The Game of Revolutionaries and Spies

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

slides available on DBW preprint page

Joint work with
Jane V. Butterfield, Daniel W. Cranston,
Gregory J. Puleo, and Reza Zamani;
plus
Clifford Smyth and David Howard

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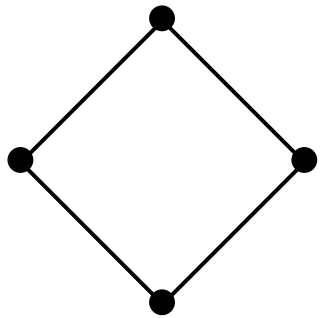

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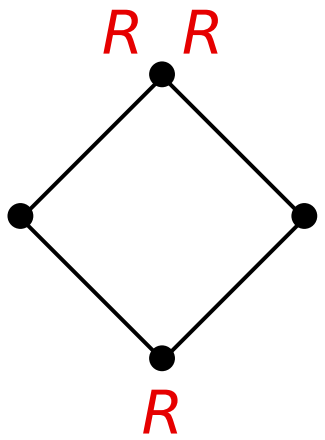

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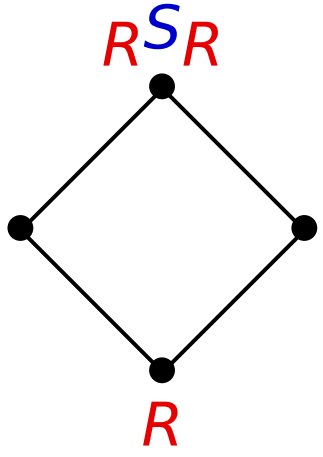

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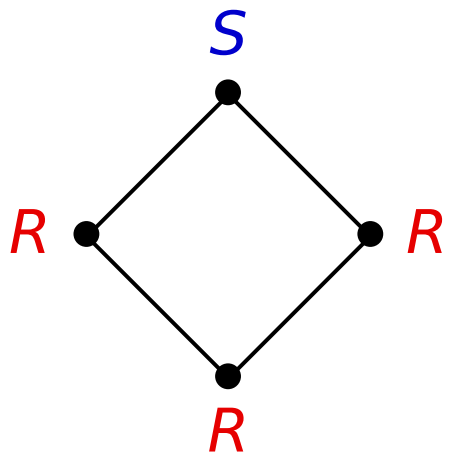

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G is spy-bad for particular (r, m): $\sigma(G, m, r) = r - m + 1$.

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- Some graphs are spy-good: $\lfloor r/m \rfloor$

Trees, dominated graphs, webbed trees,

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- Some graphs are in between: cr/m

Complete multipartite (good upper and lower bounds).
Complete bipartite (exact answers for $m \in \{2, 3\}$).

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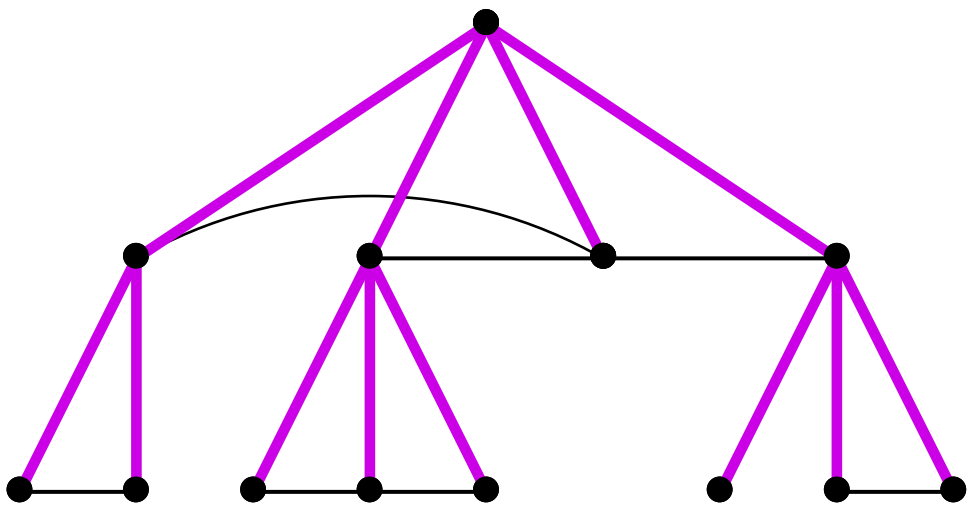

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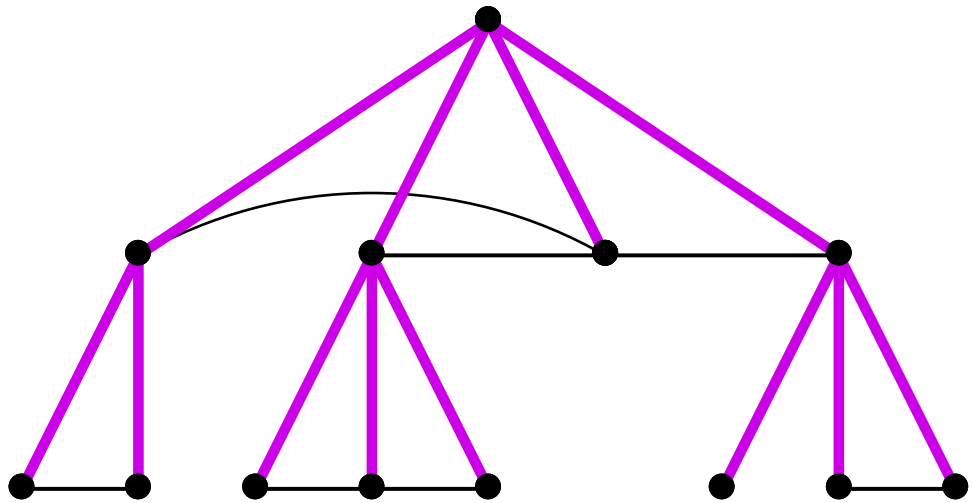

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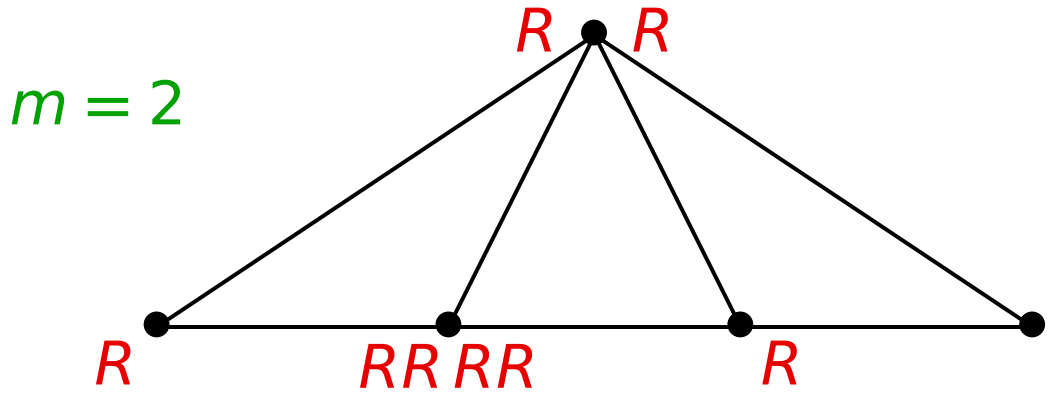

