Rainbow Matching in Edge-Colored Graphs

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Abstract

A *rainbow subgraph* of an edge-colored graph is a subgraph whose edges have distinct colors. The *color degree* of a vertex v is the number of different colors on edges incident to v. Wang and Li conjectured that for $k \geq 4$, every edge-colored graph with minimum color degree at least k contains a rainbow matching of size at least $\lceil k/2 \rceil$. We prove the slightly weaker statement that a rainbow matching of size at least $\lfloor k/2 \rfloor$ is guaranteed. We also give sufficient conditions for a rainbow matching of size at least $\lceil k/2 \rceil$ that fail to hold only for finitely many exceptions (for each odd k).

1 Introduction

Given a coloring of the edges of a graph, a *rainbow matching* is a matching whose edges have distinct colors. The study of rainbow matchings began with Ryser, who conjectured that every Latin square of odd order contains a Latin transversal [3]. An equivalent statement is that when n is odd, every proper n-edge-coloring of the complete bipartite graph $K_{n,n}$ contains a rainbow perfect matching.

Wang and Li [4] studied rainbow matchings in arbitrary edge-colored graphs. We use the model of graphs without loops or multi-edges. For a vertex v in an edge-colored graph G, the *color degree* is the number of different colors on edges incident to v; we use the notation $\hat{d}_G(v)$ for this. The *minimum color degree* of G, denoted $\hat{\delta}(G)$, is the minimum of these values. (Subgraphs whose edges have distinct colors have also been called *heterochromatic*, *polychromatic*, or *totally multicolored*, but "rainbow" is the most common term.)

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Wang and Li [4] proved that every edge-colored graph G contains a rainbow matching of size at least $\lceil \frac{5\hat{\delta}(G)-3}{12} \rceil$. They conjectured that a rainbow matching of size at least $\lceil \hat{\delta}(G)/2 \rceil$ can be guaranteed when $\hat{\delta}(G) \geq 4$. A properly 3-edge-colored complete graph with four vertices has no rainbow matching of size 2, but Li and Xu [2] proved the conjecture for all larger properly edge-colored complete graphs. Proper edge-colorings of complete graphs using the fewest colors show that the conjecture is sharp. A survey on rainbow matchings and other rainbow subgraphs appears in [1].

We strengthen the bound of Wang and Li for general edge-colored graphs, proving the conjecture when $\hat{\delta}(G)$ is even. When $\hat{\delta}(G)$ is odd, we obtain various sufficient conditions for a rainbow matching of size $\lceil \hat{\delta}(G)/2 \rceil$. Our results are the following:

Theorem 1.1. *Every edge-colored graph G has a rainbow matching of size at least $\lfloor \hat{\delta}(G)/2 \rfloor$.*

Theorem 1.2. *For an edge-colored graph G, let $k = \hat{\delta}(G)$. Each condition below guarantees that G has a rainbow matching of size at least $\lceil k/2 \rceil$.*

(*a*) *G contains more than $\frac{3(k-1)}{2}$ vertices.*

(*b*) *G is triangle-free.*

(*c*) *G is properly edge-colored, $G \neq K_4$ and $|V(G)| \neq k+2$.*

Condition (a) in Theorem 1.2 implies that, for each odd k, only finitely many edge-colored graphs with minimum color degree k can fail to have no rainbow matching of size $\lceil k/2 \rceil$, where an edge-coloring is viewed as a partition of the edge set. Condition (c) guarantees that failure for a proper edge-coloring can occur only for K_4 or a graph obtained from K_{k+2} by deleting a matching.

2 Notation and Tools

Let G be an edge-colored graph other than K_4, and let $k = \hat{\delta}(G)$. If $|V(G)| = k+1$, then G is a properly edge-colored complete graph and has a rainbow matching of size $\lceil k/2 \rceil$, by the result of Li and Xu [2]. Therefore, we may assume that $|V(G)| \geq k+2$.

