# Cycle Spectra of Hamiltonian Graphs

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**Abstract**

We prove that every Hamiltonian graph with $n$ vertices and $m$ edges has cycles of at least $\sqrt{\frac{4}{7}(m-n)}$ different lengths. The coefficient 4/7 cannot be increased above 1, since when $m = n^2/4$ there are $\sqrt{m-n+1}$ cycle lengths in $K_{n/2,n/2}$. For general $m$ and $n$ there are examples having at most $2\left\lceil\sqrt{2(m-n+1)}\right\rceil$ different cycle lengths.

**Keywords:** cycle, cycle spectrum, Hamiltonian graph, Hamiltonian cycle.

## 1 Introduction

The *cycle spectrum* of a graph $G$ is the set of lengths of cycles in $G$. A cycle containing all vertices of $G$ is a *spanning* or *Hamiltonian cycle*, and a graph having such a cycle is a *Hamiltonian graph*. An $n$-vertex graph is *pancyclic* if its cycle spectrum is $\{3, \ldots, n\}$. All our graphs have no loops or multiple edges. Let $d_G(x)$ denote the degree in $G$ of a vertex $x$ (its number of neighbors). A graph is $k$-regular if every vertex has degree $k$.

Interest in cycle spectra arose from Bondy's "Metaconjecture" (based on [3]) that sufficient conditions for existence of Hamiltonian cycles usually also imply pancyclicity, with possibly a small family of exceptions. For example, Bondy [3] showed that the sufficient condition on $n$-vertex graphs due to Ore [16] ($d_G(x) + d_G(y) \geq n$ whenever $x$ and $y$ are nonadjacent vertices) implies also that $G$ is pancyclic or is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Schmeichel and Hakimi [13] showed that if a spanning cycle in an $n$-vertex graph $G$

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has consecutive vertices with degree-sum at least $n$, then $G$ is pancyclic or bipartite or lacks only $n-1$ from the spectrum, with the latter cases occurring only when the degree-sum is exactly $n$. Bauer and Schmeichel [1] used this to give unified proofs that the conditions for Hamiltonian cycles due to Bondy [4], Chvátal [5], and Fan [8] also imply pancyclicity, with a small family of exceptions. Further results about the cycle spectrum under degree conditions on selected vertices in a spanning cycle appear in [10] and [14].

At the 1999 conference "Paul Erdős and His Mathematics", Jacobson and Lehel proposed the opposite question: *When sufficient conditions for spanning cycles are relaxed, how small can the cycle spectrum be if the graph is required to be Hamiltonian?* For example, consider regular graphs. Bondy's result [3] implies that $\lceil n/2 \rceil$-regular graphs other than $K_{\frac{n}{2},\frac{n}{2}}$ are both Hamiltonian and pancyclic. On the other hand, 2-regular Hamiltonian graphs have only one cycle length. For $3 \le k \le \lceil n/2 \rceil - 1$, Jacobson and Lehel asked for the minimum size of the cycle spectrum of a $k$-regular $n$-vertex Hamiltonian graph, particularly when $k = 3$.

Let $s(G)$ be the size of the cycle spectrum of a graph $G$. At the SIAM Meeting on Discrete Mathematics in 2002, Jacobson announced that he, Gould, and Pfender had proved $s(G) \ge c_k n^{1/2}$ for $k$-regular graphs with $n$ vertices. Others later independently obtained similar bounds, without seeking to optimize $c_k$. For an upper bound, Jacobson and Lehel constructed the 3-regular example below with only $n/6+3$ distinct cycle lengths (when $n \equiv 0 \bmod 6$ and $n > 6$), and they generalized it to the upper bound $\frac{n}{2}\frac{k-2}{k} + k$ for $k$-regular graphs.

**Example 1** When $k = 3$ and 6 divides $n$, take $n/6$ disjoint copies of $K_{3,3}$ in a cyclic order, with vertex sets $V_1, \ldots, V_{n/6}$. Remove one edge from each copy and replace it by an edge to the next copy to restore 3-regularity. A cycle of length different from 4 or 6 must visit each $V_i$, and in each $V_i$ it uses 4 or 6 vertices. Hence the cycle lengths are 4, 6, and each even integer from $2n/3$ through $n$. For the generalization, use $K_{k,k}$ instead of $K_{3,3}$. □

