

Summer 2007 REGS in Combinatorics — D. B. West

Overview

Our REGS project in 2007 was quite large and quite successful. Progress was made on the usual half-dozen or so problems, with several papers already submitted and several still in preparation. For a general description of the format and rationale of the program, see the reports from 2006 or 2005.

A distinctive feature for 2007 was the visit during the first week by André Kézdy and four of his students from the University of Louisville. Like our students, these students shared problems with the group and participated in research. Their group continued to work on the problems in regular meetings after they returned to Louisville. Kevin Milans made a trip to Louisville during the year to give a talk and continue work with Lesley Wiglesworth on one of the problems.

One minor difficulty arose due to the large number of people involved. Several of the easier problems attracted large groups of interested students. The resulting large number of names on a paper is perhaps not so desirable. In terms of the aims of REGS as research training, this is perhaps not a big problem, but it is a phenomenon that should be kept in mind. I made attempts during the summer to divide a couple of the groups by splitting the topics into smaller problems; this was reasonably successful.

Participants

Current students: Michael Barrus, Jan Butterfield, Jeong Ok Choi, Kyle Jao, Younjin Kim, Tim LeSaulnier, Suil O, Lale Ozkahya, Noah Prince, Naeem Sheikh, Chris Stocker, Ida Svejdarova, Paul Wenger, Hehui Wu.

CS students: Tracy Grauman, Bill Kinnersley, Kevin Milans, Erin Chambers, Pratik Worah.

Former students: Dan Cranston, Gexin Yu, Seog-Jin Kim, Qi Liu.

Post-Doc: Stephen Hartke.

Faculty: Alexandr Kostochka and Chandra Chekuri dropped by to give us open problems, as did Michael Ferrara from Akron.

Louisville students and faculty: Ben Allgeier, Tim Brauch, Adam Jobson, Max Leidner, Lesley Wiglesworth, André Kézdy.

Results

Hub number. One of the earliest problems attacked came from a talk by Peter Hamburger at the CanaDAM meeting in Banff in May 2007. The *hub number* is a very domination-like parameter, and indeed we proved that it cannot differ from the connected domination number by more than 1. In essence, our effect here was aimed at killing off study of this parameter. This was a prime example of a discussion with too many people involved. A short paper on the relationship to domination and some algorithmic aspects has been submitted to *Information Processing Letters* by Grauman, Hartke, Jobson, Kinnersley, Wiglesworth, West, Worah, and Wu.

Gap-free lists. This problem originated in email sent to me by Yair Caro. It is an elementary exercise that every graph has two vertices of the same degree. We defined the *repetition number* of a graph to be the maximum multiplicity in the degree list. Caro and I wrote a paper on this that has been submitted. REGS students thought some about it, but their success was on an offshoot of the problem. A *gap-free list* is a list of consecutive positive integers, with multiplicities. Fixing the largest and smallest values, the question is how long (counting multiplicity) the list must be to guarantee that it is realizable as the degree list of some graph (assuming the sum is even). This problem was solved in REGS; there is now a draft of a paper by Barrus, Hartke, Jao, and West.

Acquisition number. Each vertex initially has weight 1. When adjacent vertices u and v both have positive weight, u can absorb (all) the weight from v if the weight on u is currently at least that on v . This continues until the vertices with positive weight form an independent set. The *acquisition number* of the graph is the smallest possible number of vertices in the final independent set. We studied acquisition number of trees, graphs with acquisition number 1, proved that always a graph or its complement has acquisition number 1, etc. There is now a draft of a paper by LeSaulnier, Prince, Wenger, West, and Worah.

Game acquisition number. This was an instance of successfully splitting a large group into two smaller investigations. In the game acquisition problem, acquisition moves are made alternately by two players; one wants to minimize the size of the final independent set, the other to maximize it. The *game acquisition number* is the result of optimal play. The problem is considerably more difficult than the ordinary acquisition number. There is now a draft about the game acquisition number of complete bipartite graphs by Milans, Stocker, West, and Wiglesworth.

