Bounds on the k-dimension of products of special posets

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July 30, 2008

Abstract

Trotter conjectured that $\dim P \times Q \geq \dim P + \dim Q - 2$ for all posets P and Q. To shed light on this, we study the k-dimension of products of finite orders. For fixed k, the value $2\dim_k(P) - \dim_k(P \times P)$ is unbounded when P is an antichain, and $2\dim_2(mP) - \dim_2(mP \times mP)$ is unbounded when P is a fixed poset with unique maximum and minimum. For products of the "standard" orders S_m and S_n of dimensions m and n, $\dim_k(S_m \times S_n) = m + n - \min\{2, k-2\}$. For higher order products of "standard" orders, $\dim_2\left(\prod_{i=1}^{t} S_{n_i}\right) = \sum n_i$ if each $n_i \geq t$.

1 Introduction

The dimension of a poset P is a natural measure of its complexity. Written $\dim P$, the *dimension* is the least t such that P embeds as a subposet of $\mathbb{R}^t$ under the componentwise ordering. Note that $\mathbb{R}^t$ is the product of t copies of $\mathbb{R}$, where the *product* $P \times Q$ of posets P and Q is the poset on $\{(p,q)\colon\ p \in P, q \in Q\}$ given by $(p,q) \leq (p',q')$ if and only if $p \leq p'$ in P and $q \leq q'$ in Q.

The behavior of poset dimension under the product operation is not fully understood. If P embeds in $\mathbb{R}^s$ and Q embeds in $\mathbb{R}^t$, then assigning (p,q) the concatenation of the images of p and q under these embeddings shows that $P \times Q$ embeds in $\mathbb{R}^{s+t}$, and hence $\dim(P \times Q) \leq \dim(P) + \dim(Q)$. Since P and Q both appear in $P \times Q$, the trivial lower bound is $\max\{\dim(P), \dim(Q)\}$.

The upper bound is tight when P and Q are *bounded*, meaning that each has a unique maximal element and a unique minimal element. This equality was proved first by Baker [2]

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and later more simply by Kelly [9]. It is thought that the upper bound is never far from the truth. Trotter's conjecture that $\dim(P \times Q) \geq \dim P + \dim Q - 2$ appears in [10].

The fundamental result in this direction is that of Trotter [15], proving that for $n \geq 2$ the product of the smallest n-dimensional poset with itself has dimension $2n - 2$. Let $[n] = \{1, \ldots, n\}$. The *standard example* S_n is the containment poset on the family of sets consisting of the n singleton sets and their complements in $[n]$. Hiraguchi [7] showed that S_n is the smallest n-dimensional poset. Reuter [13] extended Trotter's result (using "concept analysis") to show that $\dim(S_m \times S_n) = m + n - 2$ (for $\min\{m, n\} \geq 2$). Reuter also showed that $\dim(P \times P) \geq 4$ whenever $\dim P = 3$. Lin [11] gave another proof of Reuter's result for the product of standard examples and proved that $\dim(S'_m \times S_n) = m + n - 2$, where S'_m is the poset obtained from the standard example S_m by adding unique maximal and minimal elements. This shows that Trotter's Conjecture cannot be refined by improving the bound to $\dim(P \times Q) \geq \dim P + \dim Q - 1$ when P or Q is bounded.

In this paper we extend Reuter's result on the standard examples. Let $\underline{k}$ denote the poset that is a chain of k elements. The *k-dimension* of a poset P is the minimum t such that P embeds in the product poset $\underline{k}^t$. Reducing k restricts the flexibility of embeddings for P, so $\dim_2 P \geq \dim_3 P \geq \cdots \geq \dim_\infty P = \dim P$. The concatenation argument again shows that $\dim_k(P \times Q) \leq \dim_k(P) + \dim_k(Q)$.

For $n \geq 3$, the poset S_n is a subposet of the containment poset on the family of all subsets of $[n]$. Thus S_n embeds in $\underline{2}^n$, and the dimension hierarchy collapses to yield $\dim_k S_n = n$ for $n \geq 3$ and $k \geq 2$. By a more detailed look at Trotter's counting argument for the lower bound, we determine $\dim_k(S_m \times S_n)$ whenever $\min\{m, n\} \geq 3$ and $k \geq 2$. Our main result in this direction is that $\dim_k(S_m \times S_n) = m + n - \min\{2, k - 2\}$.

Our result suggests an extension of Trotter's conjecture. A poset is *connected* if its diagram is connected.

Conjecture 1.1 *For connected posets P and Q with dimension at least 2,* $\dim_k(P \times Q) \geq \dim_k P + \dim_k Q - \min\{2, k - 2\}$. *In particular,* $\dim_2 (P \times Q) = \dim_2 (P) + \dim_2 (Q)$.

However, we shown in Section 2 that for disconnected posets no such conjecture holds, and the difference between $\dim_k P + \dim_k P$ and $\dim_k (P \times P)$ can be large.

The definition of dimension via embedding in products of chains is due to Hiraguchi [8] and to Ore [12]. It is equivalent to the earlier and more familiar definition of Dushnik and Miller [3]. An *extension* Q of a poset P is a partial order on the same set such that $x \leq y$ in P implies $x \leq y$ in Q. A *realizer* of P is a set S of extensions of P such that $x \leq y$ in P if and only if $x \leq y$ in each member in S. A *linear extension* is an extension that is a chain (a total order). The *dimension* of P is the minimum size of a realizer of P consisting of linear extensions.

The concept of k-dimension for general k was introduced by Trotter [14]. We need the equivalent realizer phrasing. A *ranking* or *weak order* with k levels is a poset whose elements partition into k sets $X_1, \ldots, X_k$ such that $a < b$ if and only if $a \in X_i$ and $b \in X_j$ with $i < j$. A *k-extension* of P is an extension that is a ranking with k levels. The *k-dimension* of P is the minimum size of a realizer of P consisting of k-extensions. The coordinates in an embedding of P in $\underline{k}^t$ yield a realizer of size t for P by k-extensions, and vice versa.

Following Trotter [15], we count what can be accomplished by a single k-extension. A *critical pair* is an ordered pair (x, y) of incomparable elements in P such that $(z < x \Rightarrow z < y)$ and $(z > y \Rightarrow z > x)$. In other words, no relation can be added to P that implies $x > y$. A critical pair (x, y) is *reversed* in an extension E if $x > y$ in E. A set of extensions of P is a realizer of P if and only if every critical pair is reversed in at least one extension.