A related problem is the conjecture of Erdős [6] that $s(G) \ge \Omega\left(d^{\lfloor (g-1)/2 \rfloor}\right)$ when $G$ has girth $g$ and average degree $d$. Erdős, Faudree, Rousseau, and Schelp [7] proved the conjecture for $g = 5$. Sudakov and Verstraëte [15] proved the full conjecture in a stronger form, obtaining $\frac{1}{8}\left(d^{\lfloor (g-1)/2 \rfloor}\right)$ consecutive even integers in the cycle spectrum for graphs with fixed girth $g$ and average degree $48(d+1)$. Gould, Haxell, and Scott [11] proved a similar result: for $c > 0$, there is a constant $k_c$ such that for sufficiently large $n$, the cycle spectrum of every $n$-vertex graph $G$ having minimum degree $cn$ and longest even cycle length $2l$ contains all even integers from 4 up to $2l - k_c$ (see also [2]).

Prior arguments for lower bounds on $s(G)$ when $G$ is regular and Hamiltonian used only the number of edges, $m$, not regularity. The complete bipartite graph $K_{n/2,n/2}$ shows that the coefficient $c$ in a lower bound of the form $\sqrt{c(m-n)}$ cannot exceed 1. We give constructions for general $m$ and $n$ when $m$ is above or below $n^2/4$; they are far from regular.

**Example 2** For $m \le n^2/4$, consider the graph $G$ formed by replacing one edge of $K_{t,t}$ with a path having $n-2t$ internal vertices; there are $n$ vertices and $t^2-2t+n$ edges. The cycle spectrum of $G$ consists of the $t-1$ even numbers $\{4,\dots,2t\}$ and the $t-1$ numbers from $n-2t+4$ to $n$ having the same parity as $n$. Letting $m$ be the number of edges, this construction yields $s(G) \le 2(t-1) = 2\sqrt{m-n+1}$.

Deleting edges cannot enlarge the cycle spectrum. Hence when we specifiy $m$ as the number of edges, we can let $m'$ be the next larger value such that $m'-n+1$ is a square and apply the construction above for $m'$ edges to obtain an upper bound. After discarding $m'-m$ edges, we obtain $s(G) \le 2\left\lceil\sqrt{m-n+1}\right\rceil$.

The two parts of the spectrum remain separate when $n > 4(t-1)$, which holds for $m < n^2/16+n-1$. When $n$ is even and $m \ge n^2/16+n$, the two parts overlap; in fact, $s(G) = \sqrt{m-n+1}$ when $m = n^2/4$.

For $m > n^2/4$, consider the graph $G$ formed by replacing one edge of $K_t$ with a path having $n-t$ internal vertices; there are $n$ vertices and $t(t-3)/2+n$ edges. The cycle spectrum of $G$ consists of the $t-2$ numbers $\{3,\dots,t\}$ and the $t-2$ numbers $\{n-t+3,\dots,n\}$. In terms of the number of edges, $m$, we have $t = \sqrt{2(m-n)+9/4}+3/2$. Thus $s(G) \le 2(t-2) < 2\sqrt{2}\sqrt{m-n+1}$. The coefficient declines from $2\sqrt{2}$ toward $\sqrt{2}$ as $m$ increases toward $\binom{n}{2}$. The worst case for the construction, in terms of $m-n$, occurs when $m = n^2/4+1$. Again we can interpolate for other values of $m$ by inserting a ceiling function on the expression for $t$ in terms of $m$ and $n$, obtaining a general construction with $s(G) \le 2\left\lceil\sqrt{2(m-n+1)}\right\rceil$. □

Our main result is that $s(G) \ge \sqrt{\frac{4}{7}(m-n)}$ when $G$ is an $n$-vertex Hamiltonian graph with $m$ edges. A crucial tool is a lemma of Faudree, Flandrin, Jacobson, Lehel, and Schelp [9, Lemma 3]. We need a stronger version than their proof yields; we obtain this in Section 2. Section 3 applies it to $s(G)$, and Section 4 further pursues the problem from Section 2.