On-line degree Ramsey number. In graph Ramsey theory, “Painter” 2-colors the edges of a graph presented by “Builder”. Builder wins if every coloring has a monochromatic copy of a fixed graph G . In the on-line version, Builder presents edges one by one, and Painter must color each edge without seeing later edges. Builder must keep the presented graph in a class \mathcal{H} . Builder wins the game (G, \mathcal{H}) if a monochromatic copy of G can be forced. The *on-line degree Ramsey number* $\text{odr}(G)$ is the least k such that Builder wins (G, \mathcal{H}) when \mathcal{H} is the class of graphs with maximum degree at most k .

There is now a partial draft of a paper by Butterfield, Grauman, Kinnersley, Milans, Stocker, and West. Among the results are the following. 1) $\text{odr}(G) \leq 3$ if and only if each component of G is a path or each component is a subgraph of the claw $K_{1,3}$ (and $\text{odr}(K_{1,m}) = m$). 2) $\text{odr}(G) \geq 2\Delta(G) - 1$ if G has two adjacent vertices of maximum degree. 3) $\text{odr}(G) \leq 2\Delta(G) - 1$ if G is a tree. 4) $\text{odr}(G)$ is bounded if $\Delta(G) \leq 2$. 5) $4 \leq \text{odr}(C_n) \leq 5$. 6) $\text{odr}(C_n) = 4$ if n is even or 3 or at least 689.

The lower bounds come from strategies for Painter that keep the red graph in a specified class. The upper bounds come from a result showing that Builder may assume that Painter plays “consistently”, which uses techniques from logic.

Meanwhile, Milans has gone on to study an off-line version of degree Ramsey number. Here Builder present a graph and Painter colors it. The problem is to find the smallest maximum degree of a graph Builder can present to force Painter to produce a monochro-

matic copy of G . Milans has proved singly-authored results about this.

Domination. The previous summer's results on Roman domination were submitted in a paper by Chambers, Kinnersley, Prince, and West. Prince generalized these techniques to prove bounds on the various types of domination parameters for a graph and its complement. These results now appear in his thesis.

In a separate investigation, Suil O and others studied the domination number of Kneser graphs.

Injective Coloring. An *injective coloring* uses each color at most once in the neighborhood of any one vertex; it need not be a proper coloring. The problem is to minimize the number of colors use. Hahn, Kratochvil, Siran, and Sotteau initiated the study of this parameter in a 2002 paper. Our study of it was prompted by a talk by Raspaud at CanaDAM. Cranston, Kim, and Yu have submitted a paper on their results.

Thickness and Arboricity The *thickness* of a graph is the minimum number of planar graphs needed to cover it. The *arboricity* of a graph is the minimum number of forests needed to cover it. The delegates from Louisville were interested in the thickness of "sphere-of-influence graphs". Their question evolved into a question about the ratio between thickness and arboricity. Every planar graph has arboricity at most 3, so the thickness is at least $1/3$ of the arboricity. We conjectured that the ratio is close to $1/3$ when the graph is dense, enforced for example by large minimum degree. This problem may receive further attention in 2008.

Representation number and product dimension The *representation number* of a graph is the minimum N such that the graph can be represented by assigning each vertex an integer so that vertices are adjacent if and only if their labels are relatively prime modulo N . When there are no vertices with identical neighborhoods, the notion is equivalent to *product dimension* (also called *Prague dimension*, which is the minimum d enabling an encoding of the graph by d -tuples assigned to the vertices so that vertices are adjacent if and only if their codes differ in every coordinate. Ida Svejdarova brought this problem to the group, and she obtained new results on this very difficult parameter, improving various bounds for trees. Her paper is now submitted.

Steiner trees. Chandra Chekuri presented to the group several problems about connectivity conditions for Steiner trees in graphs, meaning trees joining a specified set S of vertices. Hehui Wu thought further about these problems and has continued working on them during the year, obtaining several results.

Participant Reports (alphabetical order)

Michael Barrus

I participated in the Combinatorics REGS held June-August 2007. I participated in discussions on several problems; the following are problems to which I devoted significant attention, and a summary of my activities:

Gap-free sequences and the repetition number.

