New Proofs for Strongly Chordal Graphs and Chordal Bipartite Graphs

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August 2, 2004

Abstract

We give new proofs of well-known characterizations of strongly chordal graphs and chordal bipartite graphs. The key ingredient is the dual hypertree structure for totally balanced hypergraphs. We also consider split graphs and threshold graphs.

1 Introduction

A *hole* in a graph is an induced subgraph that is a cycle of length at least 4. A *chordal graph* is a graph with no hole; such graphs have been studied extensively since their introduction in [21] almost 50 years ago. Farber [15] introduced a subclass; a graph is *strongly chordal* if it is chordal and every even cycle of length at least 6 has a *strong chord*, meaning a chord joining vertices whose distance along the cycle is odd. Farber's motivation was a polynomial-time algorithm for the weighted dominating set problem on strongly chordal graphs; the problem is NP-hard for chordal graphs in general. Farber and others explored graph, hypergraph, and matrix characterizations of strongly chordal graphs (for example, see [15, 1, 4, 7, 37]). Many results are collected in [6].

Chordal and strongly chordal graphs are characterized by the existence of vertices with special properties. A *simplicial vertex* in a graph is a vertex whose neighbors form a clique (a *clique* is a set of pairwise adjacent vertices). A *simple vertex* in a graph is a vertex such that the closed neighborhoods of its neighbors form a chain under inclusion. (The *closed*

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neighborhood $N[v]$ of a vertex v is $N(v) \cup \{v\}$, where $N(v) = \{u : uv \in E(G)\}$.) Note that a simple vertex is also simplicial.

Dirac [14] proved that a graph is chordal if and only if every induced subgraph has a simplicial vertex. The analogue by Farber [15] is that a graph is strongly chordal if and only if every induced subgraph has a simple vertex. Since the families are hereditary (closed under taking induced subgraphs) they are thus the families of graphs having simplicial elimination orderings or simple elimination orderings, respectively. By *elimination ordering*, we mean an ordering of the vertices so that at each step the vertex about to be deleted has the desired property in the subgraph induced by the vertices that remain. Such characterizations facilitate fast algorithms and inductive proofs of graph properties for hereditary families.

Dirac's characterization of chordal graphs has a short inductive proof; Farber's characterization of strongly chordal graphs is more subtle. If every induced subgraph has a simple vertex, then the graph is chordal, and a simple vertex in the subgraph induced by the vertices of an even cycle (of length at least 6) leads to a strong chord of the cycle. The converse involves a characterization of strongly chordal graphs by forbidden induced subgraphs, and this is what is widely known as Farber's Theorem.

A *sun* [28, p145] is a graph obtained from a cycle whose length is even and at least 6 by adding edges to make the even-indexed vertices pairwise adjacent. Others [15, 10] call this a *complete sun* or a *complete trampoline* and use "sun" more generally for a graph obtained by placing any chordal graph on the even-indexed vertices, but a chordal graph contains an induced subgraph of that type if and only if its smallest such subgraph is what we have called here a sun. We only need the more restrictive concept and hence define it as in [28].

A sun with $2k$ vertices is a *k-sun*. A graph is *sun-free* if has no sun as an induced subgraph. A strongly chordal graph must be sun-free, because the original even cycle in the definition of an induced sun has no strong chord. Farber [15] and Chang [9] proved that a graph is strongly chordal if and only if it has no hole and no induced sun. To prove both this and the characterization in terms of simple vertices, it suffices to prove that every sun-free chordal graph has a simple vertex.

In Section 2 we translate this claim into hypergraph language and reduce it to a statement about "totally balanced hypergraphs", which we prove in Section 3. The reduction uses several observations that have been proved in other contexts; we include the proofs here for completeness. With these included, the proof is still shorter than that of Farber, which used four lemmas and nine nontrivial claims. A possible reason for the simplicity of the hypergraph approach may be that it puts the needed statement into a more general setting that facilitates an inductive proof using smaller changes.

Farber actually defined strongly chordal graphs in another way. A *strong elimination ordering* is an ordering $v_1, \ldots, v_n$ of the vertices such that if $i < j$ and $k < l$, and $v_k, v_l \in N[v_i]$ and $v_j \in N[v_k]$, then $v_l \in N[v_j]$. Farber defined strongly chordal graphs to be graphs with strong elimination orderings. It is easy to show that every strong elimination ordering is a simple elimination ordering, and Farber [15] proved that a strongly chordal graph as we have defined it has a strong elimination ordering (see Section 4).

A bipartite graph is *chordal bipartite* if it contains no induced cycle of length more than 4. This bipartite analogue of chordal graphs has been studied in [20, 19, 2, 23, 25, 24]; page 121 of [28] lists some of the results). Many hard problems can be solved quickly on this class.

From the forbidden subgraph characterization of chordal graphs, it is easy to show that a bipartite graph is chordal bipartite if and only if the graph obtained by adding edges to turn one partite set into a clique is strongly chordal. In Section 4 we explore how this and the existence of strong elimination orderings of chordal graphs relates to known characterizations of chordal bipartite graphs. In Section 5 we consider split graphs and threshold graphs and their relation to strongly chordal and chordal bipartite graphs.

After completing this paper, we learned that A. Brandstädt [5] also has a short proof of Farber's Theorem via hypergraph methods, using similar prior results in a different way.

2 Dual hypertrees, suns, and simple vertices

In this section we rephrase in a hypergraph context the concepts of "chordal", "sun-free", and "simple vertex". This leads to the statement we will prove to obtain Farber's Theorem.

Definition 2.1 A *hypergraph* H is a pair $(V(H), E(H))$, where $V(H)$ is a finite set called the *vertex set* of H, and the *edge set* $E(H)$ is a family of distinct nonempty subsets of $V(H)$. A *subhypergraph* of a hypergraph H is a hypergraph H' such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. The *2-section* of a hypergraph H, denoted G_H, is the graph with vertex set $V(H)$ such that u and v are adjacent if and only if they lie in a common edge in H.