# 2 Chords of a Spanning Path

A path with endpoints $x$ and $y$ is an *$x,y$-path*. A *chord* of a path (or cycle) $P$ in a graph is an edge of the graph not in $P$ whose endpoints are in $P$, and the *length* of the chord is the distance in $P$ between its endpoints. Throughout this section, and in the hypotheses of its results, we let the graph $G$ consist of the spanning $x,y$-path $P$ plus $q$ chords of the same length $l$, and we let $r$ be the number of different lengths of $x,y$-paths in $G$. The vertices of $P$ are $v_1,\dots,v_n$ in order, with $v_1 = x$ and $v_n = y$. As defined above, the *length* of a chord $v_iv_j$ of $P$ is $|j-i|$. Two chords $v_av_c$ and $v_bv_d$ *overlap* if $a < b < c < d$. When $v_a$ and $v_b$ are vertices of $P$, we use $P[v_a,v_b]$ to denote the $v_a,v_b$-path contained in $P$.

Lemma 3 in [9] claims that in this setting, always $r \ge q/3+1$. However, the argument in [9] produces only $q/6+1$ path lengths in the following example.

**Example 3** Let $G$ be obtained from the 3-regular graph in Example 1 by deleting one edge that lies in no 6-cycle. Here $n = 6k = 2q$, and the endpoints $x$ and $y$ of the spanning path $P$ are the endpoints of the deleted edge. The graph is bipartite, so all $x,y$-paths have odd length. All $q$ chords have length 3, and they group into $k$ sets of three overlapping chords. Each such set occupies six consecutive vertices along $P$, and an $x,y$-path visits exactly four or six vertices in each such group. Hence the lengths of $x,y$-paths are all odd numbers from $n-1$ down to $2n/3-1$; there are $q/3+1$ of them. The argument in [9] discards half of these groups of three chords on six vertices and thus guarantees only $q/6+1$ path lengths. □

Theorem 6 below will provide a lower bound on $r$ that is always at least as large as $q/3+1$. The graph in Example 3 demonstrates sharpness.

**Lemma 4** *If the chords other than the one nearest $v_n$ pairwise overlap, then $r \ge q-1$, with equality possible only when $l$ is odd.*

**Proof.** Let the chords be $e_1, \ldots, e_q$, indexed in order of the indices of their lower endpoints among $v_1, \ldots, v_n$ ($e_q$ is nearest to $v_n$. Suppose first that $e_q$ overlaps $e_1$. For $2 \le j \le q$, let $P_j$ be the $v_1, v_n$-path using the chords $e_1$ and $e_j$ and no other chords. There is exactly one such path; if $e_1 = v_a v_c$ and $e_j = v_b v_d$, then $P_j$ contains $P[v_1, v_a]$, $P[v_b, v_c]$, and $P[v_d, v_n]$. The length of $P_j$ is $n+1-(b-a)-(d-c)$. As $j$ increases, $b$ and $d$ increase, so $P_2, \ldots, P_q$ have distinct lengths. Furthermore, the $v_1, v_n$-path $Q$ that contains $e_1$ and no other chord has length $n-l$. Since $P_2, \ldots, P_q$ have distinct lengths, $r \ge q-1$. If $l$ is even, then the length of $Q$ has opposite parity from the lengths of $P_2, \ldots, P_q$ (since $(b-a)+(d-c)$ is even), and hence $r \ge q$.

Now suppose that $e_q$ does not overlap $e_1$. Let $P'$ be the $v_1, v_n$-path using $e_1$ and $e_q$ and no other chords; it has length $n+1-2l$. Furthermore, this path is shorter than any of the paths $P_2, \ldots, P_{q-1}$ or $Q$ constructed as in the previous case for $G - e_q$. □

**Proposition 5** *Let $G$ be a graph with a distinguished spanning $x,y$-path $P$, let $Q_1, \ldots, Q_t$ be pairwise edge-disjoint subpaths of $P$, and let $H_j$ be the subgraph of $G$ induced by $V(Q_j)$. If $H_j$ has $r_j$ lengths of paths joining the endpoints of $Q_j$, then $G$ has $x,y$-paths with at least $1 + \sum_{j=1}^{t}(r_j - 1)$ different lengths.*

**Proof.** Starting with a $v_1, v_n$-path that uses $Q_1, \ldots, Q_t$ (actually, this is $P$), one can iteratively shorten the path $\sum_{j=1}^{t}(r_j - 1)$ times. □

**Theorem 6** *If $G$ is a graph consisting of a spanning path $P$ with vertices $v_1, \dots, v_n$ and $q$ chords of length $l$, then the number $r$ of $v_1, v_n$-paths in $G$ is at least*

$$\max\left\{\frac{q}{2} - \frac{n-1}{2l} + 1, \frac{q}{3} + 1\right\}.$$

*Moreover, if $l$ is even, then there are at least $q/2 + 1$ such paths.*

**Proof.** Choose chords $e_1, \dots, e_k$ as follows. Let $e_1$ be the chord with lowest-indexed endpoint. Having chosen $e_1, \dots, e_{j-1}$, let $e_j$ be the chord with lowest endpoint that overlaps none of $e_1, \dots, e_{j-1}$; do this until no further chord can be added. Note that every chord of $G$ coincides with or overlaps at least one of the chosen chords.