A list of positive integers is said to be a *gp-list* if the sum of the terms is even, and if every integer between the maximum and minimum list values also appears in the list (i.e. it is “gap-free”). Y. Caro has recently examined the *repetition number* of a degree sequence, i.e. the largest multiplicity of a term in the degree sequence of a graph. Caro determined a general bound on the repetition number of the degree sequence of a general graph; while doing so he made a conjecture that gp-lists with length at least $3k/2 - 1$ are graphic, where k is the value of the largest term of the gp-list. In our group we succeeded in proving that any gp-list with length at least $3k/2$ is graphic, and this result is sharp. Moreover, Kyle Jao, a fellow group member, provided a formula for this “graphicality threshold” on the sequence length in terms of both the maximum and the minimum values of the list. I was able to offer an alternate version of one portion of Kyle’s proof, and to adapt the method of proof to prove that any list of positive integers (not just gap-free lists), maximum k and minimum s , and length at least $(k + s + 1)^2/(4s)$ satisfies the ErdHos-Gallai inequalities for graphicality, and this result is sharp for infinitely many pairs (k, s) . One notes that graphicality thresholds are thus known for sequences where consecutive terms differ by at most 1, and for sequences where consecutive terms differ by at most $k - s$; one may ask what the graphicality threshold is for sequences whose consecutive terms differ by at most d , where $1 \leq d \leq k - s$. Independent of the group (whose attention led in another direction, as described below), I examined this problem and have obtained some partial results which may be improved in the future.

Once the gap-free questions had been successfully answered, our group moved on to examine a series of ongoing conjectures and results of Caro concerning the repetition number of degree sequences of line graphs. In particular, we have looked at finding asymptotic bounds on the repetition number of the line graph of trees, outerplanar graphs, and other classes of graphs whose minimum and average degrees are constants. In some cases our group has been able to slightly improve Caro’s proofs of bounds, and it seems that determining the exact asymptotic bound is within reach.

Graph properties, the dominance order, and induced subgraphs.

This is a problem which I posed for the class. Given two partitions π, σ , of an even number, with parts arranged in nonincreasing order, we say that π *dominates* (or *majorizes*) σ if each partial sum of π is at least as large as the corresponding partial sum of σ . The dominance relation yields a partial order on the set of partitions of any fixed even number; it is well known that the graphic partitions of a number form an ideal under this dominance order, and the threshold sequences are exactly the set of maximal graphic sequences under this order. One trivially has that if π is graphic, π majorizes σ , and σ is threshold, then π is threshold as well. I showed this summer that the same property holds if “threshold” is replaced by “split.” Split sequences, like threshold sequences, have been studied in detail. Both sequences correspond to hereditary classes of graphs; the threshold graphs are precisely the $\{2K_2, C_4, P_4\}$ -free graphs, and the split graphs are the $\{2K_2, C_4, C_5\}$ -free graphs. In our group (which, as a group, was fairly short-lived, as members’ interests drew them away) we began to examine which classes \mathcal{F} satisfied the property that

$[\pi, \sigma$ graphic with the same sum; π dominates σ ; and σ potentially (forcibly) \mathcal{F} -free]

$\Rightarrow \pi$ potentially (forcibly) \mathcal{F} -free.

We showed that this property did not hold when \mathcal{F} was set of forbidden subgraphs of the classes of perfect graphs or even chordal graphs, both of which contain the class of split graphs. I had shown, previous to posing the problem to the class, that \mathcal{F} is a singleton set in the “forcibly” version of the problem if and only if \mathcal{F} is a graph on 2 or fewer vertices. In our group we began to look at the “potentially” version of the problem, with \mathcal{F} again a singleton set. A counterexample to a given \mathcal{F} would involve a potentially \mathcal{F} -free sequence σ majorized by a forcibly \mathcal{F} -inducing sequence π . In general finding such pairs, or determining if they exist, appears to be quite difficult. We were able to show that such a pair exists when \mathcal{F} is a path on 3, 4, or 5 vertices, but we conjectured that no forcibly P_6 -inducing sequence exists, and we were not able to prove this. I continued our work somewhat past the group’s dissolution. I examined forcibly \mathcal{F} -inducing sequences for singletons \mathcal{F} where the graph involved contained 3 or fewer vertices. Most recently I have examined forcibly $3K_1$ -inducing sequences.

Besides these groups, I participated in discussions with other groups on a few other problems (especially at the beginning of the summer), and I worked with Stephen Hartke and Mohit Kumbhat on a problem (characterizing degree-sequence-forcing sets) that we have worked on in past years in REGS, and on two related research problems on my own.

