Report on REGS Program in Combinatorics — D. B. West

Activities through Fall 2005

Overview

The REGS Program in Combinatorics began officially in Summer 2004, although there were earlier semesters in which similar research groups met. The project was very successful in Summer 2004 and continued in various forms through Academic Year 2004-05 and in Summer 2005. For various reasons, we were planning not to have formal meetings this fall, but the students clamored for a continuation, and so we are still meeting.

The group involves graduate students at various levels from both the Mathematics Department and the Computer Science Department, (plus two undergraduates in Fall 2004), for a total of 15–20 students each summer. The project introduces beginning students to a research environment, develops good research habits, fosters mentoring relationships, and leads to publishable results. The beginning students benefit from the participation of the more senior students. They get started doing research faster than they otherwise would, and the hope is that this will help them get into thesis research faster and reduce their time to degree.

This report serves also as a proposal to create a course Math 596, Research Experiences, with sections as appropriate for various research groups. For that reason, it also describes briefly the activities of Summer 2004.

Combinatorial problems lend themselves particularly well to this project, because many of them can be described in the terminology of a single introductory course and understood by beginning students. Also, the questions can be studied on examples to gain understanding, and proving a hard conjecture on a significant special class is often worthwhile. Nevertheless, it should be possible to conduct similar projects in other research areas.

The appendix to this report consists of short reports from participants describing their experience over the past year and its value to them. The first report is from the post-doc, Stephen Hartke. There follow, in alphabetical order, reports from four computer science students and five mathematics students.

Modus Operandi and Rationale

In Summer 2004, the group met three times per week for 2–3 hours. During the Fall and Spring, the group met once per week for two hours, 3–5pm on Thursdays. In Fall 2004 there was no official context for the group, but in Spring 2005 it met as Math 598 REC (a section of the Literature Seminar course entitled “Research Experiences in Combinatorics”). In Summer 2005, the group met together three times per week from 3–5pm, and there were additional meetings of smaller subsets of the participants.

For about the first half of each semester, there are presentations of open problems in the first hour of each meeting. Problems are presented by both faculty and students. Students formally enrolled are responsible for presenting at least one problem area, and many other students also give problems. Problems come from conferences, from open problems on the web, from published papers in journals, from students’ own conjectures, etc. Students learn to use various resources to find problems and look up what is known about the problems. Discussion of the problems, led by faculty, develops taste for what makes a good problem and generates suggestions for how to approach the problem and for related problems to consider.

In Summer 2004, these presentations were more formal and lengthier. Since the focus is on research activity, not on learning to lecture, we have deemphasized this aspect since then. The second half of the meetings is devoted to group work on open problems, right from the beginning, and in the second half of the term the presentation of problems tapers off in favor of spending all the time on research activity and discussion.

With all students and faculty presenting problems, quite a lot of problems are proposed. Not all problems are appealing to students or lend themselves to generating ideas and results. Some problems are discarded as efforts coalesce around problems that students find appealing and accessible.

Since the research discussions involve groups of three to five students who get interested in a particular problem, the ratio of problems presented to problems studied is about right. As a result, no one needs to screen problems in advance with an eye to what is most promising for beginning students. Students are exposed to a wide variety of problems, both new and classical, and they learn how to find something they can work on.

The group work is analogous to the collaborative study sessions used in the undergraduate graph theory course (Math 412) and by some instructors in the undergraduate "transition to proofs" course (Math 347). Small groups in the same room facilitate efficient contact for the instructor/advisor, many students can be talking and thinking independently at the same time, students engaged in discussion together feel more comfortable speaking up and contributing their ideas, etc. We do expect much more in the REGS, including working on the problems outside of the meeting time and working more independently of the instructor.

Results from the research are presented in the regular seminar in Graph Theory and Combinatorics during the following semester. Part of the focus of the Fall 2004 group was to write up results from the summer research. Papers from Summer 2004 on graph pebbling and cut-and-paste sorting have already been submitted. Drafts of papers on decomposition of planar graphs and pagenumber have also been written. Two other problems from Summer 2004 may also still yield papers.