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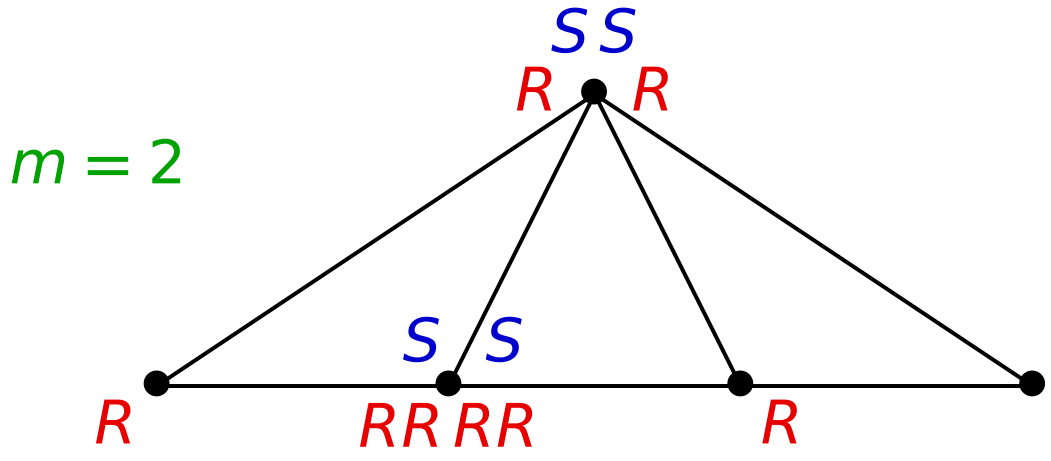

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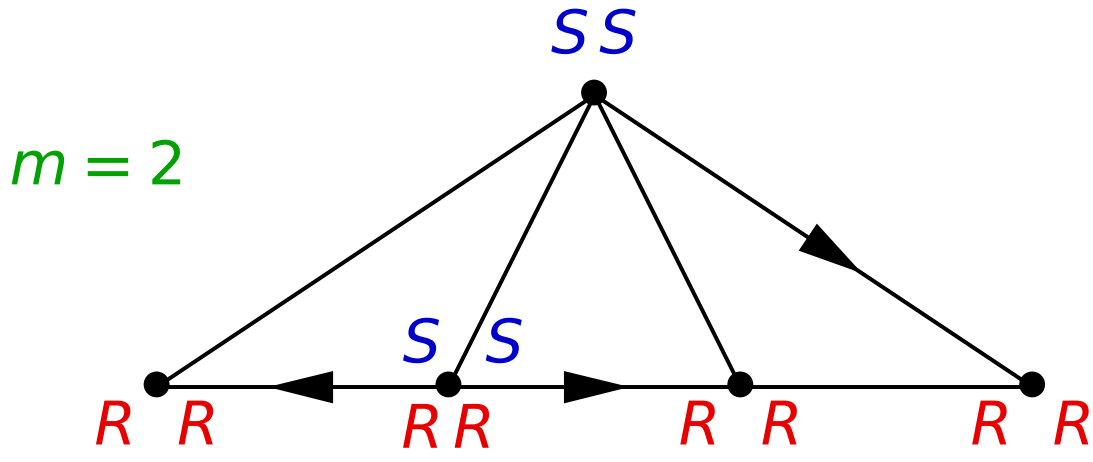

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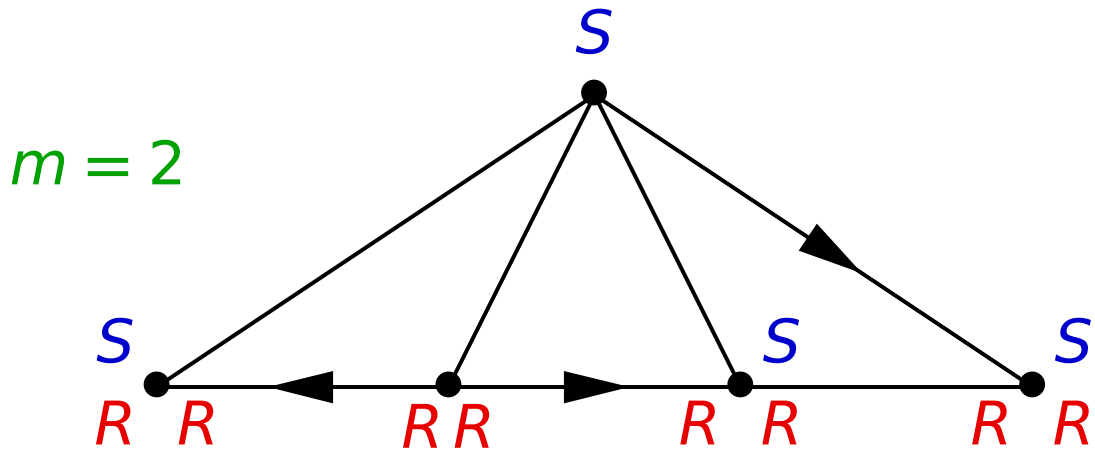

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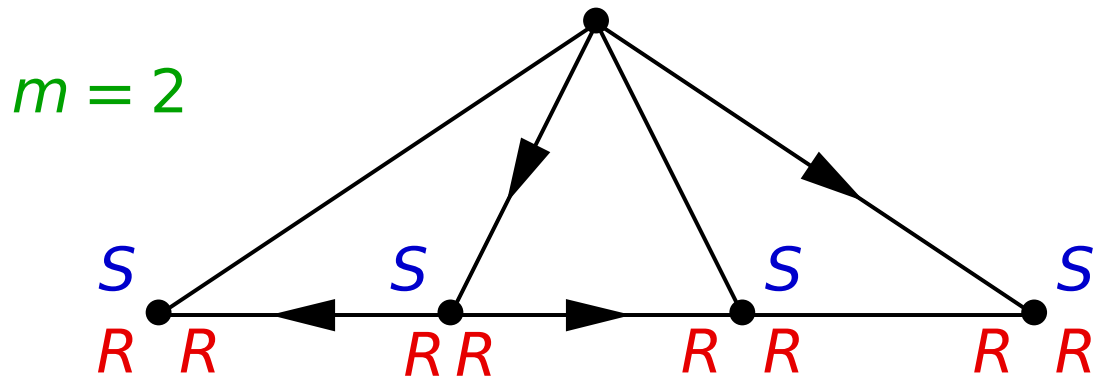

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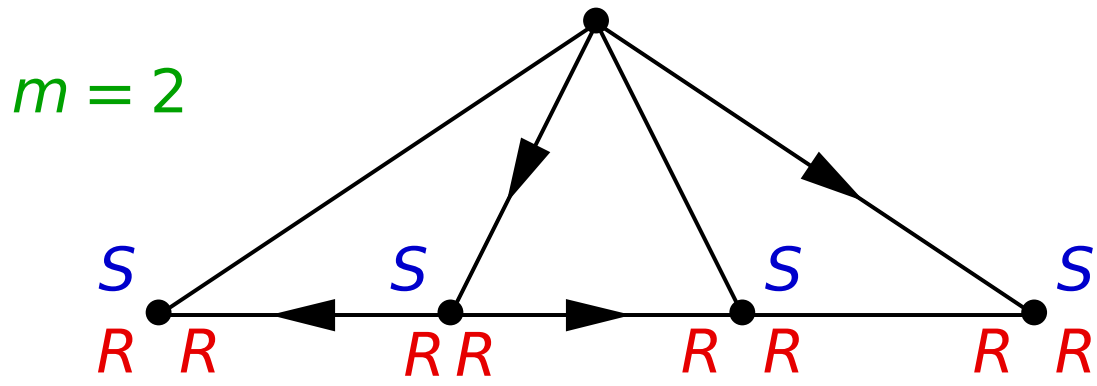

Idea: Restore stability after each round using matching in a bipartite graph.