I found the REGS group to be beneficial, for the most part, for the usual reasons of having a number of interesting problems to think about, and the availability of a variety of collaborators with different backgrounds and ideas. It was a nice addition this year to have visitors from the University of Louisville here at the beginning of the summer. I felt that their presence, together with the long days of hearing problems posed, made for more enthusiasm at the outset of the group, and many of the problems posed were interesting to me. In fact, my only frustration with this year’s experience is that as the summer progressed, I had more research projects than I had time for.

Jane V. Butterfield

I was part of the Combinatorics REGS group this summer, led by Doug West. Each of us brought a problem to present to the group, and since there were about thirty people participating there were a lot of available problems. Rather than presenting them all at the beginning and saving the research until afterwards, we divided our early sessions up between those two activities. This meant that there was not as much time to work on the later problems, but it also meant that the whole summer was active and involving.

The summer would have been a bit more manageable if there had been fewer presented problems; then we would have had more sessions devoted exclusively to research, and there would have been more time to explore the later problems. On the other hand, one thing I found beneficial about the program was the exposure I got to a large volume of open problems in the field.

The large number of problems, and the long time it took to present them all, meant that I transferred my attention between them quite a lot at first. This felt wasteful, but I think it is part of the process. I initially spent my time working on the problem I had brought, but there was little lasting interest in that problem from other people in the

group. I then switched briefly to another two problems before finally settling on one that was presented quite late in the summer, regarding on-line Ramsey theory.

Several people were involved in the Ramsey problem, which meant there was a lot of discussion and that our initial progress was fairly fast-paced. Most of the other groups were also large; in spite of the large supply, there were only a few problems that people found interesting. Perhaps if we had compiled a list of problems beforehand, people could have decided which ones interested them sooner, and divided up into more equitably sized groups. Presentations would still be useful, but we also could have come prepared for more than just our own problem that way.

The Ramsey group proved a few somewhat interesting results before I had to leave town, which I had cleared with Doug West ahead of time. Our thought was that I would be able to stay in contact fairly well through email. Since so many people were involved with the problem, however, this proved rather difficult. I kept in contact with one of them, but he did not always have access to the proofs of results that other members had brought to the sessions; consequently, of course, I had trouble attempting to build off of their results.

This was the first time I have been involved in group research, although I did research on my own as an undergraduate. Group work is a different experience, and I did not keep as well in contact with the other participants as I should have. At first, I worked alone on my own problem and even on problems I knew other people were tackling. Now, however, I am planning to continue work on the Ramsey theory question with one of the other people who were working on it, since we think the results we have so far are not quite interesting enough to motivate a paper. I think this continuation of the summer experience into the semester is a good thing and am looking forward to it.

Daniel Cranston

The REGS meetings were useful for me because they exposed me to a large number of problems. In particular, some of these problems have originated quite recently and no results have been published yet. I worked primarily on the problem of injective coloring. This problem seems to have originated in 2005 and the only papers on the topic are two preprints. REGS was particularly valuable in helping me to find two other people (Seog-Jin Kim and Gexin Yu) who were interested in the problem and experienced working with the relevant techniques. We were able to prove three theorems (each of which was an extension or strengthening of one of the results in the more recent preprint) and we expect to submit a paper within the next few months.

Fang-Kai Jao

I was very excited when I knew the REGS as I have never heard about this kind of session before, but my research experiences are not enough and I have some backgrounds needed to build up. However, Professor West inspired and encouraged me so that I decided to attend REGS with more confidence.

The hub number problem is the first one I am interested in. I was working on the hub number until the repetition numbers and gap-free degree sequences problem was proposed.

Although I didn't get any result on the hub number problem, I spend lots of time on gap-free degree sequence.

A sequence is called *gap-free* if each value between the minimum and the maximum occurs in the sequence. A *gp-list* is a gap-free sequence of positive numbers with even sum. The gap-free sequence problem is to find the maximum number of terms in a nongraphic gp-list with largest element k . We found the answer for gp-lists with minimum 1; then we generalized it to get a formula for the length that involves the maximum and minimum: $f(k, s) = k + \lceil \frac{k+s}{2s} \rceil$, where k and s are the maximum and minimum, respectively. Every gp-list with maximum k , minimum s , and length at least $f(k, s)$ is graphic. Moreover, $f(k, s)$ is sharp in that there are infinitely many pairs (k, s) for which there exist gp-lists with maximum k , minimum s , and length $f(k, s) - 1$ that are not graphic. Another beautiful sharp bound for general degree lists was found by Mike, one of our group, which is $(k + s + 1)^2/4s$.