Let $S(E)$ denote the set of critical pairs reversed in an extension E of P. A k-extension E of P is *saturated* if $S(E)$ is maximal among all the sets $S(E')$ such that E' is a k-extension of P. An extension E is *dominated* by an extension E' if $S(E) \subset S(E')$.

Lower bounds follow from studying the structure of saturated k-extensions. We count the critical pairs of various types that can be reversed by various saturated k-extensions. The requirement of reversing all the critical pairs yields a linear optimization problem to minimize the total number of k-extensions needed to provide the reversals. The numerical result of this linear program yields a lower bound that is achievable using appropriate extensions, which completes the computation of the dimension.

In Section 6, we introduce another technique to give a simple proof of a stronger result about 2-dimension of iterated products of standard examples: $\dim_2\left(\prod_{i=1}^{t} S_{n_i}\right) = \sum n_i$. Since products of standard examples are not standard examples, and since a t-fold product groups into two factors in various ways, this proves the conjecture about 2-dimension of products for a class of non-bounded posets that includes more than just standard examples.

2 Disconnected Posets

Conjecture 1.1 is restricted to connected posets for the following reason:

Proposition 2.1 *For the antichain A_n with n elements, the difference between $\dim_2 A_n + \dim_2 A_n$ and $\dim_2(A_n \times A_n)$ grows without bound.*

Proof. By Sperner's Theorem, the maximum size of an antichain of subsets of $[t]$ is $\binom{t}{\lfloor t/2 \rfloor}$. Hence $\dim_2 A_n$ is the minimum t such that $\binom{t}{\lfloor t/2 \rfloor} \geq n$. Thus $\dim_2 A_n \in \lg n + \frac{1}{2}\lg\lg n + O(\lg\lg\lg n)$, where lg denotes $\log_2$.

Note that $A_n \times A_n = A_{n^2}$. We have $\dim_2 A_n + \dim_2 A_n \in 2\lg n + \lg\lg n + O(\lg\lg\lg n)$, but $\dim_2(A_n \times A_n) \in 2\lg n + \frac{1}{2}\lg\lg n + O(\lg\lg\lg n)$. Hence the difference between $\dim_2 A_n +$

$\dim_2 A_n$ and $\dim_2 (A_n \times A_n)$ can be arbitrarily large. $\square$

We generalize Proposition 2.1 and its proof in two natural ways. First, we prove the claim also for $\dim_k$ (in fact, k may grow slowly with n). Second, we show for $k = 2$ that the computation remains valid when the antichain is replaced with multiple copies of any bounded poset. We cannot suppress k and extend the construction to ordinary dimension, because $\dim A_{n^2} = 2 = \dim A_n + \dim A_n - 2$.

For k-dimension, we mimic the argument for Proposition 2.1. As in that discussion, $\dim_k(A_n)$ is the smallest t such that $\underline{k}^t$ contains A_n. Thus we need to approximate the width of $\underline{k}^t$. Since chain-products are symmetric chain orders, the width is the size of the middle rank; that is, the number of elements of $\{0, \ldots, k-1\}^t$ whose entries sum to $\lfloor t(k-1)/2 \rfloor$. The asymptotic behavior of the width is a special case of a result of Alekseev [1] that was generalized by Engel [4] (see [5]). We give a short argument for the special case needed here.

Lemma 2.2 *For $k \in o(\log t)$, the width $w(\underline{k}^t)$ of $\underline{k}^t$ is asymptotic to $k^t / \sqrt{(k^2-1)\,\pi t/6}$.*

Proof. It suffices to show that the probability of a uniformly selected random member of $\underline{k}^t$ having rank $\lfloor t(k-1)/2 \rfloor$ is asymptotic to $1/\sqrt{(k^2-1)\,\pi t/6}$. The rank is the sum of t independent uniform random variables on $\{0, \ldots, k-1\}$; the mean μ and variance σ^2 of each are $(k-1)/2$ and $(k^2-1)/12$, respectively. By the Central Limit Theorem, the probability that the value equals x is asymptotic to $e^{-(x-t\mu)^2/(t\sigma^2)}/\sqrt{2\pi t\sigma^2}$. Setting $x = \lfloor t(k-1)/2 \rfloor$ completes the proof. $\square$

Thus the growth of the width of $\underline{k}^t$ is close to k^t. This will make the k-dimension of antichains arbitrarily subadditive. Indeed, we obtain that conclusion even when k grows (sufficiently slowly) with n.

Theorem 2.3 *The difference between $2\dim_k (A_n)$ and $\dim_k (A_n \times A_n)$ grows without bound as n grows, as long as k grows no faster than $\left(\frac{\log n}{\log\log n}\right)^{f(n)}$, where f is any function such that $\lim_{n\to\infty} f(n) = 0$.*

Proof. Recall that $\dim_k(A_n)$ is the least t such that $w(\underline{k}^t) \geq n$. By Lemma 2.2, $\dim_k (A_n) \in \log_k n + \frac{1}{2}\log_k \log_k n + O(\log_k \log_k \log_k n)$. Now $2\dim_k (A_n) - \dim_k (A_n \times A_n)$ is given by the same computation as in Proposition 2.1, with $\log_k$ in place of lg.

The resulting difference is $\frac{1}{2}\log_k \log_k n + O(\log_k \log_k \log_k n)$. This difference is bounded by a constant when k is given by any positive constant power of $\frac{\log n}{\log\log n}$. However, if k is bounded by a power of $\frac{\log n}{\log\log n}$ that tends to 0, then the difference is unbounded. $\square$

We now generalize Proposition 2.1 in a different direction by replacing the elements of the antichain A_n with copies of any bounded poset P.

Lemma 2.4 *If P is a bounded poset and* $\dim_2 P = p$*, then mP embeds in $\underline{2}^t$ if and only if $m\underline{(p+1)}$ embeds in $\underline{2}^t$.*

Proof. Consider an embedding of mP into $\underline{2}^t$. Each copy of P must embed into the interval between the images of its top and bottom elements. Each such intervals is isomorphic to $\underline{2}^s$ for some s with $p \le s \le t$. The top of copy i of P fails to be above the bottom of copy j of P if and only if no element of the ith such interval is above any element of the jth such interval. In particular, choosing a $(p+1)$-element chain from each of the intervals yields an embedding of $m\underline{(p+1)}$ in $\underline{2}^t$.

Conversely, an embedding of $m\underline{(p+1)}$ yields m intervals of length at least p in $\underline{2}^t$ that serve as the intervals into which copies of P can be embedded to embed mP into $\underline{2}^t$. $\square$

The problem of embedding $m\underline{(p+1)}$ into $\underline{2}^t$ has a well-known solution.