Summer 2005 has again been very productive, and the papers will emerge in a more timely manner thanks to the efforts of our combinatorics post-doc, Stephen Hartke. In addition to the regular meetings of the overall group, he met separately with three groups of students for continuing work on their projects. These groups have already presented their results in the seminar this fall, and he is actively prodding the students to write the papers.

The distinction between the summer and academic-year sessions is important. The consensus among the faculty and students involved is that the activities in the summer are much more efficient in accomplishing the goals of the program than those during the academic year. During the academic year, everyone has many other things to do, but students can focus on research during the summer. This holds true especially of the younger students at whom the program is aimed, since they have full course loads (including comp courses) during the academic year, and these courses are not offered during the summer. Taking REGS as a regular course during the academic year would cut down on other courses they could take.

From the administrative viewpoint, it seems that a REGS activity would not give a faculty member credit for a full course, and it would not meet more than three hours per week during the academic year. Thus even if made a regular course during the academic year it would not bring as much benefit as the summer program. The Chair has expressed concern about shifting of teaching credit to the summer, even at only a half-course. Nevertheless, if summer REGS programs do reduce time to degree, then it would be highly efficient graduate teaching.

Personnel

Summer 2004: VIGRE funds supported several graduate students, including Qi Liu, Lale Ozkahya, Jennifer Vandenbussche, Joseph Wright, and Gexin Yu. Two undergraduates, Noah Prince and Derrick Cheng, participated with REU funding. Hailong Hu, Jeong-Ok Choi, and Weiting Cao had support as graders for summer courses. Senior students participating occasionally without support included Naeem Sheikh, Kyung-Won Hwang, Hemanshu Kaul, Jeong-Hyun Kang, Kittikorn Nakprasit, and Seog-Jin Kim (the last three had defended their theses). From Computer Science, David Bunde, Kevin Milans, Erin Wolf, and Dan Cranston participated regularly. Short reports on the experience were submitted by Choi, Hu, Liu, Ozkahya, Yu, and Vandenbussche and included with VIGRE materials.

Fall/Spring 2004: In addition to myself, Prof. Kostochka and our new post-doc Stephen Hartke participated in leading the group. In the Fall, it had no official status, but most of the summer participants still on campus continued to meet. Entering students, most having taken the comp course Math 580 during the fall, joined the group in Spring 2005, when it was offered as Math 598 REC. That course had 11 students officially registered: Weiting Cao, Jeong-Ok Choi, Dan Cranston, Hailong Hu, Radoslav Kirov, Qi Liu, Kevin Milans, Lale Ozkahya, William Shanley, Ida Svejdarova, and Gexin Yu. Among these, Kirov and Svejdarova were first-year students and Shanley was an undergraduate. Again there were additional participants; meetings regularly had 15 or more students.

Summer 2005: The department provided funding for some students, mostly those at earlier stages. Again senior students participated without direct support, including Kaul, Yu, Cao, Choi, and occasionally Sheikh. First-year students included Michael Barrus, Mohit Kumbhat, and Paul Wenger (Wenger is in Math 580 only now, after starting research with the group). Second-year students included Liu, Ozkahya, and Vandenbussche. The computer scientists from Summer 2004 joined again, plus Philip Mienk. Also Noah Prince returned as a graduate student participant, and Seog-Jin Kim visited from his new position in Korea. As mentioned earlier, post-doc Stephen Hartke guided three working groups very effectively. Finally, He-Hui Wu visited for the full summer. A graduate student at West Virginia University, he was among our most active participants, presenting several of his own conjectures and coming up with the best results in the large group working on a problem West proposed. He has applied for admission to our program, and we hope that this creative and energetic researcher will be accepted for the spring. In summary, again there were more than 20 participants.

Fall 2005: Several students eager to continue and/or start research lobbied for the group to meet, even though again there is no formal framework. Included are several first-semester students taking Math 580 now: Aaron Hill, Tim LeSaulnier, Suil O, and Young-Jin Kim. We asked Jozsef Balogh to lead the group, and he presented open problems for the first two weeks. Last week problems were presented by several senior students who had attended the MIGHTY (MIdwest GrapH TheorY) Meeting the weekend before. We will follow the usual plan during the semester. Again there are regularly 15 or more participants, but this semester fewer of the computer scientists have been attending. One reason may be that they have started a similar research group for discussion of problems in theoretical computer science.