After the gap-free sequences, we moved to work on the repetition numbers from which the gap-free sequence problem came. We focused on finding the smallest repetition number over all n -vertex line graphs. Although we didn't figure out any perfect results, we still have some results that includes lower bounds for repetition numbers of line graphs of trees, maximum planar graphs and maximum outer planar graphs, respectively, and an upper bound for repetition numbers of line graphs of trees by construction. Now, we have more understanding about the line graphs, especially those of trees and maximum outerplanar graphs.

The REGS was making me more understand how to solve an open problem. The discrepancy between solving open problems and exercises is that we don't know if the statement is true or false at the beginning. We have to do it step by step, and sometimes we have to modify our hypothesis or conjecture. So I believe that REGS is a really good way to strengthen the ability of analysis and logic thinking of students on problems, and most students will get help from REGS for sure.

Mohit Kumbhat

This summer I had the opportunity to participate in the Combinatorics and Graph theory REGS sessions. Like the last two summers this summer was also very helpful for me in terms of research experience. This summer we also had visitors from University of Louisville which must have proved beneficial for most of us. For the first couple of weeks each of us presented open problems of our choice and simultaneously worked on a few of them.

I had the opportunity of working with Prof West, Prof Y. Caro (via e-mail), Mike Barrus, Stephen Hartke, Kyle Jao, Younjin Kim and Erin Wolf on the problem of "Repetition number and gap-free degree sequences". For the first few weeks we worked on finding the threshold on the length of a gap-free degree list with maximum degree k , such that beyond this threshold every such gap-free degree list is graphic. We were also able to show that the threshold is sharp. It was then generalized for lists with given maximum degree k and minimum degree s . This was also shown to be sharp.

We then moved to the original problem of repetition number of a graph which motivated the problem of gap-free sequences. Here we were interested in finding the minimum

repetition number of the line graph of certain classes of n -vertex graphs. It was obtained for trees and maximal planar and outerplanar graphs and is sharp asymptotically. Lower and upper bounds were also found in general for the line graph of an arbitrary graph.

For a while I also discussed with Naeem Sheikh about variations on the "Online Ramsey: painter-builder" problem. We made a few trivial observations and few not so trivial ones. We might continue to think more about it during the coming semesters.

Max Leidner - University of Louisville

I began the summer by introducing the Behzad-Vizing Total Coloring Conjecture (TCC: that all graphs are $(\Delta + 2)$ -total-colorable) and stating how close it was to being proved for planar graphs. Kostochka finished proving the $\Delta < 6$ cases, Vizing proved the $\Delta > 7$ cases for planar graphs, and recently Sanders & Zhao proved the planar $\Delta - 7$ case. This left the planar $\Delta = 6$ case open, which, if proved, would mean that all planar graphs obey the TCC. I like coloring problems, and I thought it would be nice to work on a problem with such specific parameters.

During the presentations in the first week of studies, I became attracted to another problem having to do with colorings of unit distance graphs. It had been proved, using a hexagonal tiling, that seven colors were sufficient to properly color vertices in a unit distance graph. I thought it would be a fun challenge to try to bring that number down to six, or at least find out why seven colors might be necessary. I thought about how to look at the problem from perspectives perhaps not yet taken, such as putting the vertices in the complex plane and iterating them in some function to approach one of six attracting points which would determine its color. I tried to come up with a game where points would randomly move around and eventually approach these six points as well. However, the hard part was always trying to show that two vertices a unit distance apart would never have the same color. It seemed that trying to determine a vertex's color by its position relative to other vertices was impossible, so I gave up on this problem and decided to go back to Total Coloring.