For a vertex v in a hypergraph H, let $E_v(H) = \{e \in E(H) \colon v \in e\}$. A hypergraph H is a *dual hypertree* if there is an associated tree T and a bijection from $E(H)$ to $V(T)$ such that for each vertex v of H, the vertices of T corresponding to $E_v(H)$ induce a subtree of T.

A *tree decomposition* of a graph G is a pair (T, H) such that T is a tree, H is a hypergraph with vertex set $V(G)$, each edge of G is contained in an edge of H, and H is a dual hypertree with associated tree T.

Note that the pair (T, H) is a tree decomposition of a graph G if and only if $G \subseteq G_H$ and H is a dual hypertree with associated tree T.

Dual hypertrees are our main structural tool (the associated tree for a dual hypertrees need not be unique). To explain the name, the *dual* of a hypergraph is obtained by interchanging the vertices and edges to transpose the vertex-edge incidence matrix. A *hypertree* (or *subtree hypergraph*) is a hypergraph H whose edges are vertex sets of subtrees of an associated tree with vertex set $V(H)$. Except for the occurrence of multiple edges, dual hypertrees are precisely the duals of hypertrees.

It is well-known [8, 18, 38] that a graph G is chordal if and only if it is the intersection graph of a family of subtrees of a tree; that is, there exists a tree T and subtrees $\{S_v : v \in V(G)\}$ such that $E(G) = \{uv : V(S_u) \cap V(S_v) \neq \emptyset\}$. Since pairwise intersecting subtrees of a tree have a common vertex, the maximal cliques of such a graph correspond to vertices in the host tree. For an intersection representation in a smallest host tree, these are the only vertices. This yields an interpretation of chordal graphs in hypergraph language.

Proposition 2.2 *A graph G is chordal if and only if there is a tree decomposition (T, H) of G such that $E(H)$ is the set of maximal cliques of G and $G = G_H$. That is, a graph is chordal if and only if its maximal cliques form the edge set of a dual hypertree.*

Such a pair (T, H), viewed as a tree decomposition or a dual hypertree, is called a *clique-tree representation* of G.

Later we will need well-known separation properties of tree decompositions, stated in the language of dual hypertrees. For a dual hypertree H and associated tree T, we let $\hat{e}, \hat{e}', \hat{f}, \hat{f}'$ denote the vertices of T corresponding to $e, e', f, f' \in E(H)$, respectively. We always designate vertices of the associated tree of a dual hypertree using this notation.

Lemma 2.3 ([13, 3]) *Let H be a dual hypertree with associated tree T. If $\hat{f}$ lies on the path from $\hat{e}$ to $\hat{e}'$ in T, then $e \cap e' \subseteq f$. If $\hat{e}'\hat{e}'' \in E(T)$, and T' and T'' are the components of $T - \hat{e}'\hat{e}''$ containing $\hat{e}'$ and $\hat{e}''$, respectively, then $\left(\bigcup_{\hat{e} \in V(T')} e\right) - (e' \cap e'')$ and $\left(\bigcup_{\hat{e} \in V(T'')} e\right) - (e' \cap e'')$ are disjoint subsets of $V(H)$. If $N_T(\hat{f}) = \{\hat{e}_1, \ldots, \hat{e}_k\}$, and T_i is the component of $T - \hat{f}$ containing $\hat{e}_i$, then the sets $\left(\bigcup_{\hat{e} \in V(T_i)} e\right) - f$ for $1 \leq i \leq k$ are pairwise disjoint subsets of $V(H)$.*

Next we introduce the concepts needed to express "sun-free" in hypergraph language.

Definition 2.4 A *strict cycle* in a hypergraph H is a subhypergraph H' with edges $\{e_1, \ldots, e_k\}$, where $k \geq 3$, containing distinct "special" vertices $v_1, \ldots, v_k \in V(H)$ such that each v_i is contained in e_i and in $e_{i(\bmod k)+1}$ but in no other edge of H'. A hypergraph is *totally balanced* if it has no strict cycle. A hypergraph is *conformal* if its maximal edges are the maximal cliques of its 2-section.

Lemma 2.5 *Every totally balanced hypergraph is conformal.*

Proof. For every hypergraph H, every edge of H is contained in a clique of G_H. To prove the claim, it thus suffices to obtain a strict cycle in H from a clique of G_H not contained in any edge of H.

Let Q be such a clique in G_H. Let e be an edge of H having maximal intersection with Q; note that $|Q \cap e| \geq 2$. Choose $x \in Q - e$. For any $y \in Q \cap e$, some edge in H contains $\{x, y\}$. Since no edge of H contains x and $Q \cap e$, we may let e_1 and e_2 be distinct edges of H that contain x and have maximal intersections with $Q \cap e$. Then there are vertices $y \in (e \cap e_1) - e_2$ and $z \in (e \cap e_2) - e_1$, and $\{e, e_1, e_2\}$ is a strict cycle with special vertices y, x, z, respectively. $\square$

Lemma 2.6 [26, 32] *A hypergraph is totally balanced if and only if every subhypergraph is a dual hypertree.*

Proof. If H is not totally balanced, then H contains a strict cycle H' with edges $e_1, \ldots, e_k$ and special vertices $v_1, \ldots, v_k$. If H' is a dual hypertree, with associated tree T, then $E_{v_i}(H') = \{e_i, e_{i+1}\}$ requires $\hat{e}_i\hat{e}_{i+1} \in E(T)$ for all i, by the definition of dual hypertree. This yields a cycle in T, which contradicts that T is a tree.

If H is a totally balanced hypergraph, then every subhypergraph is also totally balanced, so it suffices to show that H itself is a dual hypertree. Its 2-section is chordal, since edges of H containing the edges of a hole in G_H would form a strict cycle. By Lemma 2.5, the maximal cliques of G_H are maximal edges of H. Now Proposition 2.2 implies that these edges form a dual hypertree; let T be the associated tree.