Lemma 2.5 (Griggs–Stahl–Trotter [6]) *$m\underline{(p+1)}$ embeds in $\underline{2}^t$ if and only if $m \le \binom{t-p}{\lfloor (t-p)/2 \rfloor}$.*
$\square$

Theorem 2.6 *If P is a bounded poset and* $\dim_2(P) = p$*, then*

$$\dim_2 mP = \min\{t\colon \tbinom{t-p}{\lfloor (t-p)/2 \rfloor} \ge m\}.$$

In particular, $2\dim_2(mP) - \dim_2(mP \times mP)$ is unbounded as a function of m.

Proof. The equation is the immediate consequence of the two preceding lemmas. Letting $n = t - p$, where p is fixed, the subsequent conclusion follows by the same computation as in Proposition 2.1. $\square$

3 Critical Pairs in Products of Bipartite Posets

A *bipartite poset* is a poset containing no 3-element chain, so that each element is maximal or minimal in the poset (equivalently, it is a poset whose comparability graph is bipartite). In this section we develop technical properties of bipartite posets that are needed for the reductions in our dimension arguments to be valid. The standard examples have these properties. We discuss the standard examples in the following form.

Definition 3.1 *The* standard example S_n *consists of minimal elements* $\{a_1, \ldots, a_n\}$ *and maximal elements* $\{b_1, \ldots, b_n\}$*, with* $a_i < b_j$ *if and only if* $i \neq j$*.*

The critical pairs in S_n are $\{(a_i, b_i)\colon\ 1 \leq i \leq n\}$. Each linear extension reverses at most one such pair. On the other hand, $\dim_2 S_n \leq n$ for $n \geq 3$, since S_n is a subposet of $\underline{2}^n$. Thus $n \leq \dim S_n \leq \dim_k S_n \leq \dim_2 S_n \leq n$, and equality holds throughout for all $k \geq 2$ and $n \geq 3$. This yields the trivial bound $\dim_k(S_n \times S_m) \leq n + m$.

In a product $P_1 \times P_2$, an element (x_1, x_2) is minimal if and only if x_1 is minimal in P_1 and x_2 is minimal in P_2 (similarly for maximal). We write $x \| y$ to indicate incomparability of x and y.

Definition 3.2 *A bipartite poset is* nontrivial *if for every pair* (u, v) *with* $u < v$*, there exist incomparable elements* x *and* y *with* $u < x$ *and* $y < v$*, and for every ordered incomparable pair* (x, y)*, there is an element comparable to* x *but not to* y*.*

Lemma 3.3 *If* P *and* Q *are nontrivial bipartite posets, then the critical pairs in* $P \times Q$ *are the ordered incomparable pairs* (x, y) *such that* x *is minimal and* y *is maximal in* $P \times Q$*. Furthermore,* $\dim_k(P \times Q)$ *equals the* k*-dimension of the subposet of* $P \times Q$ *consisting of the maximal elements and the minimal elements.*

Proof. The condition for (x, y) to be a critical incomparable pair is that $(z < x \Rightarrow z < y)$ and $(z > y \Rightarrow z > x)$. If x is minimal and y is maximal, then these conditions hold vacuously. We show that (x, y) is not a critical pair when x is not minimal, and then by symmetry also (x, y) is not a critical pair when y is not maximal.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If x is not minimal, then x_1 or x_2 is not minimal in its factor. We may assume by symmetry that x_1 is not minimal in P. We will force x over y by putting (z_1, x_2) over (w_1, y_2) for an appropriate choice of z_1 and w_1 with $z_1 < x_1$.

If $y_1 < x_1$, then P has incomparable elements z_1 and w_1 such that $z_1 < x_1$ and $y_1 < w_1$, as desired. The only other case is $y_1 \| x_1$ (since x_1 is not minimal). We can now choose z_1 such that $z_1 < x_1$ and $z_1 \| y_1$. If we set $w_1 = y_1$, then again putting (z_1, x_2) over (w_1, y_2) forces x over y.

For the final statement, we now know that the minimal and maximal elements are the only elements of the product appearing in critical pairs. For any k, a set of k-extensions that reverses all the critical pairs reverses all incomparable pairs and yields a k-realizer. Also note that every k-extension of a subposet can be augmented by the remaining elements to form a k-extension of the full poset. Together, these observations yield the claim. □

4 Saturated k-extensions of $S_m \times S_n$

For $n \geq 3$, S_n is a nontrivial bipartite poset, so Lemma 3.3 applies to $S_m \times S_n$. In $S_m \times S_n$, we write $a_{i,j}$ for the minimal element (a_i, a'_j), where a_i and a'_j are the ith and jth minimal elements in S_m and S_n, respectively. Similarly, $b_{r,s}$ denotes the maximal element (b_r, b'_s). A min/max pair $(a_{i,j}, b_{r,s})$ is incomparable (and hence a critical pair) if and only if $i = r$ or $j = s$. That is, $a_{i,j} < b_{r,s}$ in $S_m \times S_n$ if $i \neq r$ and $j \neq s$.

We place the minimal elements in one grid and the maximal elements in another. Corresponding positions act as “pins” holding the two grids together. The incomparable pairs consist of one position from each grid, taken from the same column or from the same row.

Definition 4.1 *The critical pairs in $S_m \times S_n$, written as $(a_{i,j}, b_{r,s})$, form three classes:*
a) vertical pairs *are those with $i \neq r$ and $j = s$,*
b) horizontal pairs *are those with $i = r$ and $j \neq s$,*
c) straight pairs *(“pins”) are those with $i = r$ and $j = s$.*
There are $m^2n - mn$ vertical pairs, $mn^2 - mn$ horizontal pairs, and mn straight pairs.

Let S^* be the subposet consisting of the maximal and the minimal elements of $S_m \times S_n$. A 2-extension of S^* is an order-preserving map from S^* to $\{0, 1\}$. We represent it as two 0,1-matrices, a lower one for the minimal elements and an upper one for the maximal elements. Let $\alpha_{i,j}$ denote the value assigned to $a_{i,j}$, and let $\beta_{r,s}$ denote the value assigned to $b_{r,s}$.

The lower bound for $\dim_2(S_m \times S_n)$ follows from Theorem 6.8 without characterizing saturated 2-extensions, but including this reduces the case analysis for 3-extensions.