Now, Sanders and Zhao showed in 2001 that planar graphs with $\Delta = 7$ were Class 1, which proved a part of Vizing's planar graph conjecture, which, when combined with the Four-Color Theorem and a trick that Yap used, leads to the proof of the TCC for planar $\Delta = 7$ graphs. For some reason, I wasn't very interested in going about the problem this way; I would have liked to do it more directly. I started looking for configurations that could not appear in minimum counterexamples, in order to narrow down the types of graphs that would give me the most problems. I did make a little bit of progress here. My advisor helped me focus my efforts by showing me how reducible configurations could be combined with unavoidable sets to get unavoidable reducibility and thus desirable results. Soon afterwards, I found the paper by Sanders and Zhao in which they directly proved that a planar graph with $\Delta = 7$ is 9-[total]-colorable, and this is exactly the method they used. I observed that they used a charging argument to find an unavoidable set, and then showed that each element of this set was reducible so that it could not appear in a minimum counterexample.

This got my hopes up, because I could see how the set of unavoidable configurations could be converted into a set of configurations more suitable for the $\Delta = 6$ case, and

I had already shown that about a third of these were reducible. This excited me, and my goal then became clear; I would try to prove that *all* of the relevant configurations were reducible, then try to adapt the charging argument to show that the new set was unavoidable. This is a challenge, but I have seen hope. So far I have still not yet proved that the set I have come up with is reducible.

I feel that I could have worked harder in the first five or six weeks of the course; though I was also teaching a class in the first month, I was relatively unburdened for a week after that and got a little lazy as I tried to scrape up what freedom I could. Then I decided to spend the majority of my thoughts on this planar total coloring problems. The more I worked, the more I felt like I was making progress, which gave me more hope and motivation, which drove me to work more and allowed me to report proudly that I was trying and getting somewhere. This was different from the first few weeks of the class, when I was distracted and would only think about the problem I was working on (i.e., coloring unit distance graphs) for brief periods and never make any progress. Then I would be afraid of reporting my progress, lose hope and motivation, and even seek out new ways of distracting myself from the work. That was a bad cycle to be in. But now I am in a better cycle, and the only problem is that the course is ending and the new semester is starting soon, which presents me with an unavoidable distraction.

I don't think that much could be done to improve this summer's course, except that I would have liked to meet less often. Usually I would just briefly say what I had been working on and then want to go back to work. Sometimes I would open up to outside assistance and suggestions, but that wasn't something I needed three times a week.

Kevin Milans

About three years ago, I began my participation in Professor Douglas West's combinatorics research group. Professor West's research group continues to provide an excellent environment where graduate students of all levels of experience grow academically. Working groups of discussed open problems, possible approaches to a problem, and ultimately, solutions.

At the beginning of the program, each student presents an open problem to the group. The student becomes familiar with the relevant journals, has a chance to practice speaking before a knowledgeable but informal audience, and is introduced to the group. Simultaneously, the group is enriched by a collection of open problems to attack throughout the summer. If students are not able to make satisfactory progress on one problem, there are always others available. As the summer progresses, the emphasis shifts from the presentation of open problems to the student groups who work on solutions. As the student groups exchange ideas about their chosen problems, Professor West visits each group to offer advice, new ideas, and evaluation of student proofs.

I presented some recent results on online Ramsey theory; online Ramsey theory studies a family of games between two players: a builder and a painter. For each graph G and every class of graphs \mathcal{H} , the builder/painter game (G, \mathcal{H}) is defined as follows. In each round, the builder presents painter with two vertices $\{v_i, v_j\}$ from an infinite collection of vertices, and painter responds by coloring the edge $v_i v_j$ with one of two colors. Builder's goal is to force painter to produce a monochromatic copy of G while maintaining the in-

variant that the presented graph belongs to the class \mathcal{H} . The fundamental question, then, is to decide which of the players has a winning strategy. From classical Ramsey theory, it is known that if \mathcal{H} is the class of all graphs, then builder wins (G, \mathcal{H}) .

This summer, I and other students have examined the special case when \mathcal{H} is the class of graphs with a fixed maximum degree. Let $\mathcal{S}_k = \{H : H \text{ has maximum degree at most } k\}$. We have been able to show that if G is a cycle, then painter wins (G, \mathcal{S}_3) . If G is an even cycle, the triangle, or a sufficiently large odd cycle, builder wins (G, \mathcal{S}_4) . Although we have found that builder wins (G, \mathcal{S}_5) for every cycle G , research continues on whether or not builder has a winning strategy for (G, \mathcal{S}_4) when G is a small odd cycle on at least five vertices. Other continuing research focuses on this question: is it true that for each l , there exists a constant $k = k(l)$ such that for each graph G with maximum degree at most l , builder wins (G, \mathcal{S}_k) ? If $l = 1$, the question is easily resolved in the affirmative; if $l = 2$, we are confident that our techniques on cycles will provide another positive answer. Nothing is known if $l \geq 3$.