A nonmaximal edge e in H lies in some maximal edge e' corresponding to a vertex $\hat{e}' \in V(T)$. Modify T by adding a leaf $\hat{e}$ with neighbor $\hat{e}'$ for each such e. The resulting tree T' expresses H as a dual hypertree, since each vertex v of an added edge e also lies in e', and adding $\hat{e}$ simply grows the subtree corresponding to $E_v(H)$. $\square$

The following theorem is now a fairly straightforward translation of a graph condition into a hypergraph condition. Farber [15] obtained this as a corollary of the forbidden subgraph characterization, showing that each condition is equivalent to being strongly chordal. Using hypergraphs, it is now easy to show directly that these are equivalent, and we then use the equivalence to prove Farber's Theorem.

Theorem 2.7 [15] *If G is a chordal graph, with clique-tree representation (T, H), then G is sun-free if and only if H is totally balanced.*

Proof. If G has an induced k-sun G' with vertex set U, where $U = \{u_1, \ldots, u_{2k}\}$, then for $1 \le i \le k$ the vertex u_{2i-1} belongs to a unique maximal clique Q_i in G', given by $Q_i = \{u_{2i-2}, u_{2i-1}, u_{2i}\}$ (with subscripts modulo $2k$). A maximal clique in G containing Q_i is an edge e_i in H, and $e_i \cap U = Q_i$. Hence $e_1, \ldots, e_k$ is a strict cycle in H with special vertices $u_2, \ldots, u_{2k}$, and H is not totally balanced.

Conversely, consider a smallest strict cycle in H, with edges $\{e_1, \ldots, e_k\}$ and special vertices $v_1, \ldots, v_k$. Since each e_i is a clique in G, the special vertices in order form a cycle in G. Since G is chordal, there exists $e_0 \in H$ containing nonconsecutive special vertices. Since we chose a smallest strict cycle, e_0 contains all k special vertices. This holds for every edge of H containing nonconsecutive special vertices.

If $\hat{e}_i$ lies on the path from $\hat{e}_0$ to $\hat{e}_j$ in T, where $i, j \in [k]$, then by Lemma 2.3 e_i contains both special vertices of e_j, which is a contradiction. Thus the neighbor $\hat{e}'_i$ of $\hat{e}_i$ on the $\hat{e}_i, \hat{e}_0$-path is also the neighbor of $\hat{e}_i$ on the $\hat{e}_i, \hat{e}_j$-path for all $j \in [k] - \{i\}$. Since e_i is a maximal clique in G, there is a vertex $u_{2i-1} \in e_i - e'_i$. Also let $u_{2i} = v_i$. By Lemma 2.3, $N_G(u_{2i-1}) \cap \{u_1, \ldots, u_{2k}\} = \{u_{2i-2}, u_{2i}\}$. Hence $\{u_1, \ldots, u_{2k}\}$ induces a k-sun in G. $\square$

For chordal graphs, we have translated the "sun-free" condition into hypergraph language. To prove that a sun-free chordal graph has a simple vertex, we need a hypergraph condition that yields simple vertices. In a chordal graph, a clique-tree representation (T, H) yields simplicial vertices. For a leaf $\hat{e}$ of T, the clique e in G contains a vertex v in no other maximal clique. Thus $N[v] = e$, and v is simplicial in G. We want to use clique-tree representations similarly to obtain simple vertices in strongly chordal graphs.

Definition 2.8 In a hypergraph H, an edge e is a *simple edge* if the intersections of e with the other edges of H form a chain under inclusion.

Proposition 2.9 *If H is the hypergraph of maximal cliques in a graph G, then every simple edge of H contains a simple vertex of G.*

Proof. Let e be a simple edge of H. Since e is a maximal clique of G, no other edge of H contains e. Also its intersections with other edges are ordered by inclusion. Hence there is a vertex $v \in e$ that lies in no other edge of H. Hence $N[v] = e$, and $N[v]$ is a clique in G. If v is not simple, then v has neighbors u_1 and u_2 such that there exist $w_1 \in N(u_1) - N[u_2]$ and

$w_2 \in N(u_2) - N[u_1]$. Let e_1 and e_2 be maximal cliques in G containing $\{u_1, w_1\}$ and $\{u_2, w_2\}$, respectively. Since $u_1w_2, u_2w_1 \notin E(G)$, we have $u_1 \in e \cap e_1 - e \cap e_2$ and $u_2 \in e \cap e_2 - e \cap e_1$, which contradicts e being a simple edge. Therefore v is a simple vertex of G. □

3 Proof of Farber's Theorem

In light of Theorem 2.7 and Proposition 2.9, it suffices to show that a totally balanced hypergraph H has a simple edge. To do this, we will use dual hypertree structure of H (Lemma 2.6) and find a simple edge of H corresponding to a leaf of the associated tree T.

For technical reasons in the proof, we must enlarge the scope of our argument to permit multiple copies of edges in our hypergraphs. That is, a *multihypergraph* H consists of a *vertex set* $V(H)$ and a multiset of subsets of $V(H)$ comprising the edges $E(H)$. The set of incidence matrices of multihypergraphs is the set of all 0,1-matrices. The *underlying hypergraph* of a multihypergraph is the hypergraph with the same vertices formed by keeping one copy of each edge.

Using the same definition that forbids strict cycles, a multihypergraph is totally balanced if and only if its underlying hypergraph is totally balanced. Similarly, a multihypergraph is a dual hypertree if and only if its underlying hypergraph is a dual hypertree, by subdividing or contracting edges of the tree incident to vertices corresponding to duplicated edges. In particular, Lemmas 2.5 and 2.6 hold also for multihypergraphs, and it suffices to show that a totally balanced multihypergraph has a simple edge.