Let M be a subgraph of G whose edges form a largest rainbow matching. Let $c = k/2 - |E(M)|$, and let the edges of M be $e_1, \ldots, e_{k/2-c}$, with $e_j = u_j v_j$. We may assume throughout that $c \geq 1/2$, since otherwise G has a rainbow matching of size $\lceil k/2 \rceil$. Let H be the subgraph induced by $V(G) - V(M)$, and let $p = |V(H)|$. Note that $|V(G)| = |V(M)| + |V(H)| = k - 2c + p$. Since $|V(G)| \geq k+2$, we conclude that $p \geq 2c+2$.

Let A be the spanning bipartite subgraph of G whose edge set consists of all edges joining $V(M)$ and $V(H)$ (see Figure 1). We say that a vertex v is *incident* to a color if some edge

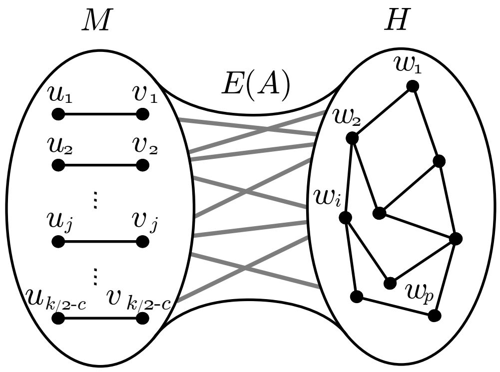

Figure 1: $V(M)$ and $V(H)$ partition $V(G)$.

incident to v has that color. A vertex $u \in V(M)$ is incident to at most $|V(M)| - 1$ colors in the subgraph induced by $V(M)$, so u is incident to at least $2c + 1$ colors in A. That is,

$$\hat{d}_A(u) \geq 2c + 1. \tag{1}$$

We say that a color appearing in G is *free* if it does not appear on an edge of M. Let B denote the spanning subgraph of A whose edges have free colors. We prove our results by summing the color degrees in B of the vertices of H. Consider $w \in V(H)$. There are only $k/2 - c$ non-free colors, so w is incident to at least $k/2 + c$ free colors. By the maximality of M, no free color appears in H, so the free colors incident to w occur on edges of B. That is, $\hat{d}_B(w) \geq k/2 + c$. Summing over $V(H)$ yields

$$\hat{d}_B(V(H)) \geq p(k/2 + c), \tag{2}$$

where $f(S) = \sum_{s \in S} f(s)$ when f is defined on elements of S.

For $1 \leq j \leq k/2 - c$, let E_j be the subset of $E(B)$ incident to $u_j v_j$. Let B_j be the graph with vertex set $V(H) \cup \{u_j, v_j\}$ and edge set E_j. Note that $\hat{d}_{B_j}(w) \leq 2$ for $w \in V(H)$.

Lemma 2.1. *If at least three vertices in $V(H)$ have positive color degree in B_j, then only one such vertex can have color degree 2 in B_j. Furthermore,*

$$\hat{d}_{B_j}(V(H)) \leq p + 1. \tag{3}$$

Proof. Let w_1, w_2, and w_3 be vertices of H such that $\hat{d}_{B_j}(w_1) = \hat{d}_{B_j}(w_2) = 2$ and $\hat{d}_{B_j}(w_3) \geq 1$. By symmetry, we may assume that $w_3 v_j \in E(B_j)$. Maximality of M requires the same color on $u_j w_1$ and $v_j w_2$. Since $\hat{d}_{B_j}(w_2) = 2$, the color on $u_j w_2$ differs from this. Now $u_j w_1$ or $u_j w_2$ has a color different from $v_j w_3$, which yields a larger rainbow matching in G.

Now consider $\hat{d}_{B_j}(V(H))$. Since $p \geq 2c + 2$, we have $p \geq 3$. If $\hat{d}_{B_j}(V(H)) \geq p + 2$, then $\hat{d}_B(w) \leq 2$ for all $w \in V(H)$ requires three vertices as forbidden above. $\square$

If $p = 3$, then color degrees $2, 2, 0$ for $V(H)$ in B_j do not contradict Lemma 2.1. For $p \geq 4$, the next lemma determines the structure of B_j when $\hat{d}_{B_j}(V(H)) = p + 1$. Let $N_G(x)$ denote the neighborhood of a vertex x in a graph G.