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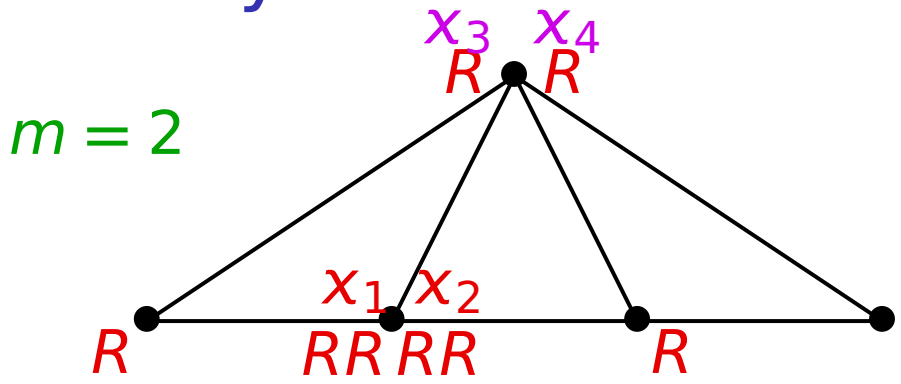

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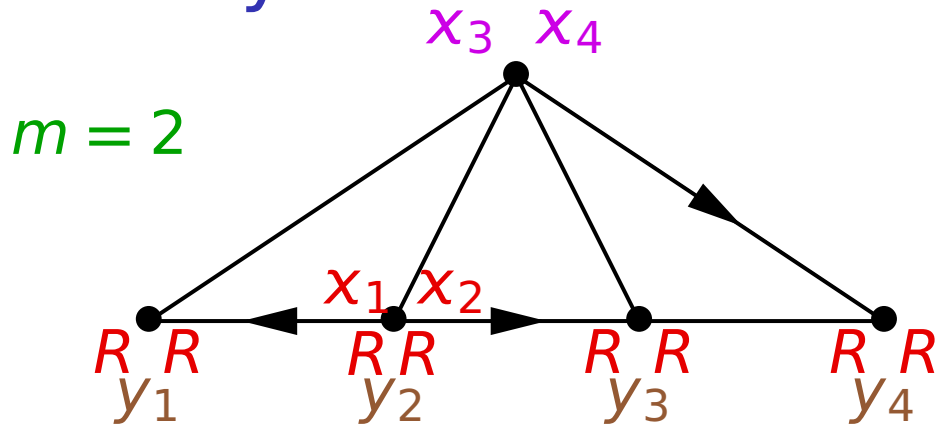

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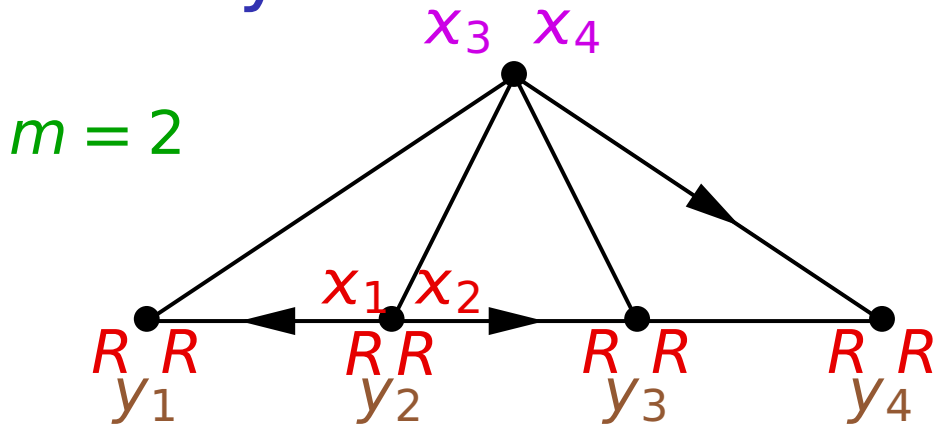

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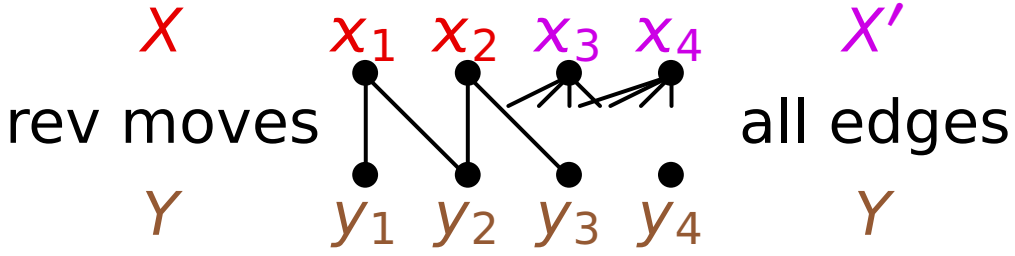

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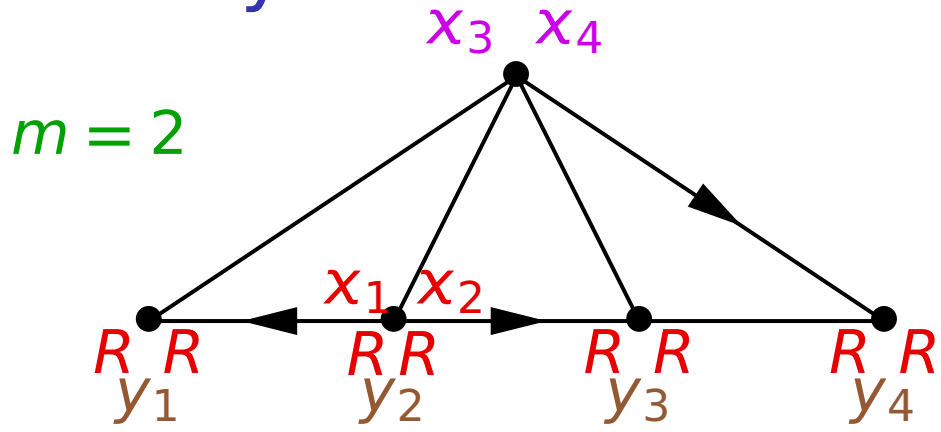

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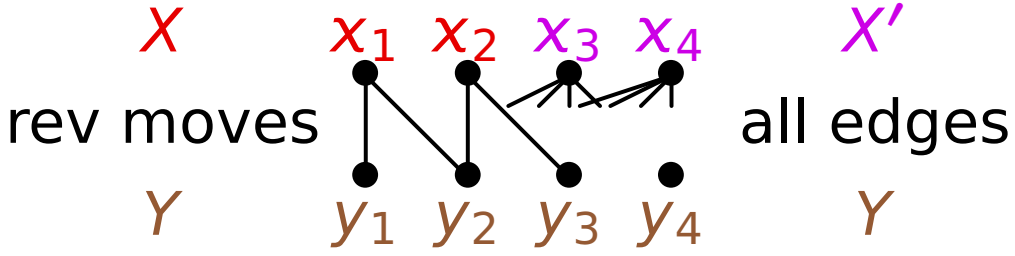

Hall's Condition: For $T \subseteq Y$, the $m|T|$ revs in these meetings came from $N(T) \cap X$ or from no meeting in X, so $m|T| \leq m|N(T) \cap X| + r - m|X|$.