The maximal cliques of a sun-free chordal graph form a totally balanced hypergraph. Hypertree structure applies to all totally balanced hypergraphs, including those with non-maximal edges. Now we further enlarge the class to allow multiple edges. This enlargement of the class of structures being considered allows us to take smaller steps in an inductive proof. This is perhaps the reason why our approach yields a simple constructive argument.

We need a lemma establishing a hereditary property for the dual hypertree representation of a totally balanced multihypergraph. A graph is *nontrivial* if it has an edge.

Lemma 3.1 *Let H be a totally balanced multihypergraph with associated nontrivial tree T. Given $\hat{e} \in V(T)$, let $\tilde{T}$ be an associated tree for the subhypergraph $\tilde{H}$ of H whose edges correspond to the neighbors of $\hat{e}$ in T, with $V(\tilde{T}) = N_T(\hat{e})$. The hypergraph $H - e$ is a dual hypertree with associated tree T', where $T' = (T - \hat{e}) \cup \tilde{T}$. Moreover, each leaf of T' is also a leaf of T, except possibly the neighbor of $\hat{e}$ when $\hat{e}$ is a leaf of T.*

Proof. Since every subhypergraph of H is also totally balanced, an associated tree $\tilde{T}$ for $\tilde{H}$ exists. If $\hat{e}$ is a leaf of T, then $\tilde{T}$ is a single vertex and $T' = T - \hat{e}$, and the lemma is clearly true. Hence we may assume that $\hat{e}$ is not a leaf of T.

Let $N_T(\hat{e}) = \{\hat{e}_1, \ldots, \hat{e}_k\}$ with $k \geq 2$, and let T_i be the component of $T - \hat{e}\hat{e}_i$ that contains $\hat{e}_i$, for $1 \leq i \leq k$. For $v \in V(H)$, the vertices of T_i corresponding to edges of H containing v induce a subtree of T_i, since the vertices corresponding to $E_v(H)$ induce a subtree of T. Also $E_v(\tilde{H})$ induces a subtree of $\tilde{T}$. If each of these subtrees contains a vertex other than $\hat{e}_i$, then v lies in edges of H corresponding to vertices in both $T_i - \hat{e}_i$ and $N_T(\hat{e}) - \{\hat{e}_i\}$. By Lemma 2.3, this requires $v \in e_i$. Hence the union of these subtrees of T_i and $\tilde{T}$ is a subtree of T'. This proves that T' is an associated tree for the dual hypertree $H - e$.

Since $k \geq 2$, every vertex of $\tilde{T}$ has a neighbor in $\tilde{T}$. Thus if a vertex of $\tilde{T}$ is a leaf in T', then it is a leaf in T. Every vertex of T' not in $\tilde{T}$ has identical neighborhoods in T and in T'. Therefore, each leaf of T' is also a leaf of T. □

If a chordal graph has only one maximal clique, then it is a complete graph, and all vertices in all its induced subgraphs are simple vertices. To complete the proof of Farber's Theorem, it thus suffices to show that a totally balanced multihypergraph with associated nontrivial tree T has a simple edge corresponding to some leaf of T. We obtain this from the next theorem.

Definition 3.2 A *simple ordering* of a multihypergraph H is an ordering $e_1, \ldots, e_m$ of $E(H)$ such that each e_i is a simple edge in the spanning subhypergraph with edges $e_1, \ldots, e_i$. A simple ordering $e_1, \ldots, e_m$ of a dual hypertree H with associated tree T is *T-simple with root r* if $r = e_1$ and $\hat{e}_1, \ldots, \hat{e}_i$ induces a subtree of T, for each i.

Theorem 3.3 *If H is a totally balanced multihypergraph with associated tree T, and r is a maximal edge in H, then H has a T-simple ordering with root r.*

Proof. We use induction on $\sum_{e \in E(H)} |e|$, denoted by $\sigma(H)$. When $|E(H)| = 2$, the chain condition holds trivially, so both edges are simple and the only ordering starting with r is T-simple. Hence we may assume that $|E(H)| > 2$.

Case 1: *H has at least two maximal edges.*

Let U be the subset of $V(T)$ corresponding to the maximal edges of H. Let T^* be the minimal subtree of T containing U. Let e be an edge of H other than r such that $\hat{e}$ is a leaf of T^*. Let $\hat{e}'$ be the neighbor of $\hat{e}$ in T^*. Let T_0 be the component of $T - \hat{e}\hat{e}'$ containing $\hat{e}$.

If e and e' are identical vertex sets, then obtain H' from H by deleting e, and let T' be the tree obtained by contracting $\hat{e}\hat{e}'$ in T. Now $\sigma(H') < \sigma(H)$, and H' is a dual hypertree with associated tree T'. In the T'-simple ordering with root r provided by the induction hypothesis, add e immediately after e' to obtain the desired ordering for H.

Hence we may assume that e and e' are distinct sets. We obtain H^* by adding (a copy of) the edge $e \cap e'$ to H. Subdivide $\hat{e}\hat{e}'$ in T with a new vertex $\widehat{e \cap e'}$; this expresses H^* as a dual hypertree. Finally, we obtain H' from H^* by deleting e.

By Lemma 3.1 (applied to H^*), H' is a totally balanced hypergraph with some associated tree T' such that every leaf of T' is a leaf of T, except for possibly $\widehat{e \cap e'}$ when $\hat{e}$ is a leaf of T. Since $e \not\subseteq e'$, we have $\sigma(H') < \sigma(H)$. Also r is a maximal edge in H', so applying the induction hypothesis to H' produces a T'-simple ordering π' of H' with root r. We claim that replacing $e \cap e'$ with e in π' yields a T-simple ordering π of H with root r.

Both π and π' start with r. Since $e \cap e'$ precedes the edges corresponding to $N_T(\hat{e}) - \{\hat{e}'\}$ in π', initial segments of π correspond to subtrees of T. It remains to check that π is a simple ordering.