Lemma 2.2. *For $p \geq 4$, if $\hat{d}_{B_j}(V(H)) = p + 1$, then u_j or v_j is adjacent in B_j to $p - 1$ vertices of $V(H)$ via edges of the same color.*

Proof. Since $p + 1 \geq 5$, at least three vertices of H have positive color degree in B_j. Now Lemma 2.1 permits only one vertex w such that $\hat{d}_{B_j}(w)$=2, while $\hat{d}_{B_j}(w') = 1$ for each other vertex w' in $V(H)$. Let λ_1 and λ_2 be the colors on $u_j w$ and $v_j w$, respectively. Partition $V(H) - \{w\}$ into two sets by letting $U = N_{B_j}(u_j) - \{w\}$ and $V = N_{B_j}(v_j) - \{w\}$. By the maximality of M, all edges joining u_j to U have color λ_2, and all edges joining v_j to V have color λ_1. If U and V are both nonempty, then replacing $u_j v_j$ with edges to each yields a larger rainbow matching in G. Hence U or V is empty and the other has size $p - 1$. □

Lemma 2.3. *The following imply that $\hat{d}_{B_j}(V(H)) \leq p$ for each j.*

(*a*) *$c \geq 1$.*

(*b*) *G is triangle-free.*

(*c*) *G is properly edge-colored and $p \geq 4$.*

Proof. (a) Since $p \geq 2c+2$, condition (a) implies $p \geq 4$. If $\hat{d}_{B_j}(V(H)) = p+1$, then Lemma 2.2 applies, and u_j or v_j is adjacent via the same color to all but one vertex of H. Now $\hat{d}_A(u_j) \leq 2$ or $\hat{d}_A(v_j) \leq 2$, which contradicts (1) when $c \geq 1$.

(b) If G is triangle-free, then no vertex of H is adjacent to both endpoints of an edge in M. Hence, $\hat{d}_{B_j}(w) \leq 1$ for each $w \in V(H)$.

(c) If $p \geq 4$ and $\hat{d}_{B_j}(V(H)) = p + 1$, then Lemma 2.2 applies again and implies that the edge-coloring is not proper. □

3 Proof of the Main Results

Theorem 1.1. *Every edge-colored graph with minimum color degree k has a rainbow matching of size at least $\lfloor k/2 \rfloor$.*

Proof. If the maximum size of a rainbow matching is $k/2-c$, with $c \geq 1$, then Lemma 2.3(a) yields $\hat{d}_B(V(H)) \leq \sum_{j=1}^{k/2-c} \hat{d}_{B_j}(V(H)) \leq p(k/2 - c)$, which contradicts (2). □

Theorem 1.2. *Let G be an edge-colored graph such that $\hat{\delta}(G) = k$. Each of the following guarantee that G contains a rainbow matching of size at least $\lceil k/2 \rceil$.*

(a) G has more than $\frac{3(k-1)}{2}$ vertices.

(b) G is triangle-free.

(c) G is properly edge-colored, $G \neq K_4$, and $|V(G)| \neq k+2$.

Proof. If G has no rainbow matching of size $\lceil k/2 \rceil$, then for a largest one Theorem 1.1 yields $c = 1/2$. Now (3) implies $\hat{d}_B(V(H)) \leq \sum_{j=1}^{k/2-c} \hat{d}_{B_j}(V(H)) \leq (p+1)(k/2 - 1/2)$. Combining this with (2) yields $p(k/2 + 1/2) \leq (p+1)(k/2 - 1/2)$, which simplifies to $p \leq (k-1)/2$. Hence $|V(G)| \leq 3(k-1)/2$.

If G is a properly edge-colored complete graph other than K_4, then the result of Li and Xu [2] suffices. If G is triangle-free or properly edge-colored with at least $k+3$ vertices, then Lemma 2.3 yields $\hat{d}_B(V(H)) \leq p(k/2 - c)$, which again contradicts (2). □

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