Lemma 4.2 *Every saturated 2-extension of S^* reverses precisely all vertical and straight pairs in one column (Type V), or all horizontal and straight pairs in one row (Type H), or all critical pairs using one element (Type B), or the two horizontal and two vertical pairs formed from two minimal and two maximal elements in each of S_m and S_n (Type C).*

Proof. Below are examples of the types of 2-extensions described in the statement, when $m = 3$ and $n = 4$. The two grids for a single 2-extension appear in the same column.

$$\begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 0&1&1&1\\\hline 0&1&1&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1&1&1&1\\\hline 0&0&0&0\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1&1&1&1\\\hline 0&1&1&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 0&1&1&1\\\hline 0&1&1&1\\\hline 0&0&0&0\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 1&0&1&1\\\hline 0&1&1&1\\\hline\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 1&0&0&0\\\hline 1&0&0&0\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 0&0&0&0\\\hline 1&1&1&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 1&0&0&0\\\hline 1&0&0&0\\\hline 1&1&1&1\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 0&0&0&0\\\hline 1&0&0&0\\\hline\end{array}\qquad
\begin{array}{|c|c|c|c|}\hline 0&0&0&0\\\hline 1&0&0&0\\\hline 0&1&0&0\\\hline\end{array}$$

Type V Type H Type B Type B Type C

Since extensions are order-preserving,

$$\alpha_{i,j} \le \beta_{r,s} \quad \text{when} \quad i \ne r \quad \text{and} \quad j \ne s. \tag{$*$}$$

A critical pair is reversed when $\alpha_{i,j} = 1$ and $\beta_{r,s} = 0$ with $i = r$ or $j = s$; it is horizontal (and *in row* r) when $i = r$, vertical (and *in row* s) when $j = s$, and straight when both equalities hold. By "greediness", we mean that we have reached a point where we can assign 1 to all remaining positions in the bottom grid and 0 to all remaining positions in the top grid, maximizing the number of reversed pairs. We index grid positions from the lower left.

Let E be a saturated 2-extension of S^*. If no horizontal or vertical pair is reversed, then a reversed pair has the form $(a_{i,j}, b_{i,j})$, with $\alpha_{i,j} = 1$ and $\beta_{i,j} = 0$. Since no horizontal or vertical pair is reversed, by $(*)$ no other pair is reversed. Now E is dominated by extensions of all types listed and is not saturated.

Suppose that E reverses a horizontal pair; by symmetry, we may let it be $(a_{1,2}, b_{1,1})$. Thus $\alpha_{1,2} = 1$ and $\beta_{1,1} = 0$. By $(*)$, we have $\beta_{i,j} = 1$ for $i > 1$ and $j \ne 2$, and $\alpha_{i,j} = 0$ for $i > 1$ and $j \ne 1$. If there is another reversed horizontal pair in the first row of the form $(a_{1,s}, b_{1,t})$ with $s \ne 2$ and $t \ne 1$, then the upper grid is all 1 outside row 1 and the lower grid is all 0 outside row 1. By greediness, saturation requires that E is Type H, reversing all horizontal and straight pairs in a single row. Similarly, if disjoint vertical pairs in the same column are reversed, then E is Type V.

If there is another reversed horizontal pair outside the first row, then the constraints imposed by the first pair require it to be $(a_{r,1}, b_{r,2})$ with $r > 1$. By symmetry, let $r = 2$, so we have $\alpha_{2,1} = 1$ and $\beta_{2,2} = 0$.) Now $(*)$ forces E to be Type C. The same conclusion holds when E reverses vertical pairs in distinct columns.

In the remaining case, all reversed horizontal pairs lie in the same row, and all reversed vertical pairs lie in the same column. Suppose first that there is at least one of each type. By symmetry, we may assume that $(a_{1,2}, b_{1,1})$ is a reversed horizontal pair. For each column s, either $\alpha_{r,s} = 0$ for $r > 1$ or $\beta_{r,s} = 1$ for $r > 1$, or both if $s > 2$. Hence a reversed vertical pair must be in column 1 and use $b_{1,1}$ or be in column 2 and use $a_{1,2}$. In the first case, greediness now forces E to be Type B, reversing all critical pairs using $b_{1,1}$. In the second case, again E is Type B, reversing all critical pairs using $a_{1,2}$.

Finally, we have horizontal pairs in one row and no vertical pairs, or vertical pairs in one column and no horizontal pairs. By greediness, E is Type H or Type V, respectively. □

In a k-extension, we partition the elements of S^* into k sets $X_0, \ldots, X_{k-1}$ We still represent such extensions using two grids, but with entries in $\{0, \ldots, k-1\}$. Still $(*)$ holds: $\alpha_{i,j} \le \beta_{r,s}$ when $i \ne r$ and $j \ne s$.

Lemma 4.3 *Every saturated* 3*-extension of* S^* *reverses precisely all vertical and straight pairs in one column and the horizontal pairs using one element in that column (Type* V*), or all horizontal and straight pairs in one row and the vertical pairs using one element in that row (Type* H*), or all critical pairs using the elements in one critical pair (Type* B*), or all critical pairs using one element plus one more horizontal and one more vertical pair (Type* C*).*

Proof. Below we show examples of the types of 3-extensions described in the statement (except Type C), when $m = 3$ and $n = 4$. Compared to Type B_1, Type B_2 reverses one less horizontal pair and one more straight pair.

0	2	2	2
0	2	2	2
0	1	1	1

1	0	0	0
1	0	0	0
2	0	0	0

Type V

1	2	2	2
1	2	2	2
0	2	2	2

2	0	0	0
2	0	0	0
2	1	1	1

Type V

1	2	2	2
1	2	2	2
0	0	0	0

0	0	0	0
0	0	0	0
2	1	1	1

Type H

2	2	2	2
2	2	2	2
0	1	1	1

1	0	0	0
1	0	0	0
2	2	2	2

Type H

1	2	2	2
1	2	2	2
0	1	1	1

1	0	0	0
1	0	0	0
2	1	1	1

Type B_1

1	2	2	2
1	2	2	2
1	0	1	1

0	1	0	0
0	1	0	0
2	1	1	1

Type B_2

Let E be a saturated 3-extension of S^*. If E puts no 2 in the bottom grid, then changing every 2 into a 1 in the top grid yields a 2-extension with the same critical pairs. However, each 2-extension in Lemma 4.2 is strictly dominated by some 3-extension shown above. Therefore, E puts a 2 in the bottom grid. Similarly, E puts a 0 in the top grid. By ($*$), a 2 in the bottom grid and 0 in the top grid must be in the same row or in the same column and correspond to a reversed critical pair. We call such a reversed pair an *extreme* pair.