If $\hat{f} \in V(T_0) - \{\hat{e}\}$, then $f \subseteq e$, because f is contained in some maximal edge e^* of H, and $\hat{e}$ lies on the path in T from $\hat{f}$ to $\hat{e}^*$. Also $e \cap e'$ precedes f in π'. Since $f \subseteq e$, replacing $e \cap e'$ with e preserves the chain property for intersections of f with its predecessors; the intersection with $e \cap e'$ grows to become all of f. Thus f is a simple edge when reached in π.

By Lemma 2.3, every edge of H that intersects $e - e'$ corresponds to a vertex of T_0 and hence is contained in e. For $\hat{f} \in V(T') - V(T_0)$, we therefore have $f \cap (e - e') = \emptyset$. Hence $e \cap f = (e \cap e') \cap f$. Since we merely replace $e \cap e'$ with e in the ordering, the set intersections of f with its predecessors are unchanged, so f remains simple when reached. Similarly, the predecessors of e in π contain no elements of $e - e'$, so the fact that $e \cap e'$ is simple when reached in π' implies that e is simple when reached in π.

Case 2: *H has only one maximal edge.* The maximal edge must be r itself. Obtain H' from H by deleting the edge r. For $e \in E(H) - \{r\}$, let e' be the edge of H such that $\hat{e}'$ is the neighbor of $\hat{r}$ on the path from $\hat{r}$ to $\hat{e}$ in T. By Lemma 2.3, $e' \supseteq e \cap r = e$. Thus (some copy of) every maximal edge in $E(H) - \{r\}$ corresponds to a neighbor of $\hat{r}$ in T. Let r' be a maximal edge of H' such that $\hat{r}\hat{r}' \in E(T)$.

Let T' be the associated tree for H' constructed by applying Lemma 3.1 to H. By the induction hypothesis, H' has a T'-simple ordering with root r'. Prepending r at the front yields a T-simple ordering of H with root r, since every edge of H is contained in r. □

Farber's proof required showing that a nontrivial strongly chordal graph actually has more than one simple vertex. This is now an easy corollary.

Corollary 3.4 *If H is a totally balanced hypergraph with associated nontrivial tree T, then H has two simple edges whose corresponding vertices in T are leaves. In particular, a sun-free chordal graph that is not complete has two nonadjacent simple vertices.*

Proof. With $r \in E(H)$ as any maximal edge, we apply Theorem 3.3. The final edge in the resulting ordering is a simple edge e of H corresponding to a leaf of T. Now form H' by adding a new vertex used to enlarge e to e'. Now e' is a maximal edge in H', and H' is totally balanced with associated tree T. Applying Theorem 3.3 to H' using e' as the root yields a simple edge of H' corresponding to a leaf of T different from $\hat{e}$, and this edge is also simple in H.

If (T, H) is a clique-tree for a sun-free chordal graph G, then H is totally balanced, by Lemma 2.7. By the first statement here, H has two simple edges corresponding to leaves of T. By Proposition 2.9, such edges contain simple vertices of G that are not contained in the edges corresponding to the neighboring vertices of T. Hence these vertices are nonadjacent simple vertices of G. □

The fastest known algorithms to recognize a strongly chordal graph (and produce a simple elimination ordering) run in time $O(\min\{m \log n, n^2\})$ [1, 27, 29, 34], where m and n are the numbers of edges and vertices of the input graph. It is not known whether this can be done in linear time, which means $O(m + n)$. A linear-time algorithm proposed in [37] was later withdrawn, as noted in [33]. Chordal graphs can be recognized (producing a simplicial elimination ordering) in linear time ([31, 35]). Also, Sawada and Spinrad [33] showed that a simple elimination ordering can be converted to a strong elimination ordering in linear time.

From a simplicial elimination ordering of G, the maximal cliques and the clique-tree representation (T, H) can be produced in linear time. From the reverse of a T-simple ordering of H in Theorem 3.3, a simple elimination ordering of G can be produced in linear time by iteratively extracting one vertex from the first edge e until its vertices not in other edges have been exhausted, at which point e vanishes from the ordering (and $\hat{e}$ vanishes from T). Hence it would be interesting to determine whether a T-simple ordering can be produced in linear time.

Another question is whether dual hypertrees directly yield a strong elimination ordering. As mentioned in [33], an elimination ordering $v_1, \ldots, v_n$ is a strong elimination ordering if and only if (for each i) when v_i is eliminated, what remains of the closed neighborhoods of its remaining neighbors form an inclusion chain indexed in nondecreasing order.

4 Chordal bipartite graphs

Related to strongly chordal graphs is a bipartite analogue of chordal graphs. All cycles in bipartite graphs are even and the shortest has length 4, so the following definition is natural.

Definition 4.1 A *chordal bipartite graph* is a bipartite graph having no induced cycle of length more than 4 (called a *long hole*). A *totally balanced matrix* is a 0,1-matrix having no square submatrix without repeated columns in which every row and column has exactly two 1s. The *biadjacency matrix* (or *reduced adjacency matrix*) of a bipartite graph is the submatrix of its adjacency matrix indexed by the rows of one partite set and the columns of the other.

Note that "chordal bipartite graph" differs from "bipartite chordal graph". Hence *bi-chordal* might be a better term, but *chordal bipartite* is standard, and we use it here.

Every bipartite graph G is the incidence bigraph of some multihypergraph H. The biadjacency matrix of G is simply the incidence matrix of H. Interchanging the partite sets transposes the biadjacency matrix, and hence H and its dual have the same incidence bigraph.

In discussing chordal bipartite graphs, characterizations of the biadjacency matrix are important, and these provide the first connection to strongly chordal graphs. The following proposition is immediate from the definition of a strict cycle in a hypergraph and the description of its incidence matrix.

Proposition 4.2 *A hypergraph is totally balanced if and only if its incidence matrix is a totally balanced matrix, and its incidence matrix is totally balanced if and only if its incidence bigraph is chordal bipartite.*

Theorem 2.7 now immediately yields one of Farber's characterizations.