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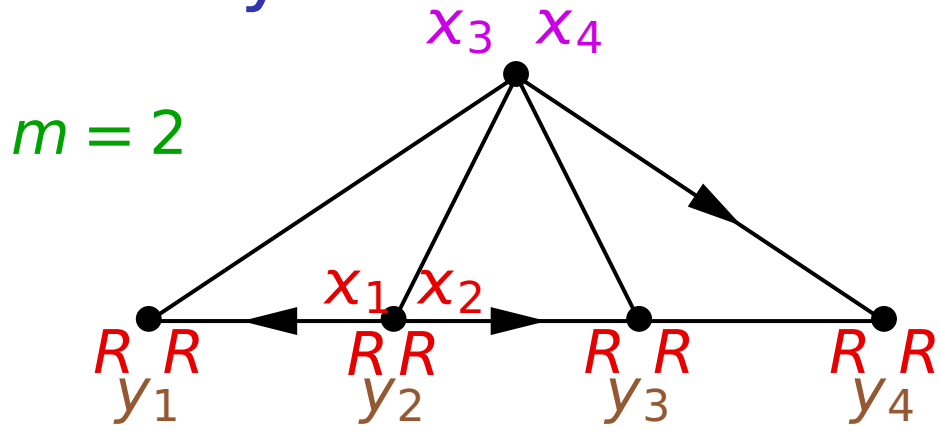

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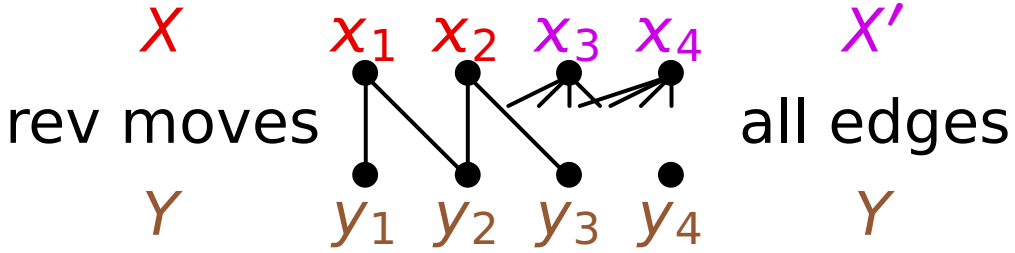

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$$|N(T)| = |N(T) \cap X| + |X'| \ge |T| - (\lfloor r/m \rfloor - |X|) + |X'| = |T|.$$

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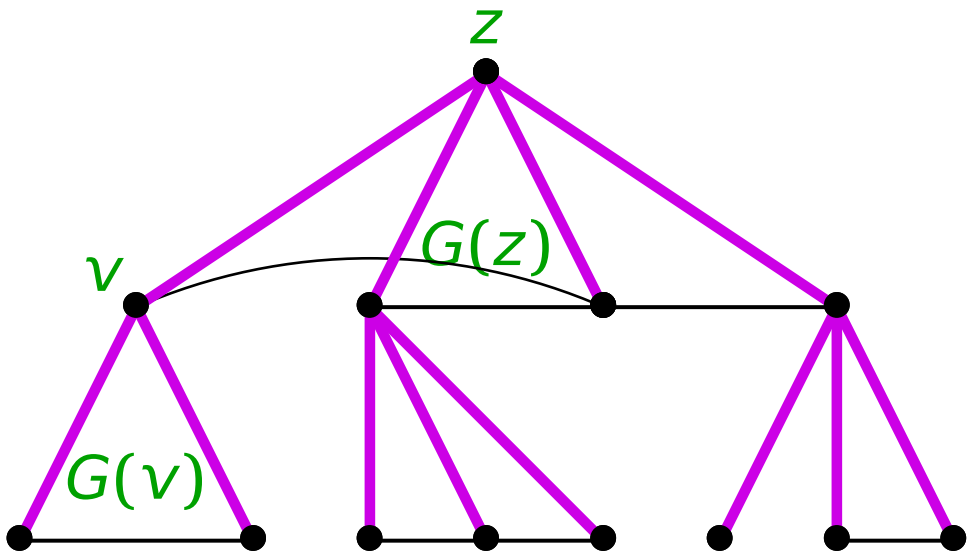
Idea: v dominates the subgraph $G(v)$ induced by $\{v\} \cup C(v)$. Spies play on these subgraphs independently to reestablish the **Rule**.

Split into Subgames (sketch)



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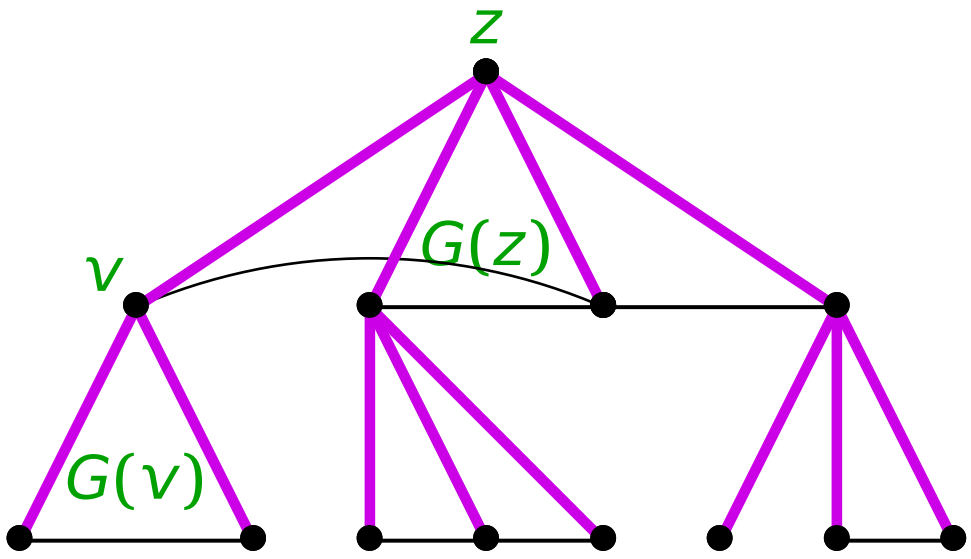

The $s(v)$ spies at v split into $\check{s}(v)$ playing in $G(v)$ and $\hat{s}(v)$ playing in the parent's graph. Let

$\check{s}(v) = \left\lfloor \frac{w^*(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w(x)}{m} \right\rfloor$ and $\hat{s}(v) = \left\lfloor \frac{w(v)}{m} \right\rfloor - \left\lfloor \frac{w^*(v)}{m} \right\rfloor$.

Here $w^*(v) = w(v) - \#$revs counted by $w(v)$ that are in the parent's graph after the revs next move.