Let $z_j$ and $z'_j$ be the lower and upper endpoints of chord $e_j$, respectively. Let $z_0 = v_1$ and $z'_{k+1} = v_n$. For $0 \le j \le k$, let $Q_j = P[z_j, z'_{j+1}]$, let $H_j$ be the subgraph of $G$ induced by $V(Q_j)$, and let $r_j$ be the number of $z_j, z'_{j+1}$-paths in $H_j$. Note that $Q_j$ is a spanning path in $H_j$. Since $e_j$ is a chord in $H_j$ and $H_{j-1}$ for $1 \le j \le k$ and every other chord of $G$ belongs to exactly one of these subgraphs, $\sum_{j=0}^{k} q_j = q + k$, where $q_j$ is the number of chords in $H_j$. Each $H_j$ has the form discussed in Lemma 4; hence $r_j \ge q_j - 1$ for $1 \le j \le k$. For $H_0$ we need the better value $r_0 = q_0 + 1 = 2$ (there is only one chord).

The odd-indexed graphs in $H_0, \dots, H_k$ are pairwise disjoint, as are the even-indexed graphs. By applying Proposition 5 separately to the even and odd pieces and summing the resulting two inequalities, we obtain

$$2r \;\ge\; 2 + \sum_{j=0}^{k}(r_j - 1) \;\ge\; 2 + q_0 + \sum_{j=1}^{k}(q_j - 2) \;=\; 2 + q - k,$$

and thus $r \ge (q-k)/2 + 1$. Since $e_1, \dots, e_k$ are pairwise non-overlapping, $n - 1 \ge kl$, so

$$r \ge \frac{q}{2} - \frac{n-1}{2l} + 1. \tag{1}$$

Furthermore, the chords $e_1, \dots, e_k$ by themselves yield $r \ge 1 + k$, and hence

$$r \ge \max\left\{1 + k, \frac{q-k}{2} + 1\right\}.$$

Optimizing $k$ yields $r \ge q/3 + +11$.

If $l$ is even, then Lemma 4 yields $r_i \ge c_i$ for $1 \le i \le k$, and hence

$$2r \;\ge\; 2 + \sum_{i=0}^{k}(r_i - 1) \;\ge\; 2 + c_0 + \sum_{i=1}^{k}(c_i - 1) \;=\; 2 + q.$$

Thus $r \ge q/2 + 1$ in this case. □

Example 3 shows that the inequality $r \geq q/3 + 1$ is sharp; the example having $n - 3$ chords of length 3 also achieves equality. Similarly, a path with $q+2$ vertices having $q$ chords of length 2 shows that $r \geq q/2 + 1$ is best possible when $l$ is even.

**Corollary 7** *In the setting of Theorem 6, if $l \leq n/2$, then*

$$r \geq \frac{q}{3}\left(1 + \frac{l}{n}\right).$$

**Proof.** By using (1) to refine the second bound in Theorem 6, we have $r \geq \max\{f_1(q), f_2(q)\}$, where $f_1(x) = \frac{x}{2} - \frac{n}{2l} + 1$ and $f_2(x) = \frac{x}{3} + 1$.

Note that $f_1$ and $f_2$ are linear functions of $x$ that intersect at a point $(x_0, y_0) = \left(\frac{3n}{l}, \frac{n}{l} + 1\right)$. Since $f_1(0) < 0 < f_2(0)$, the line $y = \frac{x}{3}\left(1 + \frac{l}{n}\right)$ that passes through $(0,0)$ and $(x_0, y_0)$ provides a uniform lower bound on $\max\{f_1(x), f_2(x)\}$. $\square$

# 3 Cycle Lengths in Hamiltonian Graphs

In this section, $G$ is a graph $G$ with $n$ vertices, $m$ edges, and a distinguished Hamiltonian cycle $C$. A *chord of length $l$* is an edge $uv$ in $E(G) - E(C)$ such that $d_C(u,v) = l$. The length of any chord is at least 2 and at most $\lfloor n/2 \rfloor$. Let the *normalized length* of a chord $uv$ be $\frac{d_C(u,v)}{n/2}$. Two chords $uv$ and $xy$ *cross* if their endpoints are ordered $u, x, v, y$ along $C$.