Corollary 4.3 ([15]) *A chordal graph is strongly chordal if and only if its vertex-clique incidence matrix is totally balanced; that is, if and only if the incidence bigraph of its hypergraph of maximal cliques is chordal bipartite.*

Theorem 2.7 also yields a different sort of characterization of chordal bipartite graphs using strongly chordal graphs.

Corollary 4.4 (Dahlhaus [12]) *A bipartite graph is chordal bipartite if and only if the graph obtained by completing one partite set is strongly chordal, where* completing *a vertex set means adding edges to make it into a clique.*

Proof. Let G be a bipartite graph with partite sets X and Y, and let G' be the graph obtaining by completing X. If G has a long hole, then in G' its vertices induce a sun.

Conversely, suppose that G' has an induced sun with clique W and independent set U of the same size. Since U is independent, at most one vertex of U is in X. Since each vertex of W has two neighbors in U, all of W lies in X. Since each vertex of U has a non-neighbor in W, all of U lies in Y. Now deleting the edges within X yields a long hole in G. □

Definition 4.5 (Uehara [37]) A vertex v in a graph G is *weakly simplicial* if its neighborhood is an independent set and the neighborhoods of its neighbors form a chain under inclusion.

This is the natural analogue of "simple vertex" for bipartite graphs. Corollary 4.4 makes the concepts essentially equivalent. The next characterization comes for free from our earlier results, although in [37] it is proved using Γ-free orderings (Definition 4.7). There is an equivalent earlier result ([22]; see also [6, p78]). Note that we could obtain two weakly simplicial vertices in each partite set of size at least two, if we so desired.

Corollary 4.6 *A graph is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex. Furthermore, a nontrivial chordal bipartite graph has a weakly simplicial vertex in each partite set.*

Proof. A k-cycle has no weakly simplicial vertex unless $k = 4$. Hence the condition is sufficient; if every induced subgraph of a graph G has a weakly simplicial vertex, then G has no shortest odd cycle and no long hole.

An induced subgraph G of a chordal bipartite graph is also chordal bipartite. Let X and Y be its partite sets. By Corollary 4.4, the graph G' obtained from G by completing X is strongly chordal. Hence G' has a simple vertex v. If $v \in Y$, then v is weakly simplicial in G. If $v \in X$, then every neighbor of v is adjacent to all of X and is weakly simplicial in G. If $v \in X$ and v has no neighbors in Y, then every vertex of Y is weakly simplicial in G. By completing Y instead, we obtain a vertex of X that is weakly simplicial in G. $\square$

Hoffman, Kolen, and Sakarovitch [23] characterized chordal bipartite graphs using a special ordering of the rows and columns of the biadjacency matrix. This ordering leads to easy proofs of other properties, just as the forbidden subgraph characterization does so for strongly chordal graphs.

Definition 4.7 A *Γ-free ordering* of a 0,1-matrix is an ordering of its rows and columns (independently) that does not have $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ as a submatrix with the 0 in the lower right. A *doubly-lexical ordering* is an ordering of the rows and columns (independently) such that for any two distinct rows, the row listed first is the one having a 0 in the last position where the two rows differ, and similarly for columns.

In [23], it is shown that a finite bipartite graph is chordal bipartite if and only if its biadjacency matrix has a Γ-free ordering, by showing that every doubly-lexical ordering of a finite totally balanced matrix is a Γ-free ordering, and by producing a doubly-lexical ordering of an m-by-n binary matrix in $O(m^2n)$-time. These two steps take about a page each, but the characterization also follows immediately from Corollary 4.4 and Farber's characterization of strongly chordal graphs by the existence of strong elimination orderings. Since those can be produced in $O(\min\{m \log n, n^2\})$-time, this gives an algorithm to construct Γ-free orderings that is faster than that of [23]. Our proof of Farber's characterization is not new, but we include it for completeness. We first recall the definition.

Definition 4.8 A *strong elimination ordering* of a graph is an ordering $v_1, \ldots, v_n$ of the vertices such that if $i < j$ and $k < l$, and $v_k, v_l \in N[v_i]$ and $v_j \in N[v_k]$, then $v_l \in N[v_j]$.

To motivate the next proof, we note that not every simple elimination ordering is a strong elimination ordering. Consider the 4-vertex graph consisting of a triangle on $\{v_1, v_3, v_4\}$ plus the edge v_2v_3. The ordering v_1, v_2, v_3, v_4 is a simple elimination ordering, but the quadruple $(i, j, k, l) = (1, 2, 3, 4)$ fails the condition for a strong elimination ordering, since although $v_3, v_4 \in N[v_1]$ and $v_2 \in N[v_3]$, the vertices v_2 and v_4 are nonadjacent.

Nevertheless, the proof shows that some simple elimination ordering is a strong elimination ordering. In the proof, the auxiliary orientation H is needed to "break ties"; simply choosing a simple vertex with smallest neighborhood can lead to problems.

Theorem 4.9 ([15]) *A graph is strongly chordal if and only if it has a strong elimination ordering.*

Proof. *Sufficiency.* Suppose that G has a strong elimination ordering σ but has an induced subgraph G' with no simple vertex. Let v_i be the first vertex of G' in σ. Since v_i is not simple in G', there are vertices $x, y \in N_{G'}(v_i)$ such that $N_{G'}[x] - N_{G'}[y]$ and $N_{G'}[y] - N_{G'}[x]$ are nonempty. Let v_k be the first of $\{x, y\}$ in σ, and let v_l be the other. Choose $v_j \in N_{G'}[v_k] - N_{G'}[v_l]$. Now i, j, k, l contradicts the definition of strong elimination ordering.