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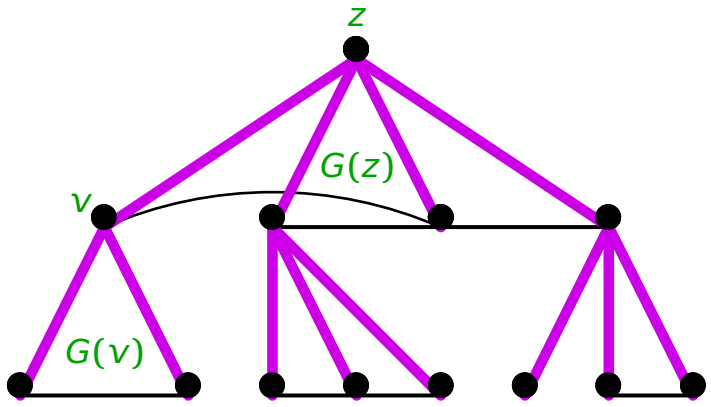

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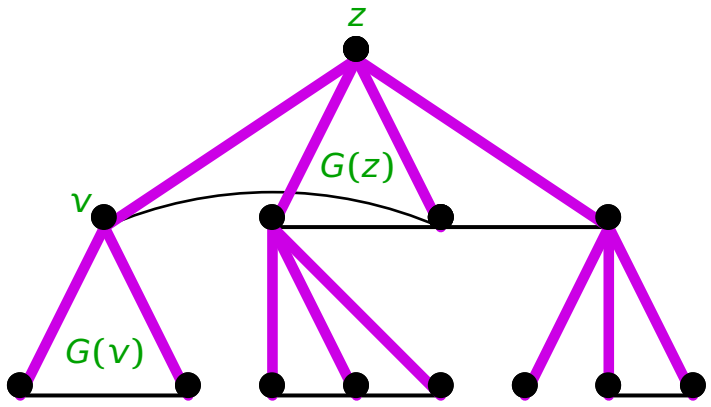

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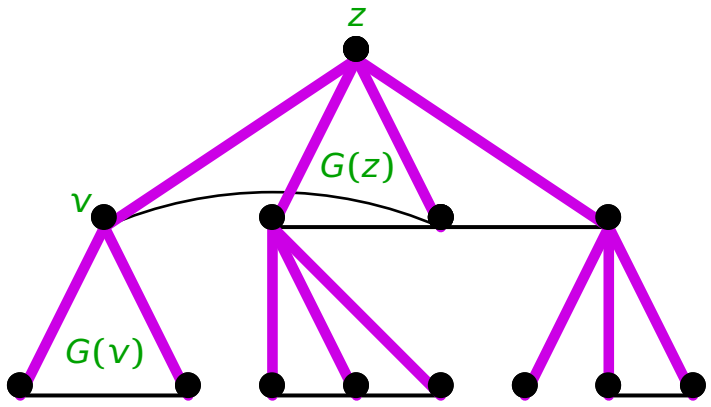

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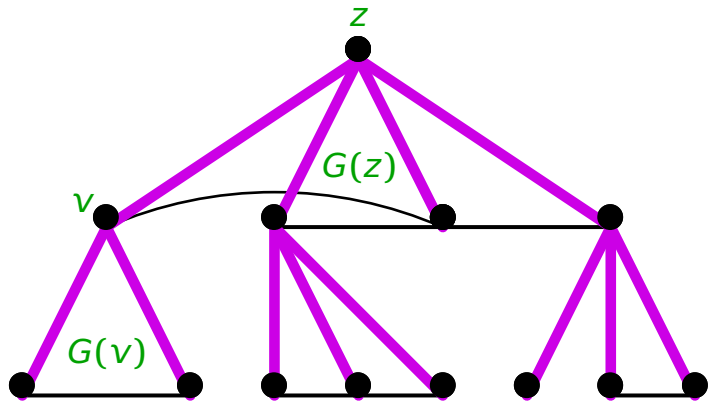

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The resulting new spy distributions restore the **Rule**:

$$s'(v) = \check{s}'(v) + \hat{s}'(v) = \left\lfloor \frac{w'(v)}{m} \right\rfloor - \sum_{x \in C(v)} \left\lfloor \frac{w'(x)}{m} \right\rfloor .$$

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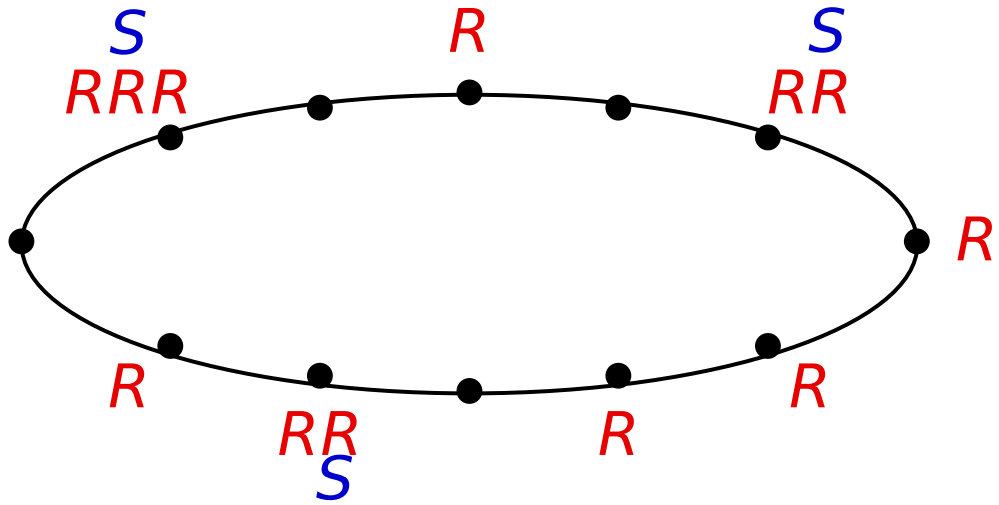

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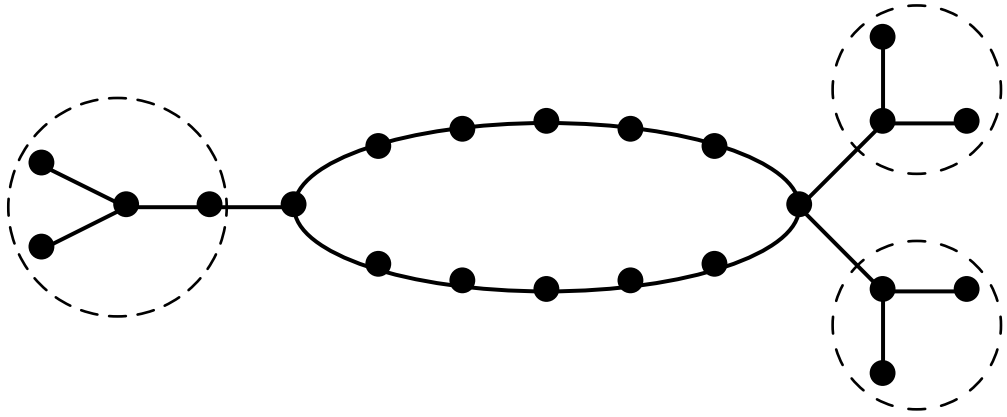
Maintain the cycle condition by keeping "fake" revs at a cycle vertex until an attached tree has enough revs to demand a spy according to the tree strategy. ■