The next two lemmas give lower bounds on $s(G)$ (the size of the cycle spectrum of $G$). The first is strong when the average length of the chords is large, and the second is strong when the average length is small. The two bounds together imply our main result.

Let $\omega(G)$ and $\alpha(G)$ denote the *clique number* and *independence number* of a graph $G$; these are the maximum sizes of sets of pairwise adjacent or pairwise nonadjacent vertices, respectively. A graph $H$ is *perfect* if for every induced subgraph $H'$ the vertices can be partitioned into $\omega(H')$ independent sets (and hence $\omega(H)\alpha(H) \geq |V(H)|$).

**Lemma 8** *Let $G$ be an $n$-vertex graph having $m$ edges and a Hamiltonian cycle $C$. If the average normalized length of the chords of $C$ is $\beta$, then $s(G) \geq \sqrt{\beta(m-n)}$.*

**Proof.** We seek a large set of chords in one of two special configurations. If $C$ has $q$ pairwise noncrossing chords, then $s(G) \geq q + 1$ (starting with $C$, we can iteratively obtain a shorter cycle $q$ times). If $C$ has $q$ pairwise crossing chords, then $s(G) \geq q - 1$ (starting with a short cycle using two "closest" among the $q$ crossing chords, we can iteratively obtain a longer cycle $q - 2$ times by replacing one of them).

To obtain a large set of pairwise noncrossing chords or a large set of pairwise crossing chords, we seek a large set of chords crossing a single diameter. With $C$ drawn on a circle, let $S$ be the vertex set of a path along $C$, with $\overline{S} = V(G) - S$. Let $p$ be the number of chords of $C$ having endpoints in $S$ and $\overline{S}$. Let $H$ be the graph whose vertex set is this set of $p$ chords, with vertices being adjacent when the chords cross. It is well known ([12], for example) that a graph generated in this way is a perfect graph (specifically, a "permutation graph"). Since $H$ is perfect, $\omega(H)\alpha(H) \geq p$. Hence $\omega(H) \geq \sqrt{p}+1$ or $\alpha(H) \geq \sqrt{p}-1$.

Choose a random set $S$ of $\lfloor n/2 \rfloor$ consecutive vertices along $C$, with the $n$ such sets being equally likely. The probability that a chord of length $l$ has exactly one endpoint in $S$ is exactly $\frac{2l}{n}$, which equals the normalized length of the chord. The expected number of chords with one endpoint in $S$ is thus the sum of the normalized lengths, which equals $\beta(m-n)$. For some choice of $S$, there are at least $\beta(m-n)$ such chords. As argued above, there are thus at least $\sqrt{\beta(m-n)}$ distinct cycle lengths in $G$. □

**Lemma 9** *Let $G$ be an $n$-vertex graph having $m$ edges and a Hamiltonian cycle $C$. If the average normalized length of the chords of $C$ is $\beta$, then*

$$s(G) \geq \sqrt{\frac{2}{3}\left(1-\frac{\beta}{4}\right)(m-n)}.$$

**Proof.** Every chord of length $l$ completes with $C$ a cycle of length $l+1$ and a cycle of length $n-l+1$; these are both shorter than $C$, and they have the same length if and only if $l = n/2$. Letting $t$ be the number of different lengths of chords of $C$, we find that $C$ together with such cycles yields $s(G) \geq 2t$.

When $t$ is small, many chords have equal length. Let $w(l) = (1+l/n)/3$. To make use of the extra factor $w(l)$ in Corollary 7, assign each chord of $C$ with length $l$ the weight $w(l)$. For an edge $uv$ of $C$ chosen uniformly at random, let $P = C - uv$; we treat $P$ as a distinguished Hamiltonian path of $G$. A chord of $C$ is also a chord of $P$; let the *$P$-length* of a chord $xy$ of $C$ be $d_P(x,y)$. For a chord of length $l$, the $P$-length is equal to $l$ with probability $1-l/n$. Let $W$ be the expectation (over the choice of $uv$) of the total weight of all chords whose length and $P$-length coincide. Letting $a_l$ be the number of chords with length $l$, the expected number of chords of length $l$ that contribute to $W$ is $a_l(1-l/n)$. Thus