Suppose first that there is an extreme horizontal pair; we may assume by symmetry that $\alpha_{1,1} = 2$ and $\beta_{1,2} = 0$. By ($*$), we have $\beta_{r,s} = 2$ for $r \neq 1$ and $s \neq 1$, and we have $\alpha_{i,j} = 0$ for $i \neq 1$ and $j \neq 2$. This implies that the only vertical pairs that can be reversed are those using $a_{1,1}$ or $b_{1,2}$. Also, the only horizontal pairs that can be reversed are those in the first row and $(a_{i,2}, b_{i,1})$ with $i > 1$.

If $2 \geq \alpha_{i,2} > \beta_{i,1} \geq 0$ with $i > 1$, then $\alpha_{i,2} = 2$ or $\beta_{i,1} = 0$. The former forces the first column (except $\beta_{i,1}$) and first row (except $\beta_{1,2}$) in the top grid to be 2, while the latter forces the second column (except $\alpha_{i,2}$) and first row (except $\alpha_{1,1}$) in the bottom grid to be 0. If $\alpha_{i,2} = 2$, then the only possible reversed pairs now are those using $b_{1,2}$ plus two pairs using $b_{i,1}$. If $\beta_{i,1} = 0$, then the only reversed pairs are those using $a_{1,1}$ plus two pairs using $a_{i,2}$. By greediness, the extension now has Type C. Compared to Type B_2, we gain one horizontal pair, but we lose $n - 2$ horizontal pairs, $m - 2$ vertical pairs, and one straight pair. Type C reverses only m vertical pairs, n horizontal pairs, and one straight pair.

Hence we may assume that no horizontal pair outside row 1 is reversed, so $\beta_{i,1} \geq \alpha_{i,2}$ for $i \neq 1$. If vertical pairs using both $a_{1,1}$ and $b_{1,2}$ are reversed, by $\alpha_{1,1} > \beta_{i,1}$ and $\alpha_{r,2} > \beta_{1,2}$ (possibly $i = r$), then $2 \geq \alpha_{1,1} > \beta_{i,1} \geq \alpha_{r,2} > \beta_{1,2} \geq 0$. Hence $\beta_{i,1} = \alpha_{r,2} = 1$. Now $\beta_{i,1} = 1$ forces the first row (except $\alpha_{1,1}$) and second column (except $\alpha_{i,2}$) of the bottom grid to be at most 1, and $\alpha_{r,2} = 1$ forces the first row (except $\beta_{1,2}$) and first column of the top grid (except $\beta_{r,1}$) to be at least 1. By greediness, we set all these values to 1, reversing precisely all critical pairs using $a_{1,1}$ and $b_{1,2}$. Now E is Type B_2 (with two reversed straight pairs).

If reversed vertical pairs use at most one of $\{a_{1,1}, b_{1,2}\}$, then we may assume that no reversed vertical pair uses $b_{1,2}$. Since $\beta_{1,2} = 0$, this implies that $\alpha_{i,2} = 0$ for $i \neq 1$. Now the only pairs that can be reversed are those in the first row and those in the first column that use $a_{1,1}$. We can reverse all of them by setting $\alpha_{1,j} = 1$ for $j \neq 1$ and $\beta_{r,1} = 1$ for $r \neq 1$. The result is a 3-extension of Type H.

This completes the analysis for the case of an extreme horizontal pair. For an extreme vertical pair, the argument is symmetric to this. Hence we may assume that there is an extreme straight pair and no other extreme pair (two extreme straight pairs would force an extreme horizontal or vertical pair).

By symmetry, we may assume that $\alpha_{1,1} = 2$ and $\beta_{1,1} = 0$. By $(*)$, the top grid is 2 outside the first row and column, and the bottom grid is 0 outside the first row and column. Since there are no extreme horizontal or vertical pairs, the remaining entries are at most 1 in the bottom grid and at least 1 in the top grid. Greediness sets them all equal to 1 and yields an extension of Type B_1 (with one reversed straight pair). □

We treat all k with $k \geq 4$ simultaneously. This means that we analyze what sets of critical pairs can be reversed for linear extensions, and we show that the maximal such sets can be achieved using 4-extensions. This yields $\dim_k(S_m \times S_n) = \dim(S_m \times S_n)$ for all $k \geq 4$. The analysis of linear extensions follows the approach of Trotter [15]; the main difference is that we separate horizontal and vertical pairs.

Lemma 4.4 *Every saturated linear extension of S^* (or saturated k-extension for $k \geq 4$) reverses precisely all vertical and straight pairs in one column and all horizontal pairs using two fixed elements (one maximal and one minimal) in that column (Type V), or all horizontal and straight pairs in one row and all vertical pairs using two fixed elements (one maximal and one minimal) in that row (Type H), or all critical pairs using the elements of one vertical pair or of one horizontal pair (plus one more critical pair) (Type B).*

Proof. Below are examples of 4-extensions of these types, when $m = 3$ and $n = 4$. For Type H, the minimal and maximal elements that cover the vertical pairs may form a straight pair or a horizontal pair; both variations are shown. Analogous variations exist for Type V.

In the examples for Type B, all critical pairs using $a_{1,1}$ are reversed, along with those using $b_{2,1}$ in the first example and those using $b_{1,2}$ in the second example, and the extra reversed pair puts $a_{2,2}$ over $b_{1,2}$ or $b_{2,1}$, respectively.

1	3	3	3
1	3	3	3
0	2	2	2

2	0	0	0
2	0	0	0
3	1	1	1

Type V

2	3	3	3
2	3	3	3
0	1	1	1

1	0	0	0
1	0	0	0
3	2	2	2

Type H

2	3	3	3
2	3	3	3
1	0	1	1

0	1	0	0
0	1	0	0
3	2	2	2

Type H

2	3	3	3
0	3	3	3
2	1	2	2

1	0	0	0
1	2	1	1
3	0	0	0

Type B

2	3	3	3
1	3	3	3
2	0	2	2

0	1	0	0
0	2	0	0
3	1	1	1

Type B

The above examples generalize for all m and n to show that extensions for $k \geq 4$ do exist that reverse the pairs as described in the statement.

Let E be a saturated linear extension of S^*. A saturated extension must reverse some critical pair. By symmetry, we may assume that $a_{1,1}$ is the highest element on the extension that belongs to any reversed critical pair. Let $b_{r,s}$ be the lowest element in any reversed critical pair. Since $\alpha_{1,1} > \beta_{r,s}$, by $(*)$ we conclude that $r = 1$ or $s = 1$.

