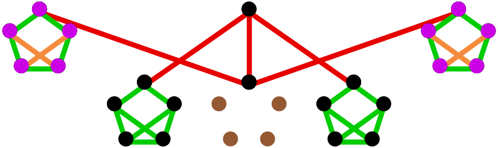

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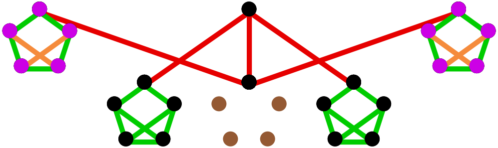

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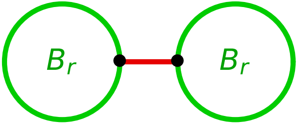
Pf. Basis



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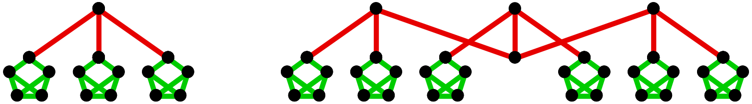

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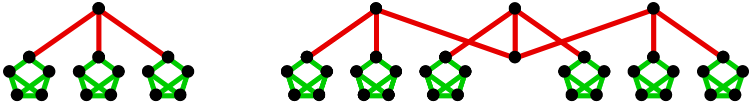

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Other Properties of $\mathcal{H}_r$ (O–West [2010])

Def. balloon = a maximal 2-edge-connected subgraph incident to one cut-edge.

- B_r = smallest balloon among $(2r+1)$-regular graphs.

For $G \in \mathcal{H}_r$, #balloons = $2 + \frac{(2r-1)(n-4r-6)}{2r(2r+2-1/r)}$.

Def. $\mathcal{F}_n^r$ = $(2r+1)$-regular n-vertex connected graphs.

Lem. In $\mathcal{F}_n^r$, #balloons maximized $\Leftrightarrow$ $G \in \mathcal{H}_r$. $\frac{n+2}{6}$

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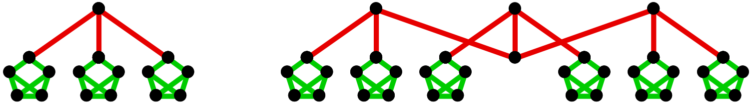

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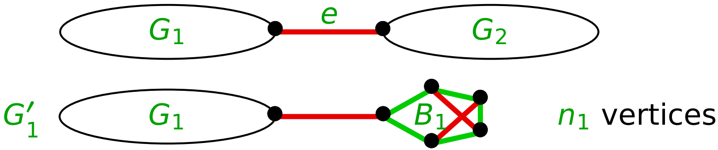

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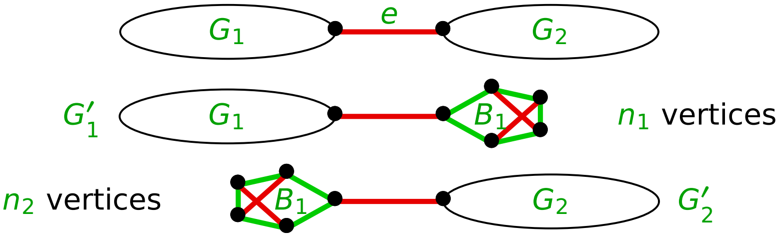

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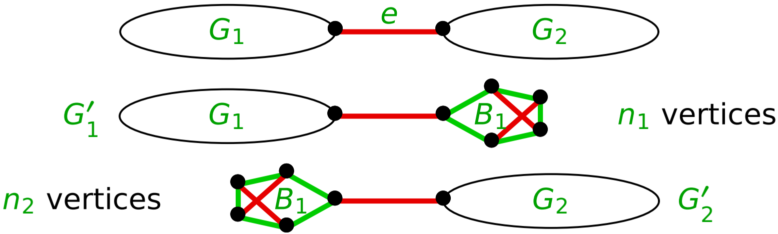