$$\begin{aligned} W &= \sum_{l\geq 2}\frac{1}{3}\left(1+\frac{l}{n}\right)a_l\left(1-\frac{l}{n}\right) = \frac{1}{3}\sum_{l\geq 2}a_l\left(1-\frac{l^2}{n^2}\right) \\ &= \frac{1}{3}\left(\sum_{l\geq 2}a_l-\frac{1}{4}\sum_{l\geq 2}a_l\left(\frac{l}{n/2}\right)^2\right) \geq \frac{1}{3}\left((m-n)-\frac{1}{4}\sum_{l\geq 2}a_l\frac{l}{n/2}\right) \\ &= \frac{1}{3}\left((m-n)-\frac{1}{4}\beta(m-n)\right) = \frac{1}{3}(m-n)\left(1-\frac{\beta}{4}\right). \end{aligned}$$

For some choice of $uv$ along $C$, the actual total weight of the chords whose length and $P$-length coincide is at least $W$. With $t$ different chord lengths, some particular length contributes at least $W/t$ to this total. Let $l$ be this length. We have $a_l \ge W/(tw(l))$, and hence by Corollary 7 the chords of this length contribute at least $W/t$ lengths of cycles.

We now have

$$s(G) \ge \max\left\{2t, \frac{(m-n)}{3t}\left(1-\frac{\beta}{4}\right)\right\} \ge \sqrt{\frac{2}{3}\left(1-\frac{\beta}{4}\right)(m-n)},$$

where the final inequality chooses $t$ to minimize the maximum. □

**Theorem 10** *If $G$ is a $n$-vertex Hamiltonian graph with $m$ edges, then $s(G) \ge \sqrt{\frac{4}{7}(m-n)}$. Furthermore, there is such a graph $G$ with $s(G) \le 2\left\lceil\sqrt{2(m-n+1)}\right\rceil$, and when $m = n^2/4$ there is such a graph with $s(G) = \sqrt{m-n+1}$.*

**Proof.** By Lemmas 8 and 9, $s(G) \ge \left[(m-n)\max\{\beta, \frac{2}{3}(1-\frac{\beta}{4})\}\right]^{1/2}$. Choosing $\beta = 4/7$ minimizes the larger lower bound. Example 2 provides the construction. □

# 4 Spanning Paths with Long Chords

In this section, we improve the result of Section 2 when $l$ is small compared to $q$. Again we maintain the setting of an $n$-vertex graph $G$ having a distinguished spanning $x,y$-path $P$ plus $q$ chords of length $l$, and $r$ denotes the number of distinct lengths of $x,y$-paths.

Say that a chord $uv$ is *independent* of a segment $P[a,b]$ if $P[u,v]$ shares no edges with $P[a,b]$, and $uv$ *interferes* with $P[a,b]$ otherwise. More generally, a chord $uv$ is independent of any subgraph of $P$ that shares no edges with $P[u,v]$. Our strategy will use the following proposition to generate many path lengths.

**Proposition 11** *Let $F$ be a subgraph of $P$ such that there are $x,y$-paths having $t$ different lengths modulo $l-1$ and each containing all edges of $P$ not in $F$. If there are $s$ pairwise non-overlapping chords that are all independent of $F$, then $r \ge t(s+1)$.*

**Proof.** Each path in the initial set having $t$ different lengths modulo $l-1$ contains the path $P[u,v]$ for each chord $uv$ that is independent of $F$. By using the chord $uv$, we can shorten the path by $l-1$ edges. Since the $s$ chords are pairwise non-overlapping, for each of the original paths we can do this with each of the chords successively to obtain $t(s+1)$ paths of different lengths. □

For some highly structured graphs, the general bound of Theorem 14 below does not hold. The *chord overlap graph* of $G$, denoted $C(G)$, is the graph whose vertices are the chords of $P$ and whose adjacency relation is overlap. Since the chords all have the same length, overlap is equivalent to intersection of segments along the path; hence $C(G)$ is an interval graph and is perfect. Say that $G$ is *exceptional* if $C(G)$ has maximum degree at most 1 and each edge in $C(G)$ corresponds to a pair of chords of $P$ whose lower endpoints are adjacent along $P$. If $G$ is exceptional, with $p$ pairs of overlapping chords, then there are exactly $q-p+1$ distinct lengths of paths in $G$ joining the endpoints of $P$; that is, $r \geq q/2+1$. Henceforth we study only nonexceptional graphs.