Case 1: $r = s$. That is, $r = s = 1$. By $(*)$, we have $\beta_{p,q} > \alpha_{1,1} > \beta_{1,1} > \alpha_{i,j}$ when $1 \notin \{p, q, i, j\}$. Hence all reversed critical pairs use only elements in the first row or only in the first column of the grids. If some such pair uses neither $a_{1,1}$ nor $b_{1,1}$, then by symmetry we may assume that it is $(a_{1,j}, b_{1,q})$, with $\alpha_{1,j} > \beta_{1,q}$. Since $j \neq 1$ and $q \neq 1$, by $(*)$ we have $\beta_{p,1} > \alpha_{1,j} > \beta_{1,q} > \alpha_{i,1}$ whenever $p \neq 1$ and $i \neq 1$. Hence there are no reversed pairs in the first column that avoid the lower-left elements. By greediness, we can now reverse all remaining pairs, which are the horizontal and straight pairs in the first row and all the critical pairs using $a_{1,1}$ and $b_{1,1}$. The result is an extension of Type H (Type V arises symmetrically).

Case 2: $r \neq s$. By symmetry, we may assume that the lowest element on the extension that belongs to any reversed pair is $b_{1,2}$. Again $(*)$ restricts the possible reversed pairs, since $\alpha_{1,1} > \beta_{1,2}$. All elements of the top grid not in the first row or column are above all elements of the bottom grid not in the first row or second column, so additional reversed pairs are confined to the first row or to the first two columns.

In particular, putting $a_{i,j}$ over $b_{r,s}$ requires $i = 1$ or $j = 2$, (due to $b_{1,2}$) and $r = 1$ or $s = 1$ (due to $a_{1,1}$), and $i = r$ or $j = s$. With $i = r$, the possible reversals are $\alpha_{1,j} > \beta_{1,s}$ and $\alpha_{i,2} > \beta_{i,1}$. With $j = s$, the possibilities are $\alpha_{1,1} > \beta_{r,1}$ and $\alpha_{i,2} > \beta_{1,2}$.

If $\alpha_{i,2} > \beta_{i,1}$ with $i \neq 1$, then $\beta_{p,1} > \alpha_{i,2} > \beta_{i,1} > \alpha_{p,2}$ for $p \neq i$ (although $\beta_{p,1} < \alpha_{1,1}$ and $\alpha_{p,2} > \beta_{1,2}$ remain possible), so this can occur for only one value of i. It if occurs, then $\beta_{1,s} > \alpha_{i,2} > \beta_{i,1} > \alpha_{1,j}$ for $s \neq 2$ and $j \neq 1$. Hence we obtain only one additional reversed pair $(a_{i,j}, b_{r,s})$ with $i = r$. By greediness, we may have all of the remaining possibilities $\alpha_{1,1} > \beta_{r,1}$ and $\alpha_{i,2} > \beta_{1,2}$. The result is an extension of Type B. (The argument is symmetric when the lowest element of reversed pairs is $b_{2,1}$, again yielding Type B.)

Finally, if $\alpha_{i,2} < \beta_{i,1}$ whenever $i \neq 1$, then by greediness we may have $\alpha_{1,j} > \beta_{1,s}$ for all j and s, and $\alpha_{1,1} > \beta_{r,1}$ for all r, and $\alpha_{i,2} > \beta_{1,2}$ for all i. The result is an extension of Type H (it yields Type V if the lowest element in a reversed pair is in column 1). $\square$

5 k-dimension of $S_m \times S_n$

Theorem 5.1 *If $m, n \geq 3$, then* $\dim_k(S_m \times S_n) = m + n - \min\{2, k-2\}$.

Proof. For the upper bounds, we show that all critical pairs can be reversed using the desired number of k-extensions of types described in Lemmas 4.2, 4.3, and 4.4. For $k = 2$, we use an extension of Type V in each column and an extension of Type H in each row to reverse all critical pairs using $m + n$ extensions.

For $k = 3$, we use an extension of Type H in each row and an extension of Type V in columns 2 through n. In the Type H extension for row i, we use $\alpha_{i,1} = 2$ and $\beta_{i,1} = 0$ and put $\beta_{r,1} = 1$ for all r not equal to i, thereby reversing all the vertical and straight pairs using $a_{i,1}$. Since every critical pair contained in the first column uses $a_{i,1}$ for some i, these extensions of Type H together reverse all critical pairs contained in the first column. Hence the remaining critical pairs are reversed by Type V extensions in columns 2 through n.

For $k = 4$, we use an extension of Type H in each row and an extension of Type V in columns 3 through n. In the Type H extension for row i, we use $\alpha_{i,1} = 3$ and $\beta_{i,2} = 0$, thereby reversing all the vertical and straight pairs using $a_{i,1}$ and $b_{i,2}$. Since every critical pair contained in the first column uses $a_{i,1}$ for some i, and every critical pair contained in the second column uses $b_{i,2}$ for some i, these extensions of Type H together reverse all vertical and straight pairs contained in the first two columns. Hence the remaining critical pairs are reversed by Type V extensions in columns 3 through n.

For $k > 4$, the upper bound follows from the construction for $k = 4$, using extensions of the subposets induced by the sets X_0, X_1, X_2, X_3 on 4-extensions.

Now consider the lower bounds. For $k \in \{2, 3, 4\}$, Lemmas 4.2, 4.3, and 4.4 show that saturated k-extensions are of the types we have called V, H, B, C (or when $k = 3$ reverse strictly fewer critical pairs than one of these types). The numbers of critical pairs of each class that are reversed by each type of saturated k-extension appear in Table 1. When $k = 3$, the Type B_2 variation reverses one more straight pair and one less horizontal or vertical pair than the Type B_1 listed in the table.

Table 1

	$k=2$			$k=3$			$k=4$		
	vertical	horiz.	straight	vertical	horiz.	straight	vertical	horiz.	straight
V	m^2-m	0	m	m^2-m	$n-1$	m	m^2-m	$2n-2$	m
H	0	n^2-n	n	$m-1$	n^2-n	n	$2m-2$	n^2-n	n
B	$m-1$	$n-1$	1	$2m-2$	$2n-2$	1	$2m-2$	$2n-2$	2
C	2	2	0	m	n	1			

These values yield the desired lower bounds numerically. We minimize the total number of saturated extensions subject to obtaining enough reversed critical pairs in each class. The result is a linear program. To obtain a lower bound, it suffices to exhibit a feasible solution to the dual problem that has at least the desired value. Ad hoc argument with variables for the number of extensions of each type also works.

The dual, with variables v, h, s, is to maximize the objective $nm(m-1)v + mn(n-1)h + mns$ subject to a linear constraint from each row of Table 1. For example, the constraint when $k = 2$ for row V is $m(m-1)v + 0h + ms \leq 1$ (the constant 1 comes from the objective of minimizing the sum of the number of extensions).