A core skill that Professor West's group helps to teach is what to do when a researcher becomes stuck on a problem. Perhaps a fellow student can suggest a novel line of attack, or perhaps the problem can be modified slightly to obtain partial or related results. Perhaps another student is aware of previous work that may be relevant and can provide a reference. As I observed first hand how other researchers cope with being stuck, I increasingly found myself better able to circumvent this common difficulty.

Suil O

At the REGS meetings of Professor West's research group, each student provides an open problem to the Group. The students become familiar with many topics in combinatorics and collect many problems.

An H -factor of a graph G is a spanning subgraph of G whose components are in the family H . In this summer's REGS, I presented the conjecture that every 3-connected cubic graph with the order divisible by 3 has a P_3 -factor. It is known that every connected cubic graph has $\{P_3, P_4, P_5\}$ -factor by Kaneko(2003), and every 2-connected cubic graph has $\{P_3, P_4\}$ -factor by Kawarabayashi, Matsuda, Ota, and Oda (2002).

Although I did not give a presentation about the acquisition number, the domination number of Kneser graphs, or the chromatic number of the square of Kneser graphs, I did research on these problems. I guess that the chromatic number of the square of the Kneser graph $KG(3k - 2, k)$ is $1 + \binom{2k-2}{k}$ (which is correct in the case $KG(7, 3)$), as I described a proper coloring of $KG(3k - 2, k)$ with this many colors.

Also I found a hamiltonian cycle in $KG(7, 3)$. Another open question is whether the Kneser graphs all have hamiltonian cycles.

Ida Švejdarová

This was my first year participating in the combinatorics summer research group, and I benefited greatly from it.

With several other students we chose to investigate representation numbers of graphs. It soon became clear that we should instead concentrate on product dimension, a more

interesting related graph parameter. During the course of the summer the other students lost interest in this problem, but I was too intrigued to give up thinking about it. I decided to concentrate on the family of trees, and I managed to improve the known lower and upper bounds, and to find an infinite family of trees for which the bounds match. This family includes and can be viewed as a generalization of sorts of the family of paths of length 2^n (for which the dimensions were already known). I am in the process of writing a paper now and I gave a talk about these results at Midwest Graph Theory conference in Detroit.

Since I had to miss the first two weeks of meetings, I appreciated that the presentation of open problems wasn't limited to the first week or two. This made it easier to join (or start) a group, for me as well as for people who decided to change the problem they were working on during the summer. I also appreciated the daily reports that Dr. West compiled. They were extremely useful for people who had to miss a meeting, and after the program ended, they can serve as a database of accessible open problems.

I think that the program has been very effective in getting students to work together and teaching them how to communicate about research with other mathematicians. I think that the younger students (especially those who have some problems with English) are often too shy to approach other (mostly older and more experienced) students and start working with them. A program like this, with regular meeting time and someone going around and encouraging people, goes a long way towards removing this shyness and teaching students how to be more assertive when collaborating with others. Also, it helps students to establish a certain mindset that one needs for research (and which is different from the mindset that one needs to excel in classes). I think that this is a very effective way to help students make the transition towards research and funding younger students is certainly a good use of department's money.

Lesley W. Wiglesworth – University of Louisville

I found the Research Experience for Graduate Students (REGS) Program to be very beneficial. The program allowed me to meet other graph theory graduate students and work with people from other graduate programs. It also exposed me to a broad range of open graph theory problems.

While in Illinois, I worked with students from the University of Illinois to determine the game acquisition number of complete bipartite graphs. After returning to Louisville, I worked with a student from U of L to determine the relationship between a graph's hub number, connected hub number, and the size of the graph's minimum connected dominating set. Andre Kezdy, Adam Jobson, and I have also given a bound on the thickness of sphere of influence graphs.

Most recently, I have been working with Dr. Kezdy and Adam on determining a ratio between the thickness and arboricity of a graph when arboricity is large compared to the order of the graph. Though no results have yet been obtained on this problem, I would like to continue to work on this problem as we enter the Fall Semester.