**Proposition 12** *Let $t$ be the clique number of $C(G)$. If $Q$ is a minimal subpath of $P$ such that $Q$ contains the endpoints of $t$ pairwise overlapping chords, then at most $3t-2$ chords interfere with $Q$.*

**Proof.** If a chord interferes with $Q$, then either both endpoints are contained in $Q$ or one endpoint is in the interior of $Q$ and one is outside $Q$. Exactly $t$ chords have both endpoints in $Q$. The chords that interfere with $Q$ and have their lower endpoint below $Q$ form a clique with the chord whose lower endpoint is at the beginning of $Q$; hence there are at most $t-1$ of them. Similarly, at most $t-1$ chords have their lower endpoint in the interior of $Q$ and their upper endpoint outside $Q$. □

**Lemma 13** *If $G$ is nonexceptional, $l \geq 3$, and $C(G)$ contains a triangle-free subgraph with $q'$ vertices, then $r \geq q'-2$.*

**Proof.** We claim first that $G$ contains two overlapping chords whose lower endpoints are not consecutive along $P$. Since $G$ is nonexceptional, $C(G)$ has a component of size at least 3 or an isolated edge whose corresponding chords have lower endpoints that are not consecutive along $P$. If $C(G)$ has a component $H$ of size at least 3, then let $e$ and $f$ be two chords that are adjacent in $H$. If the lower endpoints of $e$ and $f$ are not consecutive, then the claim holds. Otherwise, let $e'$ be another chord that is adjacent to $e$ or $f$ in $H$. Since $l \geq 3$, now $\{e', e\}$ or $\{e', f\}$ is the desired pair.

Let $e$ and $f$ be overlapping chords whose lower endpoints are not consecutive along $P$, let $S$ be a set of $q'$ chords whose chord overlap graph is triangle-free, and let $Q$ be the shortest subpath of $P$ that contains the endpoints of $e$ and $f$. Note that $Q$ and the path joining the ends of $Q$ that uses $e$ and $f$ (and no other chords) have different lengths modulo $l-1$.

Let $S_0$ be the set of all chords in $S$ that are independent of $Q$. By Proposition 12, $|S_0| \geq q'-4$. Because the chord overlap graph of $S_0$ is triangle-free and perfect, it is bipartite. Hence $S_0$ contains a set of $(q'-4)/2$ pairwise non-overlapping chords. Because these chords are independent of $Q$, Proposition 11 yields $r \geq 2((q'-4)/2+1) = q'-2$. □

We have shown that $r \geq q/2+1$ when $G$ is exceptional. The lower bound is generally stronger for nonexceptional graphs with $l \geq 3$.

**Theorem 14** *If $G$ is nonexceptional, then*

$$r \geq \begin{cases} \frac{l-1}{l}q - 15l & \text{if } l \text{ is even} \\ \frac{l-1}{2l}q - 15l & \text{if } l \text{ is odd.} \end{cases}$$

**Proof.** If $l=2$, then Theorem 6 (or explicit study of the components of the chord overlap graph, which are paths) yields $r \geq q/2+1$, which satisfies the claimed bound. Hence we may assume that $l \geq 3$.

Let $S_0$ be the set of $q$ chords of $P$. For $j \geq 1$, we construct $S_j$ from $S_{j-1}$ as follows. Let $t_{j-1}$ be the clique number of the chord overlap graph of $S_{j-1}$, and let $Q_{j-1}$ be a minimal subpath of $P$ that contains the endpoints of $t_{j-1}$ pairwise overlapping chords. Let $S_j$ be the set of all chords in $S_{j-1}$ that are independent of $Q_{j-1}$. By Proposition 12, we have $|S_j| \geq |S_{j-1}| - (3t_{j-1}-2)$. Thus $|S_j| \geq q - 3\sum_{i=0}^{j-1} t_i$. The iteration stops when $t_j \leq 2$ or $\sum_{i<j} \lfloor \frac{t_i-1}{2} \rfloor \geq l-1$. Let $T = \sum_{i<j} t_i$.

Our plan is to obtain lower bounds on $r$ in terms of $T$ under each of the two stopping criteria. This will yield $r \geq \min\{q-3T-2, \lambda(\frac{q-3T}{l}+1)\}$, where $\lambda = l-1$ if $l$ is even and $\lambda = (l-1)/2$ if $l$ is odd. We then complete the proof by showing that $T \leq 5(l-1)$.