Probably the students will want to continue the group in the spring; who will lead it? The course Math 598 REC tentatively appears on the schedule. West received a one-course teaching reduction for Fall 2005 in compensation for time spent leading this activity over the past year, but the Chair does not want the same person to be always leading this. Hartke and Balogh already have reduced teaching loads, Kostochka will be on sabbatical, etc.

Summary of Results

We briefly describe several new investigations from Summer 2005 that will result in papers.

Bar k-visibility graphs (Stephen Hartke, Jennifer Vandenbussche, Paul Wenger) A *bar visibility representation* of a graph G is a family of horizontal bars in the plane corresponding to the vertices of G such that two vertices are adjacent if and only if the corresponding bars can see each other along an unobstructed vertical sightline. A graph is a *bar visibility graph* if it has such a representation. Dean, Laison, Trotter et al introduced the notion of a *bar k-visibility graph*; in a bar k-visibility representation, a bar can see another bar if there are at most k intervening bars along some vertical sightline.

The earlier authors showed that the complete graph K_{4k+4} is a bar k-visibility graph and that no complete graph with more than $5k+5$ vertices is a bar k-visibility graph. We prove their conjecture, showing that a complete graph is a bar k-visibility graph if and only if it has at most $4k+4$ vertices. We also obtain forbidden induced subgraphs for bar k-visibility graphs, construct regular bar k-visibility graphs, and obtain some necessary conditions on the vertex degrees for regular bar k-visibility graphs.

Parity edge-coloring of graphs (David Bunde, Bill Kinnersley, Kevin Milans, Douglas B. West, He-Hui Wu). Starting from questions about which trees embed in hypercubes, we showed that a graph embeds in the k-dimensional hypercube if and only if it has an edge-coloring such that on every path some color appears an odd number of times, and on every cycle no color appears an odd number of times.

Define a *parity edge-coloring* of a graph to be an edge-coloring such that on every path some color appears an odd number of times. The *parity edge-chromatic number* $p(G)$ of a graph G is the minimum number of colors in a parity edge-coloring. For every graph, this parameter at least the edge-chromatic number and at most the number of edges. For the complete graph, we proved that $p(K_n) = n-1$ when n is a power of 2 and conjectured that always $p(K_n) = 2^{\lceil \log_2 n \rceil} - 1$, which we proved for $n \le 16$. Other results were proved for this parameter on other special families of graphs, and relationships between this parameter and others were explored.

This appears to be a new parameter, and we hope to obtain further results on it before presenting it at the joint AMS-Taiwan Mathematical Society meeting in December.

Graph classes characterized by both degree lists and forbidden induced subgraphs (Michael Barrus, Mohit Kumbhat, Stephen Hartke). For a family F of graphs, a graph G is F-free if G has no member of F as an induced subgraph. A class of graphs is *characterized by degree lists* if every graph having the same degree list as some graph in the class is also in the class. We study which classes of graphs are characterized both by degree lists and by a family of forbidden induced subgraphs. For example, the well-known families of split graphs and threshold graphs have both types of characterizations. We determine completely which pairs of forbidden induced subgraphs define families that are also characterized by degree lists, and we partially determine which triples of forbidden induced subgraphs do this.

Distinguishing chromatic number (Jeong-Ok Choi, Stephen Hartke, Hemanshu Kaul). Albertson and Collins defined a *distinguishing coloring* of a graph to be a vertex coloring such that no nontrivial automorphism preserves the colors of all vertices. The *distinguishing number* is the minimum number of colors in a distinguishing coloring. More than a dozen papers have been published on this topic. More recently, Collins and Trenk introduced the *distinguishing chromatic number*, defined to be the minimum number of colors in a proper coloring (adjacent vertices receive distinct colors) that is also a distinguishing coloring.