When $k = 2$, we set $(v, h, s) = (\frac{1}{m(m-1)}, \frac{1}{n(n-1)}, 0)$ to obtain a feasible solution with objective value $m + n$. Hence $m + n$ such extensions are needed. Since $m, n \geq 3$, the constraints for Types B and C are not tight, corresponding to using only Types V and H in the upper bound construction.

When $k = 3$, we set $(v, h, s) = (\frac{n-1}{(mn-1)(m-1)}, \frac{m-1}{(mn-1)(n-1)}, 0)$ to obtain a feasible solution with objective value $(m+n-2)\frac{mn}{mn-1}$. Since $(m+n-2)\frac{mn}{mn-1} > m+n-2$ and the number of 3-extensions is an integer, we obtain $m + n - 1$ as a lower bound. Again the constraints for Types B_1, B_2, and C are not tight and only Types V and H are used in the construction.

When $k \geq 4$, we obtain the bound more directly. If $V + H + B < m + n - 2$, then $V \leq n - 2$ or $H \leq m - 2$. By symmetry, we may assume that $V = n - r$ with $r \geq 2$. Now reversing the remaining $m(m-1)r$ vertical pairs requires $H + B \geq rm/2 > m + r - 3$. Thus $V + H + B \geq n + m - 2$. □

6 2-dimension of t-fold Products

A different approach to 2-dimension avoids describing saturated extensions.

Definition 6.1 *Critical pairs* (x, y) *and* (x', y') *are* k-conflicted *if every* k*-extension reverses at most one of them. A set* S *of critical pairs is* k-conflicted *if every two critical pairs in* S *are* k*-conflicted.*

Remark 6.2 *If* S *is a* k*-conflicted set of critical pairs in a poset* P*, then* $\dim_k(P) \geq |S|$.

Lemma 6.3 *In a poset* P*, critical pairs* (x, y) *and* (x', y') *are 2-conflicted if* $x < y'$ *or* $x' < y$*. If* $k \geq 3$*, then* (x, y) *and* (x', y') *are* k*-conflicted if* $x < y'$ *and* $x' < y$*.*

Proof. If $x < y'$, then reversing (x, y) and (x', y') in a k-extension f requires $f(y) < f(x) \leq f(y') < f(x')$. This uses three distinct values in f, so it cannot occur when $k = 2$. If also $x' < y$, then $f(x') \leq f(y)$ is needed, and this cannot be achieved for any k. □

In dimension theory for posets, the condition $x < y'$ and $x' < y$ when $k \geq 3$ makes the pair $\{(x, y), (x', y')\}$ what is known as an "alternating cycle" of size 2. The fact that a weaker condition suffices for pairs to be 2-conflicted is the reason why this notion is effective in proving lower bounds for 2-dimension. In order to motivate the argument for higher products, we first give a short explanation of it for $S_m \times S_n$.

Example 6.4 We show that if $m, n \geq 3$, then $\dim_2(S_m \times S_n) = m + n$. It suffices to present a 2-conflicted set of size $m + n$. Note that $(a_{i,j}, b_{r,s})$ with $i = r$ or $j = s$ and $(a_{g,h}, b_{p,q})$ with $g = p$ or $h = q$ are 2-conflicted if $a_{i,j} < b_{p,q}$ or $a_{g,h} < b_{r,s}$; that is, if ($i \neq p$ and $j \neq q$) or ($g \neq r$ and $h \neq s$).

We represent the order relations in one grid using arrows: $(i, j) \to (r, s)$ represents the pair $(a_{i,j}, b_{r,s})$. The pair is incomparable if the arrow lies within one row or within one column. Two incomparable pairs are 2-conflicted if, as arrows, the tail of one is in different row and column from the head of the other. Figure 1 illustrates a 2-conflicted set of arrows. The size of the set is $6 + (m - 3) + (n - 3)$, as desired. □

Let $P = \prod_{i=1}^{t} S_{n_i}$. Let A be the set of t-tuples whose ith coordinate is minimal in S_{n_i} for all i, and let B be the set of t-tuples whose ith coordinate is maximal in S_{n_i} for all i. Since products of nontrivial bipartite posets are nontrivial, it follows by induction on t that $\dim_k P = \dim_k R$, where R is the subposet of P with elements $A \cup B$. Represent each pair $(a_{i_1,\ldots,i_t}, b_{r_1,\ldots,r_t})$ in $A \times B$ by an arrow from $(i_1, \ldots, i_t)$ to $(r_1, \ldots, r_t)$ in a t-dimensional grid. The pair is a critical pair if the head and tail agree in some coordinate. Two pairs are 2-conflicted if the tail of one differs from the head of the other in every coordinate.

As motivated by Example 6.4, the idea of the proof for $t \geq 3$ is to first study $\prod_{i=1}^{t} S_t$, presenting a cycle C_t of length t^2 in the t-dimensional grid, with the edge set forming a

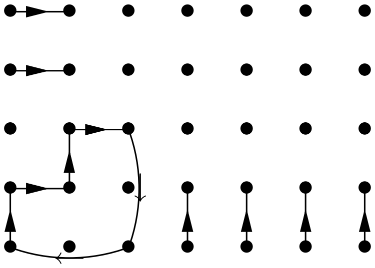

Figure 1: A 2-conflicted set for $S_5 \times S_7$

2-conflicted set. This yields $\dim_2(\prod_{i=1}^{t} S_t) = t \cdot t$. For $\prod_{i=1}^{t} S_{n_i}$ with each $n_i \geq t$, we then add $n_i - t$ arrows for extra hyperplanes in the ith direction, copying arrows on C_t. This will yield $\dim_2(\prod_{i=1}^{t} S_t) = \sum_{i=1}^{t} n_i$.

Definition 6.5 *The t-lagging cycle is the cycle C_t consisting of vectors $v_1, \ldots, v_{t^2}$ in $[t]^t$ formed by letting $v_1 = (1, \ldots, 1)$ and, for $j \geq 1$, forming v_{j+1} from v_j by adding 1 (modulo t) to each coordinate except the ith, where $i \equiv j \mod t$.*

For example, C_4 is (reading down columns):

$$\begin{array}{cccc} (1,1,1,1) & (4,4,4,4) & (3,3,3,3) & (2,2,2,2) \\ (1,2,2,2) & (4,1,1,1) & (3,4,4,4) & (2,3,3,3) \\ (2,2,3,3) & (1,1,2,2) & (4,4,1,1) & (3,3,4,4) \\ (3,3,3,4) & (2,2,2,3) & (1,1,1,2) & (4,4,4,1) \end{array}$$

Proposition 6.6 *The t-lagging cycle C_t has the following properties:*
(1) In t^2 steps of the t-lagging cycle, each entry augments $t^2 - t$ times, and C_t is a cycle.
(2) Each vector in the cycle has the form $(q-1, \ldots, q-1, q, \ldots, q)$ for some q (modulo t), every such vector arises, and each v_j agrees with v_{j+1} in the position of the first q in v_j.
(3) Each arrow $v_j \to v_{j+1}$ for successive elements of the cycle corresponds to an incomparable pair $(a_{v_j}, b_{v_{j+1}})$.