Necessity. We generate a simple elimination ordering that is also strong. Having chosen $i-1$ vertices, let G_i be the subgraph induced by the remaining vertices, and let $N_i[x]$ denote $N_{G_i}[x]$ for $x \in V(G_i)$. We maintain an orientation H of a subgraph of G_i, updated before each choice. For each edge xy of G_i that has not previously been added to H, add to H the oriented edge xy if $N_i[x] \subset N_i[y]$. Choose as v_i a vertex that is simple in G_i and is now a source in H. Delete v_i from G_i and H, increase i, and iterate.

If $xy \in E(H)$ when time i is reached, then $N_i[x] \subseteq N_i[y]$, because $N_j[x] \subset N_j[y]$ when xy was added to $E(H)$ at an earlier time j, and the intervening vertex deletions preserve weak containment. We use this twice to show that G_i has a simple vertex that is a source in H.

First, H is acyclic: if a cycle C is completed at time i, acquiring edge xy, then the vertices along the y, x-path in C satisfy $N_i[y] \subseteq \ldots \subseteq N_i[x]$ and $N_i[x] \subset N_i[y]$, a contradiction. Also, every predecessor in H of a simple vertex in G_i is simple in G_i: if y is simple in G_i and $xy \in E(H)$ at time i, then $N_i[x] \subseteq N_i[y]$, which implies that x is simple in G_i. Hence from a simple vertex of G_i we can trace back to a simple vertex of G_i that is also a source in H.

Hence the algorithm produces a simple elimination ordering; we show that it is a strong elimination ordering. Given i, j, k, l with $i < j$ and $k < l$ such that $v_k, v_l \in N[v_i]$ and $v_i, v_j \in N[v_k]$, we may assume by symmetry that $i \leq k$. Thus $v_i, v_j, v_k, v_l \in V(G_i)$. Since v_i is a simple vertex of G_i, the sets $N_i[v_k]$ and $N_i[v_l]$ are ordered by inclusion. If $v_j \in N_i[v_k] - N_i[v_l]$, then $N_i[v_l] \subset N_i[v_k]$, so $v_l \to v_k$ in H. This contradicts that v_k was chosen as a source in H while v_l was still present. Hence $v_j \in N_i[v_l] \subseteq N[v_l]$, as desired. □

Corollary 4.10 ([23]) *A bipartite graph is chordal bipartite if and only if its biadjacency matrix has a* Γ*-free ordering.*

Proof. If the biadjacency matrix has a Γ-free ordering, then there is no long hole, since the two earliest rows and two earliest columns in the ordering that correspond to vertices of the long hole would produce the forbidden submatrix Γ.

If G is chordal bipartite, then let G' be the strongly chordal graph obtained by completing partite set X. Permute the rows and columns of the biadjacency matrix of G according to a strong elimination ordering of G', extracting the row order and column order separately. If this produces Γ in rows corresponding to v_i and v_j with $i < j$ and columns corresponding to v_k and v_l with $k < l$, then we have $v_i, v_j \in N_{G'}[v_k]$ and $v_k, v_l \in N_{G'}[v_i]$, but v_j and v_l are distinct and nonadjacent. This is a contradiction. □

Strong orderings can be used in the same way to prove one direction of the following characterization; the other direction follows from the forbidden subgraph characterization.

Theorem 4.11 (Farber [15]) *A graph is strongly chordal if and only if its augmented adjacency matrix is totally balanced, where the augmented adjacency matrix is obtained from the adjacency matrix by changing each diagonal entry from 0 to 1.*

Alternatively, Sawada and Spinrad [33] note that an ordering is a strong elimination ordering if and only if using it for the rows and columns of the augmented adjacency matrix puts it in Γ-free form.

In introducing chordal bipartite graphs, Golumbic and Goss [20] were motivated by studying Gaussian elimination from sparse matrices. For this we want chordal bipartite graphs to have properties analogous to the simplicial elimination orderings and separation properties of chordal graphs. Our results so far give a short direct proof of the elimination property.

Definition 4.12 Let G be a bipartite graph. A *bisimplicial edge* of G is an edge xy such that $N[x] \cup N[y]$ induces a biclique (complete bipartite subgraph).

Theorem 4.13 (Golumbic–Goss [20]) *A bipartite graph is chordal bipartite if and only if every nontrivial induced subgraph has a bisimplicial edge.*

Proof. Let X and Y be the partite sets of a bipartite graph G. The graph G is the incidence bigraph of the multihypergraph H with $V(H) = Y$ and $E(H) = \{N(x) : x \in X\}$, so G is chordal bipartite if and only if H is totally balanced. If so, then Theorem 3.4 provides with a nonempty simple edge $e \in E(H)$, where $e = N(x)$ for some $x \in X$. Pick $y \in e$ to be in the fewest other edges. Since $\{e \cap f : f \in E(H)\}$ is ordered by inclusion, every edge in H that contains y must also contain all of e. Thus $N_G[y]$ and $N_G[x]$ form a biclique in G, and xy is a bisimplicial edge of G. Since every nontrivial induced subgraph of a chordal bipartite graph is chordal bipartite, the proof is complete.

Otherwise H has a strict cycle C of length $k \geq 3$. Then the special vertices of C (in Y) and the vertices of X corresponding to $E(C)$ induce an cycle in G of length $2k$. Such a cycle has no bisimplicial edge. □

Like the original result of Dirac [14] guaranteeing simplicial vertices and our earlier results guaranteeing simple edges, simple vertices, and weakly simplicial vertices, this result can be extended to guarantee two bisimplicial edges in chordal bipartite graphs with at least two edges.

5 Split graphs and threshold graphs

In this section we consider two special families of chordal graphs and relate them and their properties to strongly chordal graphs.

A *split graph* is a graph whose vertex set can be partitioned into a stable set S and a clique Q. Such a graph has no chordless cycle, and hence Corollary 4.4 implies that a split graph is strongly chordal if and only if the spanning subgraph whose edge set consists of the edges joining S and Q is chordal bipartite.