Albertson studied distinguishing number for cartesian products of graphs (such as the hypercube). Taking cartesian powers of graphs does not increase the chromatic number. Although the distinguishing chromatic number of a graph can be much larger than its chromatic number, we show that when d is sufficiently large, the cartesian power G^d of a graph G has distinguishing chromatic number equal to its chromatic number plus 1. For hypercubes (K_2^d), Hamming graphs (K_n^d), and cartesian powers of complete multipartite graphs, we obtain good bounds on the number of dimensions needed for this to occur.

H-linkage implications. (Qi Liu, Gexin Yu, Douglas B. West). For a fixed graph H, a graph G is *H-linked* if for every injection $f\colon V(H) \to V(G)$, there is a subdivision of H in G in which for each $v \in V(H)$ the branch vertex representing v is $f(v)$. When H is a star with k edges, a cycle with k edges, or a matching with k edges, H-linked has the same meaning as the well-studied properties k-connected, k-ordered, and k-linked, respectively.

It is well known that every k-linked graph is both k-ordered and k-connected. The converse fails, but for sufficiently large m in terms of k, every m-connected graph is both k-ordered and k-linked. A series of papers has improved the threshold; Thomas and Wollan have the best bound, showing that every $10k$-connected graph is k-linked.

We study when it holds that every H_2-linked graph is H_1-linked. This holds when H_1 is a subgraph of H_2. If I_k denotes a single edge plus $k-1$ isolated vertices and S_k denotes a star with k edges, then the properties of being H-linked are equivalent for all H such that $I_k \subseteq H \subseteq S_k$. For all these graphs, the vertex cover number is 1. (The vertex cover number $\beta(H)$ of a graph H is the minimum size of a vertex set containing at least one endpoint of every edge.) A special case of our main result is that for graphs H_1 and H_2 having the same number of vertices and having $\beta(H_1), \beta(H_2) \geq 2$, being H_2-linked implies being H_1-linked if and only if H_1 is a subgraph of H_2. This completes the analysis of the problem among graphs with the same number of vertices.

The general statement of the main result eliminates many additional implications. When H_1 has k more vertices than H_2, and H_2 has at least $k+1$ edges not in H_1, and $\beta(H_2) \geq k+2$ (with one special exception), we construct a graph that is H_1-linked but is not H_2-linked.

Roman domination. (Erin Wolf Chambers, Bill Kinnersley, Noah Prince, Douglas B. West). A *dominating set* in a graph is a set S of vertices such that every vertex outside S has a neighbor in S; the minimum size of such a set is the *domination number* $\gamma(G)$.

In the "Roman" domination model, guards may be strong or weak. Strong guards cost 2 and can protect the vertices neighboring their own location (like the vertices in a dominating set), but weak guards (costing 1) can only protect their own location. The *Roman domination* or *protection* number $\gamma_R(G)$ is the minimum cost of protecting all vertices in the graph G. Clearly $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

There have been a handful of papers on this concept, initiated by Cockayne, Dreyer, Hedetniemi, and Hedetniemi. They mostly studied $\gamma_R(G)$ for trees and studied when $\gamma_R(G)$ equals $\gamma(G)$ or $2\gamma(G)$. We study $\gamma_R(G)$ in terms of other parameters, including minimum and maximum degree, diameter, and number of vertices. We solve the extremal problem for $\gamma_R(G)$ for n-vertex trees and connected graphs; the maximum is $\lfloor 4n/5 \rfloor$, and we characterize the graphs achieving equality when n is divisible by 5. We prove better bounds for trees of small diameter or small maximum degree.

Finally, building on analogous results for $\gamma(G)$, we study the maximum values for the sum and product of $\gamma_R(G)$ and $\gamma(\overline{G})$ when G is an n-vertex graph and $\overline{G}$ is its complement.

During the summer, other groups studied a variety of other concepts. These included list edge-coloring (Weiting Cao, Jeong-Ok Choi, Seog-Jin Kim), a problem about the relationship between longest paths from a specified vertex and spanning cycles (Bill Kinnersley, Mohit Kumbhat), a problem about the Helly property for longest paths in chordal graphs (Jenn Vandenbussche, Gexin Yu), and a pursuit parameter on graphs called the "cop number" (Erin Chambers, Philip Mienk). These problems did not yet lead to publishable results.