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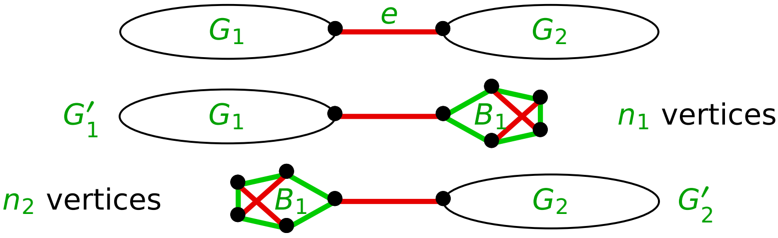

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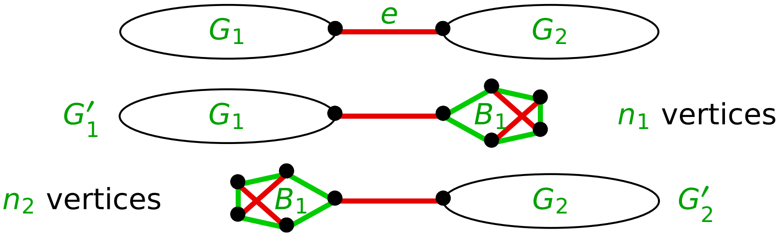

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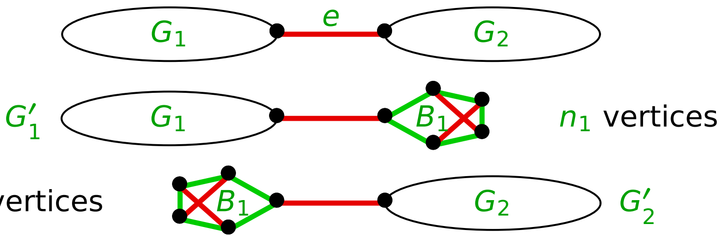

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Valid if $G_1, G_2 \neq B_1$.

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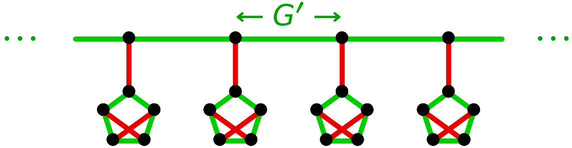

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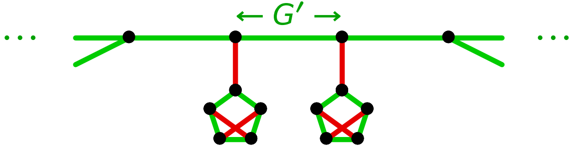

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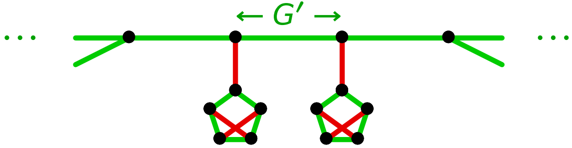

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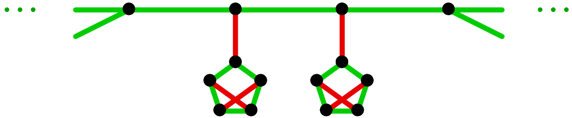

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G''

3

Idea: Combine a parity subgraph of G'' with the red edges for balloons in G.

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Pf. To cover each edge p times, $|\mathcal{M}| = p(2r+1)$. The total weight over all the matchings is pW; pigeonhole. ■

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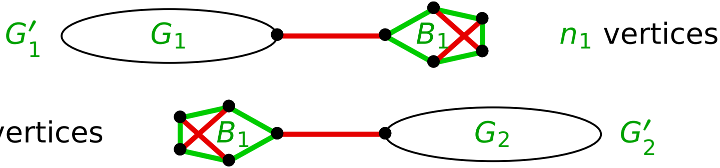

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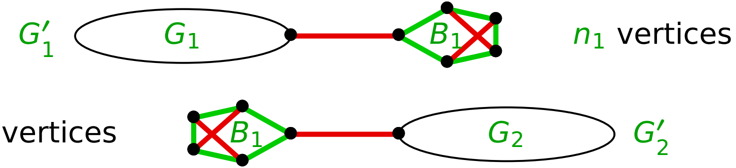

Now G also is constructed by attaching copies of B_1 to the leaves of a tree in $\mathcal{T}_1$, so $G \in \mathcal{H}_1$.