If $t_j \leq 2$, then the chord overlap graph of $S_j$ is triangle-free, and Lemma 13 implies that $r \geq |S_j| - 2 \geq q - 3T - 2$.

Otherwise, we stop because $\sum_{i<j} \lfloor \frac{t_i-1}{2} \rfloor \geq l-1$. In a segment $Q_i$, let $e_1, \ldots, e_{t_i}$ be the chords in order of the lower endpoints, and let $R_i$ be the path joining the ends of $Q_i$ that uses $e_{t_i}$ and no other chord. We find a longer path as follows. Starting from the lower endpoint of $Q_i$, which is the lower endpoint of $e_1$, we use $e_1$ and continue along $Q_i$ to the upper endpoint of $e_2$. Next we follow $e_2$ backward and continue forward along $Q_i$ to the lower endpoint of $e_{t_i}$. Now $e_{t_i}$ takes us to the upper endpoint of $Q_i$ (see Fig. 1).

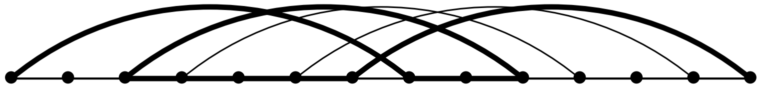

Figure 1: Increasing the length by 2

Since $t_i \geq 3$ and $e_1, \ldots, e_{t_i}$ are pairwise overlapping, what we have described is a path. This path uses the same number of edges of $Q_i$ as $R_i$ (replacing the portion joining the lower endpoints of $e_1$ and $e_2$ with the portion joining their upper endpoints), so the addition of $e_1$ and $e_2$ increases the length by 2. If $t_i \geq 5$, then we may use $e_3$ and $e_4$ to increase the length of the path again by 2 in the same way.

By construction, $Q_0, \ldots Q_{j-1}$ is a family of edge-disjoint segments of $P$. Initially, let $\rho$ be the sum of the lengths of $R_0, \ldots, R_{j-1}$. The chords allow us to increase $\rho$ repeatedly by 2, for $\sum_{i<j} \left\lfloor \frac{t_i-1}{2} \right\rfloor$ times. Because $\sum_{i<j} \left\lfloor \frac{t_i-1}{2} \right\rfloor \geq l-1$, we obtain values in every equivalence class modulo $l-1$ when $l-1$ is odd and half of the equivalence classes modulo $l-1$ when $l-1$ is even. Also, all chords in $S_j$ are independent of $Q_0, \ldots, Q_{j-1}$. Because $S_j$ contains a set of $|S_j|/l$ pairwise non-overlapping chords and $|S_j|/l \geq (q-3T)/l$, Proposition 11 yields $r \geq \lambda(\frac{q-3T}{l}+1)$.

It remains to bound $T$. Since $t_i \geq 3$ for $i < j$, each term contributes at least 1 to $\sum_{i<j} \left\lfloor \frac{t_i-1}{2} \right\rfloor$. Since $\sum_{i=0}^{j-2} \left\lfloor \frac{t_i-1}{2} \right\rfloor < l-1$, we have $j \leq l-1$. Since chords have length $l$, also $t_i \leq l$ for all $i$. We compute

$$\begin{aligned} T &= \sum_{i=0}^{j-1} t_i \leq 2\sum_{i=0}^{j-1}\left(\left\lfloor \frac{t_i-1}{2} \right\rfloor + 1\right) \\ &= 2\left(\sum_{i=0}^{j-2}\left\lfloor \frac{t_i-1}{2} \right\rfloor\right) + 2j + 2\left\lfloor \frac{t_{j-1}-1}{2} \right\rfloor \\ &\leq 2(l-2) + 2(l-1) + (l-1) < 5(l-1). \end{aligned}$$

As remarked earlier, the bound now follows under either stopping criterion. □

An $n$-vertex path has $n-l$ possible chords of length $l$. If $G$ contains them all, then $q = n-l$ and every $x,y$-path has length at least $(n-1)/l$, which is larger than $q/l$. This yields

$$r \leq \begin{cases} \frac{l-1}{l}q + O(l) & \text{if } l \text{ is even} \\ \frac{l-1}{2l}q + O(l) & \text{if } l \text{ is odd.} \end{cases}$$

Thus Theorem 14 is best possible up to the coefficient on $l$. We make no attempt to optimize this coefficient.

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