Proof. (1) Since the augmentation position cycles, the positions augment equally often. There are t^2 occasions of a position not augmenting, so each augments all but t times. Since the amount by which each position increases is a multiple of t, after t^2 steps the walk returns to the original vector. To make C_t a cycle, we need distinctness of the vertices.

(2) There are t^2 vectors of the specified form; we show that all arise. If v_j has the specified form and the position of the first q is congruent to j modulo t (which is true for $j = 1$), then those statements remain true for $j + 1$, by construction. Also the value of q increases,

except when $j \equiv t \mod t$. Therefore, letting i be the position of the first q, the pairs (q, i) determining the vector cycle through $(1, 1), (2, 2) \ldots, (t, t)$, then $(t, 1), (1, 2) \ldots, (t-1, t)$, then $(t-1, 1), (t, 2), \ldots, (t-2, t)$, and so on until $(2, 1), (3, 2), \ldots, (1, t)$ (see the example of C_4 above). Thus C_t visits all t^2 vectors of this form and is a cycle.

(3) The arrow from v_j to v_{j+1} represents the pair $(a_{v_j}, b_{v_{j+1}})$. Since their subscripts agree in coordinate i, where $i \equiv j \mod t$, they form an incomparable pair. $\square$

Lemma 6.7 *The arrows in C_t represent a set of t^2 2-conflicted pairs in $\prod_{i=1}^{t} S_t$.*

Proof. Consider distinct pairs $(a_{v_i}, b_{v_{i+1}})$ and $(a_{v_{j-1}}, b_{v_j})$; that is, with $j \neq i+1$. By Lemma 6.3, it suffices to that if a_{v_i} and b_{v_j} are incomparable, then $a_{v_{j-1}} < b_{v_{i+1}}$.

If $a_{v_i} \| b_{v_j}$, then v_i and v_j share some value q in some position. The common value q may be the "higher" (ending) value or the "lower" (beginning) value in each t-tuple.

If q is the beginning value in v_i, then the values in v_{i+1} are all at least $q+1$ (except that the first is q when $i \equiv 1 \mod t$). The corresponding values in v_{j-1} are different, since they are all at most q (and the first is $q-1$ unless $j = i+1$). Hence $a_{v_{j-1}} < b_{v_{i+1}}$. The case where the shared value is the last in v_j follows symmetrically.

The remaining case is where r is the position of the first q in v_i, s is the position of the last q in v_j, and $1 < r \leq s$, as shown below. Now r is the position of the last q in v_{i+1}, and s is the position of the first q in v_{j-1}. If $a_{v_{j-1}} \| b_{v_{i+1}}$ then $s \leq r$. Now $r = s$, and hence $j = i+1$. $\square$

$$\begin{array}{ccccc}
 & & v_i & \longrightarrow & v_{i+1} \\
q-2\ldots q-1 & & q-1\ \ldots\ q & & q\ \ \ldots q+1 \\
q-1\ldots\ \ q & & q\ \ \ldots\ q+1 & & q+1\ldots q+2 \\
v_{j-1} & \longrightarrow & v_j & &
\end{array}$$

Theorem 6.8 *If $n_1, \ldots, n_t$ are all at least t, then* $\dim_2 \prod_{i=1}^{t} S_{n_i} = \sum_{i=1}^{t} n_i$.

Proof. For $t = 2$, the claim was proved earlier, so assume $t \geq 3$.

By Lemma 6.7, the claim holds when $n_i = t$ for $1 \leq i \leq t$. When $\prod_{i=1}^{t} n_i > t^t$, begin with C_t on the subgrid $[t]^t$. To this set of arrows, add the arrow from $u(i, j)$ to $u'(i, j)$ for $1 \leq i \leq t$ and $t < j \leq n_i$, where $u(i, j)$ has j in position i and 1 everywhere else, while $u'(i, j)$ has j in position i and 2 everywhere else. Since $u(i, j)$ and $u'(i, j)$ agree in position i, the arrow represents an incomparable pair.

We claim that the resulting set is 2-conflicted. It is immediate that $u(i, j)$ and $u'(i, j')$ differ in all coordinates when $j \neq j'$, as do $u(i, j)$ and $u'(i', j')$ when $i \neq i'$, the latter since

$t \geq 3$. Hence it suffices to compare $u(i,j)$ and $u'(i,j)$ with v_p and v_{p+1} from C_t. If v_p and $u'(i,j)$ agree somewhere, then v_p begins or ends with 2. Now v_{p+1} has no 1 and no j, so it agrees with $u(i,j)$ in no coordinate. □

The same idea gives a weaker lower bound for larger k. We do not know how close it is to optimal. For $k \geq 4$, concatenating k-realizers of 2-fold products of standard examples yields $\epsilon + \sum_{i=1}^{t}(n_i - 1)$ as a trivial upper bound, where ϵ is 1 for odd t and 0 for even t.

Theorem 6.9 *If* $n_1, \ldots, n_t$ *are integers, all at least* 2, *then* $\dim_k(\prod_{i=1}^{t} S_{n_i}) \geq \sum_{i=1}^{t}(n_i - 2)$.

Proof. We produce a k-conflicted set of this size. Considering the grid $\prod_{i=1}^{t}[n]^t$, we form a set of $\sum_{i=1}^{t}(n_i - 2)$ arrows. For $1 \leq i \leq t$ and $3 \leq j \leq n_i$, add the arrow from $u(i,j)$ to $u'(i,j)$, where $u(i,j)$ has j in position i and 1 everywhere else, while $u'(i,j)$ has j in position i and 2 everywhere else.

We observed in the proof of Theorem 6.8 that $(a_{u(i,j)}, b_{u'(i,j)})$ is an incomparable pair and that $a_{u(i,j)} < b_{u'(i',j')}$ unless $i = i'$ and $j = j'$. That is, if (x,y) and (x',y') are two of these incomparable pairs, then $x < y'$ and $x' < y$. By Lemma 6.3, the set is k-conflicted. □

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