Rainbow Matching in Edge-Colored Graphs

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http://www.math.uiuc.edu/~west/pubs/publink.html

Joint work with
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Christopher Stocker
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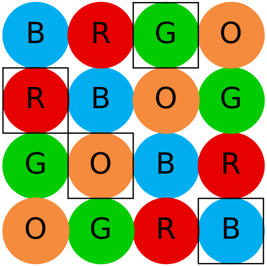

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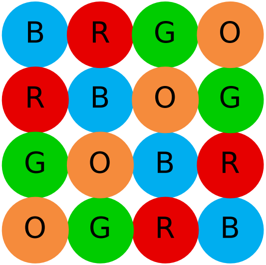

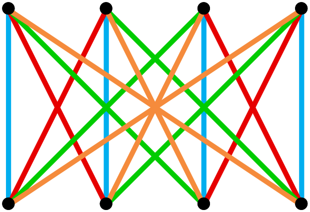

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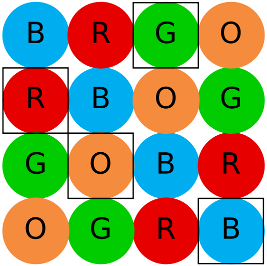

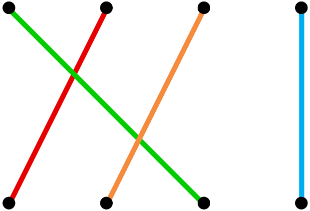

Latin square $\Leftrightarrow$ Edge-colored $K_{n,n}$

transversal $\Leftrightarrow$ Rainbow perfect matching

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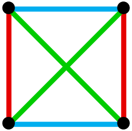
Thm. (X.Li–Z.Xu [2009]) The conjecture holds for properly edge-colored complete graphs.

New Results

Ex. A properly 3-edge-colored K_4 has no rainbow matching of size 2.



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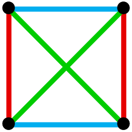

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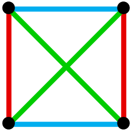

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Thm. (L–S–W–W) Each condition below yields a rainbow matching of size at least $\lceil \hat{\delta}(G)/2 \rceil$.

(a) G has more than $\frac{3(\hat{\delta}(G)-1)}{2}$ vertices.
(b) G is triangle-free.
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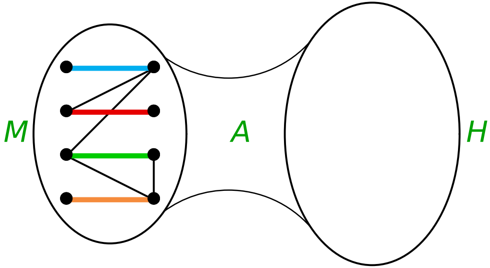

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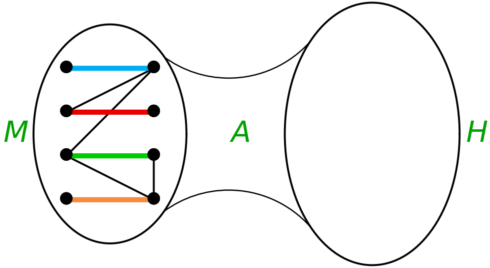

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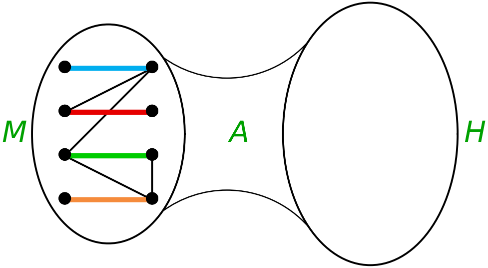

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If $u \in V(M)$, then $\hat{d}_A(u) \geq \hat{\delta}(G) - (2\,|M| - 1) = 2c + 1$. (1)

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Let $M = \{u_j v_j \colon 1 \leq j \leq |M|\}$.
Let B_j be the subgraph of B induced by $V(H) \cup \{u_j, v_j\}$.