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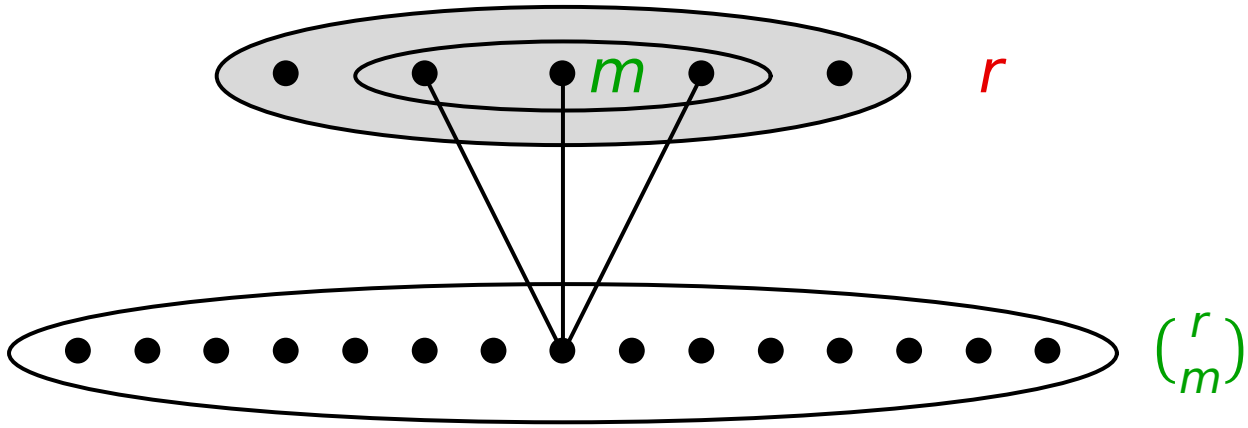

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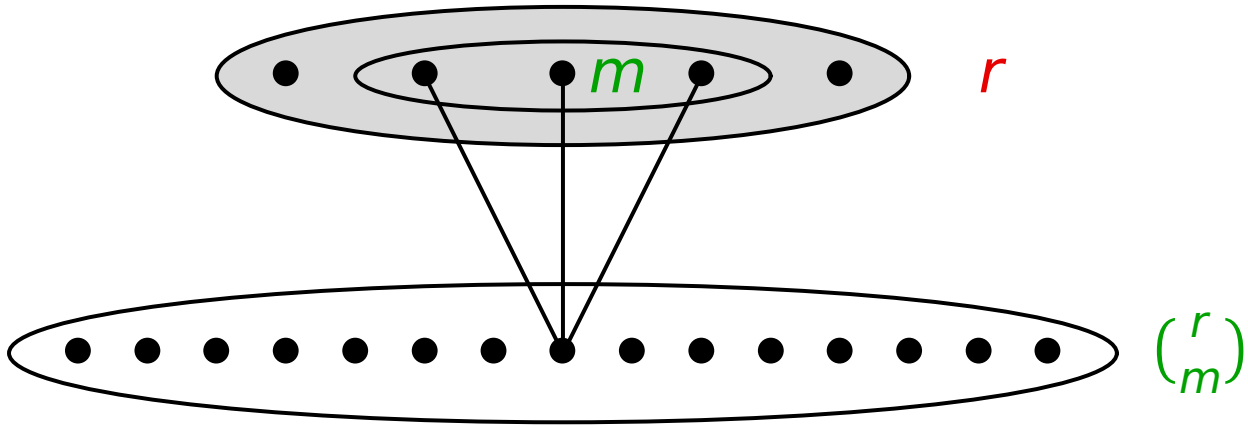

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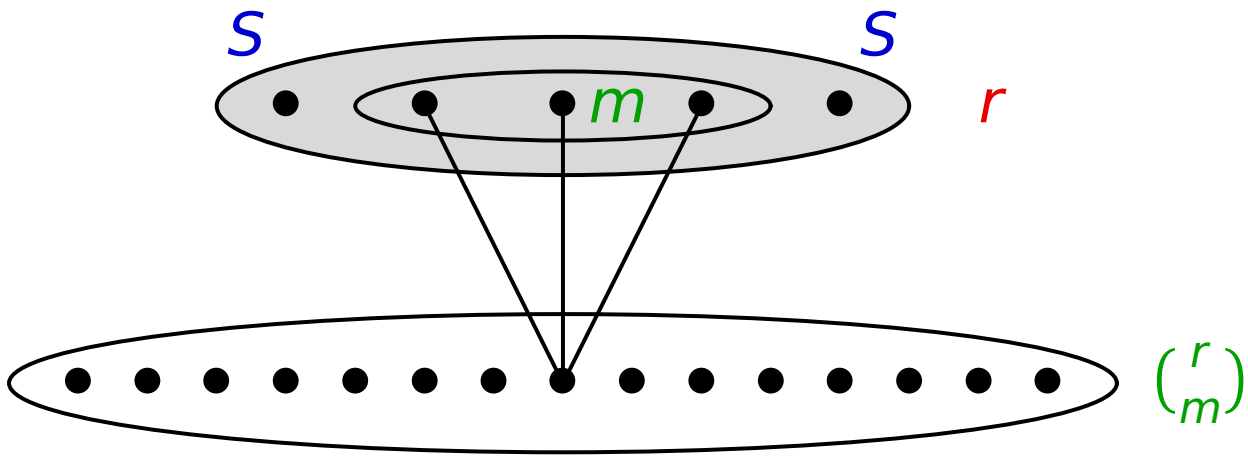

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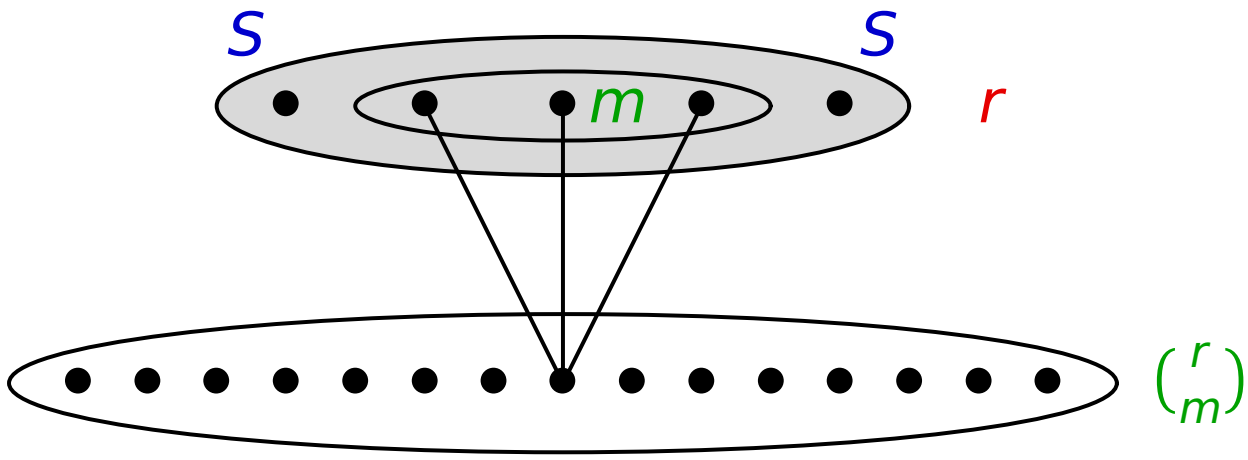

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Some m uncovered revs can meet on the first round, unreachable by spies. ■