Split graphs were introduced by Foldes and Hammer [16] (and independently in [36]) who proved a forbidden subgraph characterization. They proved the sufficiency of this characterization by choosing a largest clique whose deletion leaves the minimum number of edges and showing that any edge remaining would require the presence of a forbidden subgraph. Here we give a short proof using simplicial vertices. Note that the graph $2K_2$ (the disjoint union of two copies of K_2) is the complement of the 4-cycle C_4.

Theorem 5.1 *For a graph G, the following are equivalent:*
A) $V(G)$ can be partitioned into a clique and a stable set.
B) Both G and $\bar{G}$ are chordal graphs.
C) G has no induced C_4, $2K_2$, or C_5.

Proof. Both A $\Rightarrow$ B and B $\Rightarrow$ C are clear.

For C $\Rightarrow$ A, we use induction on the number of vertices, with trivial basis. Since an induced cycle of length at least 6 contains an induced $2K_2$, the given graph G is chordal. Let v be a simplicial vertex of G. Since condition C is hereditary, the induction hypothesis implies that $V(G-v)$ has a partition into a clique Q and a stable set S.

If v has no neighbor in S, then Q and $S \cup \{v\}$ form the desired partition. Since v is simplicial, may let thus let w be the unique neighbor of v in S; note that $N(v) \cap Q \subseteq N(w) \cap Q$. If w has two nonnneighbors in Q, then they induce $2K_2$ with w and v. If w has no nonneighbor in Q, then $Q \cup \{w\}$ and $S \cup \{v\} - \{w\}$ is the desired partition.

Hence w has exactly one nonneighbor in Q; call it x. Any neighbor of x in S induces $2K_2$ with $\{x, v, w\}$. Thus $Q \cup \{w\} - \{x\}$ and $S \cup \{x, v\} - \{w\}$ form the desired partition. □

Threshold graphs were introduced and characterized by Chvátal and Hammer [11]. A graph G is a *threshold graph* if there exists a threshold t and a function w assigning nonnegative integers to vertices such that a set S of vertices is a stable set if and only if $\sum_{x \in S} w(x) \leq t$; the pair (w, t) is a *threshold weighting*. An *edge-threshold weighting* is a threshold and function that meet this condition for pairs of vertices. We recall standard characterizations of threshold graphs (some of these equivalences were independently discovered by others).

Theorem 5.2 ([11]) *For a graph G, the following properties are equivalent.*
A) G has a threshold weighting.
B) G has an edge-threshold weighting.
C) G has no four vertices x, y, z, w such that $xy, zw \in E(G)$ and $yz, wx \notin E(G)$.
D) G does not have any of $\{P_4, C_4, 2K_2\}$ as an induced subgraph.
E) G is a split graph with no induced P_4.
F) Every induced subgraph of G has an isolated or a dominating vertex.

The implications A ⇒ B ⇒ C ⇒ D ⇒ E ⇒ F are straightforward, and F ⇒ A is proved by inductively constructing a threshold weighting using an vertex ordering $v_1, \ldots, v_n$ such that for each i, v_i is isolated or dominating in the subgraph induced by $\{v_i, \ldots, v_n\}$. We show that threshold graphs are characterized by several types of elimination orderings. (Property F also yields a characterization of threshold graphs in terms of the list of vertex degrees.)

Definition 5.3 Given a vertex ordering $v_1, \ldots, v_n$ for a graph G, let G_i be the subgraph induced by $\{v_i, \ldots, v_n\}$. An *isolated/dominating elimination ordering* is a vertex ordering such that for each i, v_i is isolated or dominating in G_i. A *simplicial/cosimplicial elimination ordering* is a vertex ordering such that for each i, v_i is simplicial in both G_i and $\bar{G}_i$. A *simple/cosimple elimination ordering* is a vertex ordering such that for each i, v_i is a simple vertex in both G_i and $\bar{G}_i$.

Theorem 5.4 *For a graph G, the following properties are equivalent.*
A) G does not have any of $\{P_4, C_4, 2K_2\}$ as an induced subgraph.
B) G has an isolated/dominating elimination ordering.
C) G has a simple/cosimple elimination ordering.
D) G has a simplicial/cosimplicial elimination ordering.

Proof. A ⇒ B (standard). Since Property A is hereditary, it suffices to show that G has an isolated or dominating vertex. Since C_5 has P_4 as an induced subgraph, G is a split graph. Let Q and S be a clique and a stable set in a partition of $V(G)$. Since G has no induced P_4, the closed neighborhoods of vertices in Q are ordered by inclusion. If the largest includes all of S, then the vertex with that closed neighborhood is dominating. Otherwise, a vertex of S omitted from the largest is isolated.

B ⇒ C. An isolated/dominating elimination ordering $v_1, \ldots, v_n$ for G works also for $\bar{G}$, so proving that v_n is a simple vertex in G shows that v_n is also simple in $\bar{G}$. Let D and I be the subsets of $V(G)$ that are designated as dominating or isolated in the elimination order, respectively. The neighbors of v_n are the earlier vertices in D. Their closed neighborhoods consist of all of D and all subsequent vertices in I. The terminal segments of I in the list are ordered by inclusion. Hence v_n is simple in G.

C ⇒ D. Every simple vertex in a graph is a simplicial vertex.

D ⇒ A. Such a vertex ordering prevents any of $\{P_4, C_4, 2K_2\}$ as induced subgraphs, because no vertex of such a subgraph can be eliminated. Suppose that v_i is the first such vertex eliminated. No vertex of an induced C_4 is simplicial in G_i, and no vertex of an induced $2K_2$ is simplicial in $\bar{G}_i$. For an induced P_4, neither middle vertex is simplicial in G_i, and neither end vertex is simplcial in $\bar{G}_i$. □

Since all threshold graphs have simple elimination orderings, they are strongly chordal.

Acknowledgement

The authors thank R. Sritharan for helpful conversations, including alerting us to a temporary gap in the main proof.

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