Slices of B

Lem. If $\hat{d}_{B_j} \geq 1$ for w_1, w_2, w_3 in $V(H)$, then only one can have $\hat{d}_{B_j} = 2$. Furthermore,

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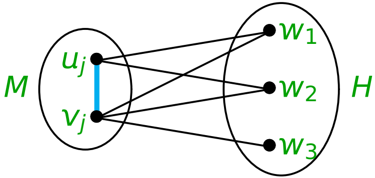

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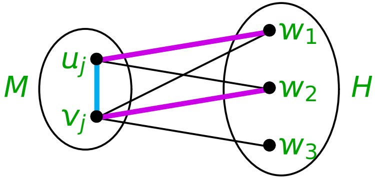

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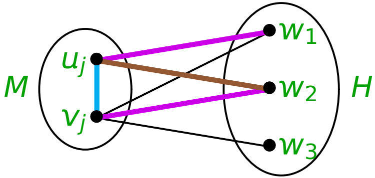

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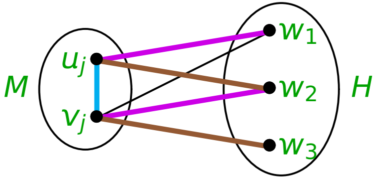

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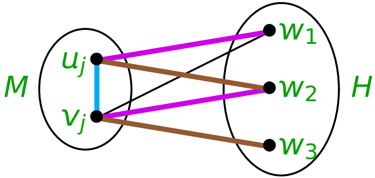

Note $p \geq 2c+2$ and $c \geq 1/2$ imply $p \geq 3$. Since $\hat{d}_{B_j}(w) \leq 2$ for $w \in V(H)$, $\hat{d}_{B_j}(V(H)) \geq p + 2$ requires a bad triple. ■

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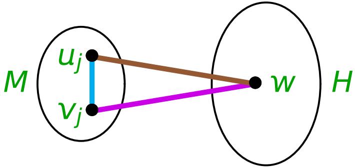

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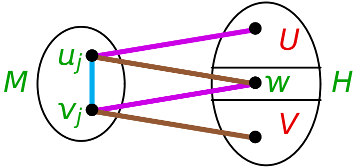

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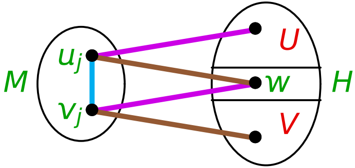

Maximality of $M \Rightarrow U = \varnothing$ or $V = \varnothing$, so $\hat{d}_A(v_j) \le 2$ or $\hat{d}_A(u_j) \le 2$, but $\hat{d}_A(u) \ge 2c+1$, so $c \le 1/2$. ∎

Conclusion

Thm. (LeSaulnier–Stocker–Wenger–West [2009+])
Every edge-colored graph G has a rainbow matching of size $\left\lfloor \hat{\delta}(G)/2 \right\rfloor$, improving to $\left\lceil \hat{\delta}(G)/2 \right\rceil$ under any of:

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(a) The stronger bound ($c \leq 0$) holds unless $\hat{d}_{B_j}(V(H)) = p + 1$ for some j, and then $c = 1/2$. Now

$$p(k/2 + c) \leq \hat{d}_B(V(H)) \leq (p+1)(k/2 - c).$$

simplifies to $2p + 1 \leq k$, or $n \leq 3(k-1)/2$.

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Thm. (LeSaulnier–Stocker–Wenger–West [2009+])
Every edge-colored graph G has a rainbow matching of size $\lfloor \hat{\delta}(G)/2 \rfloor$, improving to $\lceil \hat{\delta}(G)/2 \rceil$ under any of:

(a) G has more than $\frac{3(\hat{\delta}(G)-1)}{2}$ vertices.
(b) G is triangle-free.
(c) The coloring is proper, $G \neq K_4$, and $n \neq \hat{\delta}(G) + 2$.

Pf. When $p \geq 4$, the Lemma yields $c \leq 1/2$.
When $p \leq 3$, apply $p = n - 2|M| = n - (k - 2c) \geq 2c + 2$.

(a) The stronger bound ($c \leq 0$) holds unless $\hat{d}_{B_j}(V(H)) = p + 1$ for some j, and then $c = 1/2$. Now

$$p(k/2 + c) \leq \hat{d}_B(V(H)) \leq (p+1)(k/2 - c).$$

simplifies to $2p + 1 \leq k$, or $n \leq 3(k-1)/2$.

(b,c) For $n = k+1$, apply Li–Xu. If $n \geq k+3$, then $p \geq 4$, and the Lemma $\Rightarrow$ triangles and improper coloring. ■