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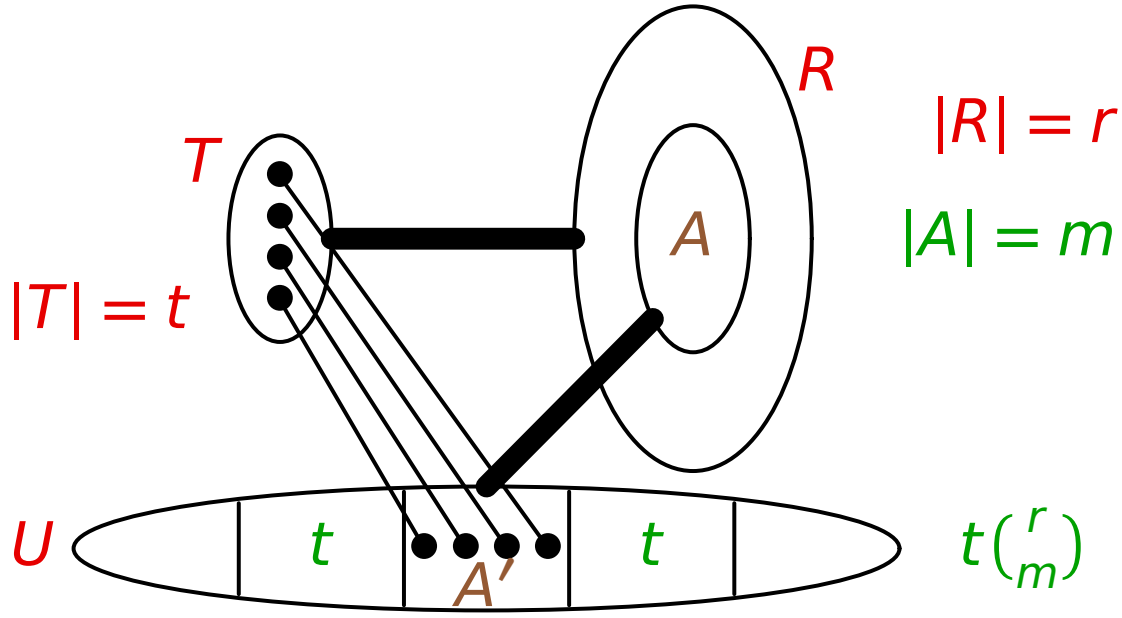

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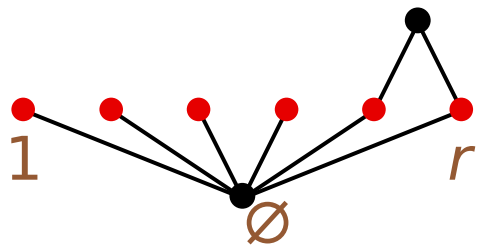

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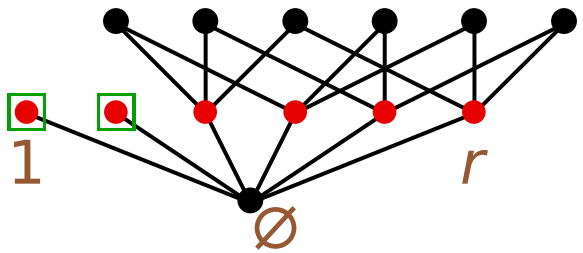

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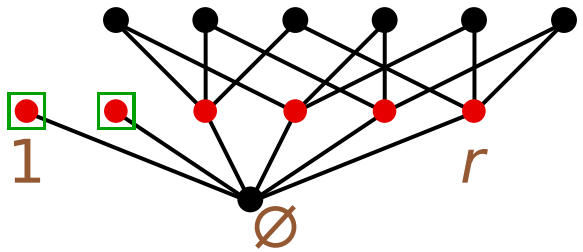

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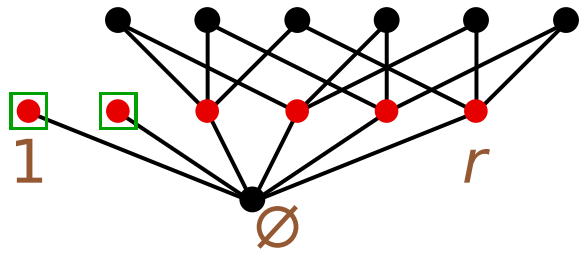

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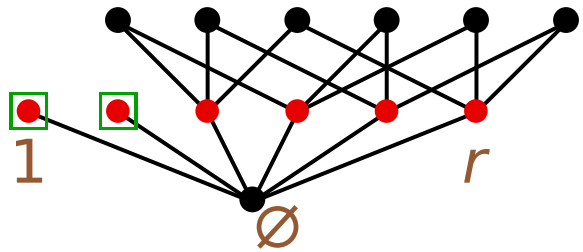

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$t = 4$ leaves six threats at doubles, not reachable by two triples (two triangles don't cover $E(K_4)$).

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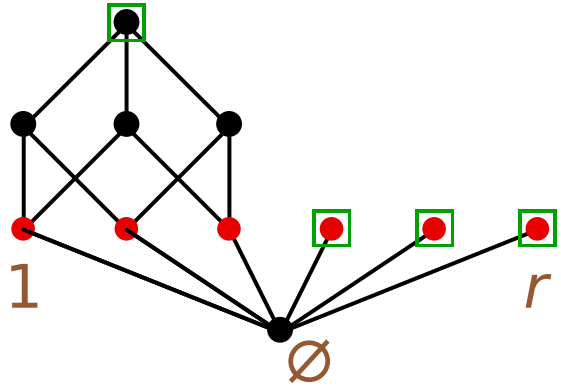
$\therefore$ $r - 3$ spies occupy singles, plus one at a triple.
By symmetry, spy is at 123, with the others at $4, \ldots, r$.



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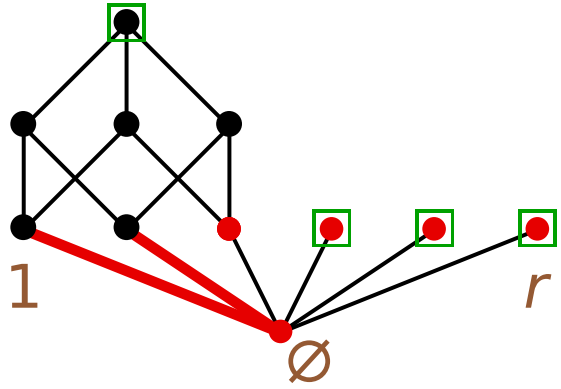

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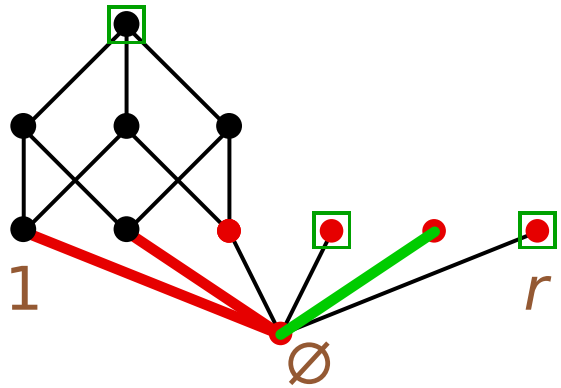

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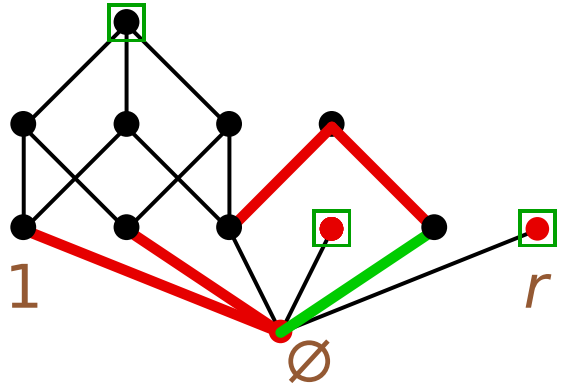

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No spy can now reach a neighbor of $\{3, j\}$.
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$\therefore$ revs win against fewer than $(d-1)\lfloor r/d \rfloor$ spies. ■

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Lem. Let X be a subset of $[d]$ with $t = |X| \geq 2m$. Let v be a vertex of weight other than 1. Within distance $m-1$ of v are at most $\binom{t-3}{m-3} + 3\binom{t-3}{m-2}$ vertices of $\binom{X}{m}$.