As a graduate student, I really enjoyed working with other people to solve open problems. While working on my doctoral thesis, I had become accustomed to doing more independent work. Traveling to Illinois to participate in this program showed me how much fun and beneficial it can be to work with others.

Pratik Worah

REGS feedback and experiences: My work in REGS: I feel that REGS is a very nice idea and thanks for spending such a lot of time and effort with the students. I tried my hand on 4-5 problems (nos. 2,4,5,13,38) and szemerédi local lemma reading group in REGS. I got some results for 3 of the problems.

I got a negative result for the $F(2,2)$ problem which used the local lemma to show that any partition of the digraph which has outdegree ≥ 2 on all vertices has to have a high ratio (although constant) between Δ^+ and δ^+ . This ruled out most of the ideas of Prof. Kezdy which tried to construct some regular structure/design to prove $F(2,2)$ is infinite. This was about all the progress I made for the general case of this problem.

I also worked on the hub number problem and I sent you a report on that with Bill. I also worked on the acquisition number problem with Noah and others, you already have the report. I mainly worked on the following subproblems in it. Graph operations that effect acq. no., upper bounding the acq. no. of diam. 2 graphs, effect of absence of C_4 on the acq. no. (polarity and Moore graphs), and "A1" trees.

On the whole it was a nice to collaborate with people. I esp. enjoyed working through the details of the regularity lemma with you, Kevin and Steven. I will be attending a mini course on additive combinatorics this fall in Princeton which will stress on similar techniques.

I never got to give my suggestions about REGS so here they are: Please reduce the number of problems from 40 to about 20, in my opinion not every one needs to present a problem. It would be really nice if the students would first send their problem to you (a few days before) and based on potential and interest you would select the problems that should be presented. A few more questions which deal with probabilistic combinatorics, algebraic methods etc should be added besides the graph questions.

Once again thanks for so much of your time and interest during the summer.

Hehui Wu

Out of the REGS support, I was able to enroll in Professor West's Combinatorics seminar this past summer. It has been a wonderful research experience. We met three times a week. There are 40 interesting problems presented in the class, among which I presented two problems: (1) the *path partition number*, and (2) *A weakening of Equitable coloring*.

Most of the time during the summer, I studied the *weak equitable coloring* problem, and I discussed this problem with Gexin Yu and Qi Liu. Since our motivation to work on this problem came from Professor Kostochka's latest research in *A Short Proof of the Hajnal-Szemerédi Theorem on Equitable Coloring*, I spent a good amount of time studying the paper carefully before trying to propose a new way to improve the proof.

After observation and contemplation, I suggested a conjecture to work on the problem. The great thing is, if the conjecture is true, the condition will be sharp. However, after spending time trying to prove the conjecture, I found a counterexample to it. This frustrating result made us look far away from our destination, which is *there is a l -partition of V , such that every part has k vertices and has at most one edge*. Luckily, we did not

have to start all the work again. We proved a weaker result, which is *there are l disjoint parts of V , such that every part has $k - 1$ vertices and without edge*. And we can easily get a corollary from this, that is *there is a l -partition of V , such that there are totally at most k edges lay in the same partition*.

I have adjusted the conjecture a little bit. And now I am trying to prove it is true for specified degree sequences. At least, it looks true so far.

While working on these two problems, I have also spent quite a bit of time working on some other problems, such as the hub number problem. Several of us proved independently that $h_c(G) \leq h(G) + 1$. Since the late part of summer till now, I have been working the *Packing Steiner Trees problem*, with focus on the case when the graph is a bipartite graph.

Looking back over the summer, I have enjoyed a lot and learned a lot from the seminar. The 40 presented topics highly extended my knowledge of the research area. I especially enjoyed the time discussing problems with the visiting students from University of Louisville. It was an enlightening experience to have them join us and discuss problems with us. I hope we could have more chances like this to communicate with peers from other university in the future. Hopefully the stimulations from the discussions for research can carry on to bear fruit in the near future.

Thanks again to the department for the REGS funding for my summer research, which made all these thinking and learning possible. I appreciate your generous support. We hope that the coming academic year is fruitful as we progress toward the solution to the problems.