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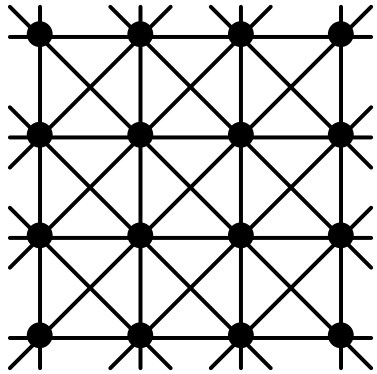
Numerically, this is less than $\binom{t}{m}$ for $c \leq t \leq r$. ■

King’s-move Grid (Howard–Smyth)



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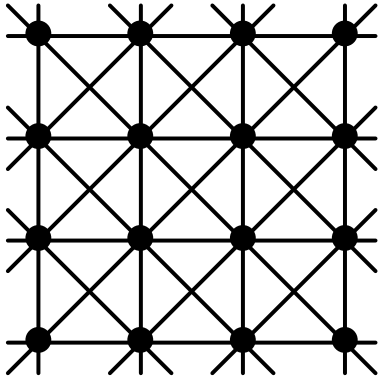

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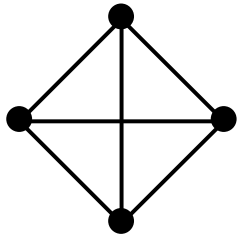

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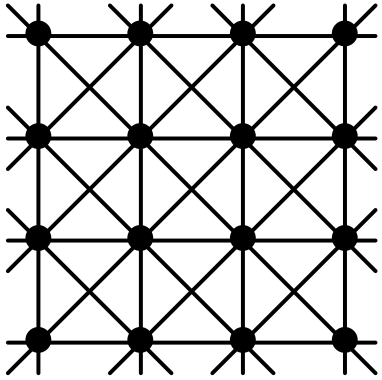

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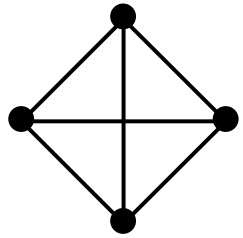

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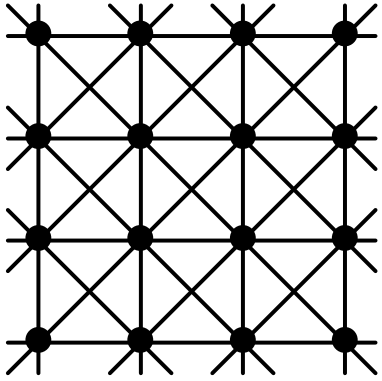

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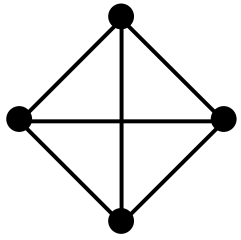

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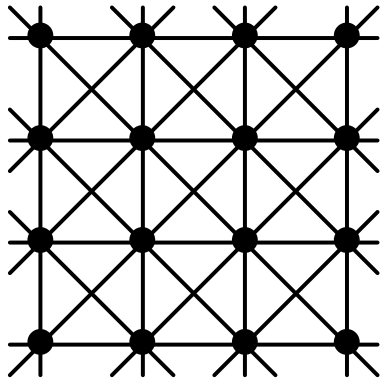

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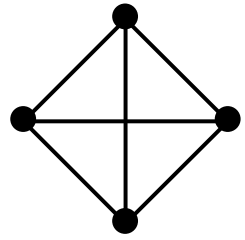

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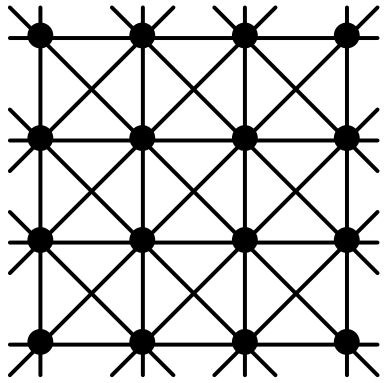

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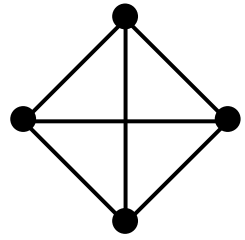

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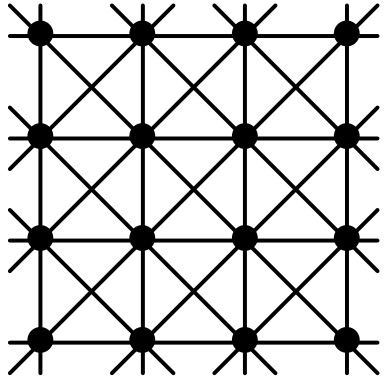

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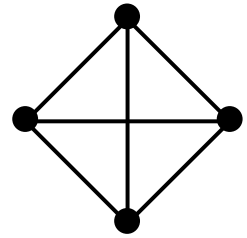

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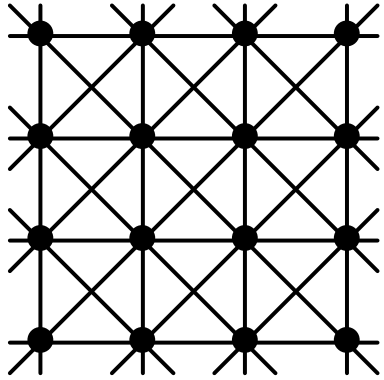

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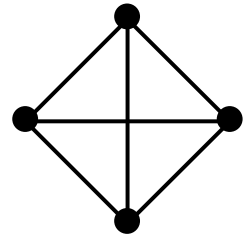

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Pf. A group of 8 revs can beat 5 spies (clever!). ■

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How many spies are needed to avoid losing by swarm?

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Case 1: $s_i > t$ for some i; revs swarm to part i.
New meetings use m incoming revs, not guardable by spies from part i. At least $\lfloor (k-1)t/m \rfloor$ additional spies must come from other parts, so

$$s \geq s_i + \left\lfloor \frac{(k-1)t}{m} \right\rfloor \geq t \left[1 + \frac{k-1}{m} \right] = \frac{k-1+m}{k} \frac{r}{m}.$$

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When $k \geq m$, the requirement from Case 2 is weaker (better for spies) than from Case 1.

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Pf. Hall's Theorem yields a matching that covers new meetings with free spies who can move there. ■

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Pf. It suffices to have $s_j \geq \frac{\hat{r}}{m(k-1)}$ for each j,

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Conj. For fixed m, the threshold for the number of spies needed to win is asymptotic to $1.5\frac{r}{m}$.

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[latter two papers (plus these slides) available at preprint page under DBW homepage]