Recall these definitions (from [2]):

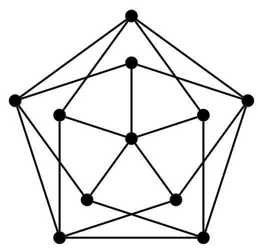
Definition (p. 116). In a graph G, a set $S \subseteq V(G)$ is a **dominating set** if every vertex not in S has a neighbor in S. The **domination number** $\gamma(G)$ is the minimum size of a dominating set in G.

Definition (p. 193). The **cartesian product** of G and H, written $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u,v) adjacent to (u',v') if and only if (1) $u = u'$ and $vv' \in E(H)$ or (2) $v = v'$ and $uu' \in E(G)$.

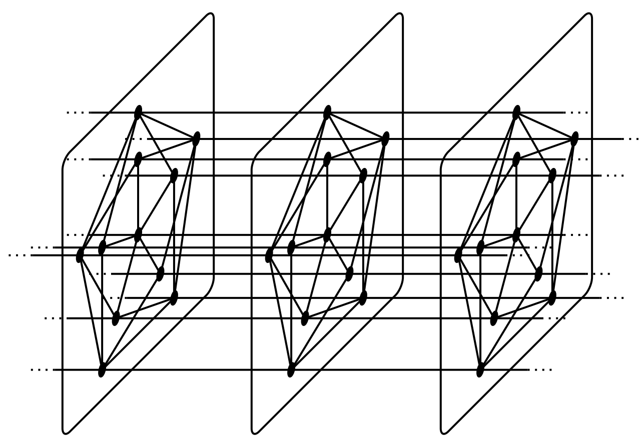
At the 2009 Cumberland Conference, Rubalcaba said that for the Petersen graph P and $k \geq 3$, $\gamma(P \,\square\, C_k) = 2k$. He also gave the following conjecture.

Conjecture. If $k \geq 3$ and $\ddot{G}$ is the Grötzsch graph then $\gamma\left(\ddot{G} \,\square\, C_k\right) = 2k$.

Update. A group at UofL proved this conjecture to be true with a construction to show that $2k$ always suffices and a counting argument to show it is also necessary.



The Grötzsch graph, $\ddot{G}$



$\ddot{G} \,\square\, C_k$

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Question. Let G be a graph with at least one edge and A the adjacency matrix of G. Is it true that there is a nonzero $\{0,1\}$-vector in the row space (over the reals) that is not a row of A?

This problem appears as problem 952 (BCC21.3) in the problems from the 21st British Combinatorial Conference [2]. The only other reference cited is [1] which is available at this web site: `http://designtheory.org/library/preprints/ranks.pdf`.

Notes:

1. It suffices to consider connected, reduced graphs (a graph is *reduced* if no two vertices have the same neighbors).
2. According to [2], the answer to the question is yes for graphs on at most 9 vertices and for line graphs.
3. Regular graphs obviously have the all ones vector in the row space, so the answer is yes also for these graphs.
4. The answer is also true for graphs with no induced P_4 (a.k.a. cographs, complement-reducible graphs) since Royle [4] proved that these graphs have the *rank property*:

   Definition. A graph has the **rank property** if the rank of its adjacency matrix equals the number of distinct nonzero rows.

   Connected, reduced graphs with the rank property must have full rank. In particular, the standard basis vectors are in the row space but are not rows of the adjacency matrix. Royle asks whether other natural families of graph have the rank property.
5. Costello and Vu [3] have shown that almost surely the rank of the random graph $G(n,p)$ equals the number of non-isolated vertices, for any $c\ln(n)/n < p < \frac{1}{2}$, where c is a positive constant. In particular, the giant component has full rank in this range which is strong evidence for an affirmative answer to the question.

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Recall the popular problem of determining the chromatic number of the plane, which can be found in [2].

Problem. Determine the minimum number of colors required to color all of the points of $\mathbb{C}$ so that no two points z and w with $d(z, w) = 1$ have the same color. This number is called the *chromatic number of the plane* and is denoted by $\chi(\mathbb{C})$ (Note: Most authors use $\chi(E^2)$ for our $\chi(\mathbb{C})$).

Theorem. $4 \le \chi(\mathbb{C}) \le 7$.

This problem was discussed at great length throughout the last century and still, it is fair to say that little progress has been made towards determining $\chi(\mathbb{C})$ exactly (at a recent conference on Ramsey Theory at DIMACS, Soifer conducted an informal survey that concluded $\chi(\mathbb{C}) \approx 4.833$). However, many other very interesting questions have arisen out of this investigation. In particular, it is believed that a countable subset of $\mathbb{C}$ exists with the same chromatic number as the entire complex plane although providing such a set remains an open problem that can be found in [2, p. 72].

Problem. Find a countable subset $C \subset \mathbb{C}$ with $\chi(C) = \chi(\mathbb{C})$.

It is well-known, due to Woodall [3], that $\chi(\mathbb{Q}^2) = 2$, so certainly the rational plane will not suffice. Such a set "certainly" exists in view of the following well-known theorem due to de Bruijn and Erdös (found in [2]). We will not prove this result here, although we mention that it assumes the axiom of choice.

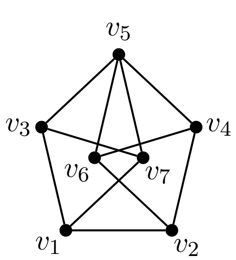
Theorem. The chromatic number of the plane is the maximum chromatic number of its finite subsets.

Let $\mathcal{A}$ denote the set of algebraic numbers, the subset of $\mathbb{C}$ whose members are the zeroes of polynomials with integer coefficients. It is not difficult to see that $\mathcal{A}$ is countable and that $\mathcal{A}$ forms a field. In fact, one can also show that $\mathcal{A}$ is closed under radicals. Lehmer [1] illustrates a well-known result that a trigonometric function of $r\pi$ is algebraic for any rational r. Using this fact, we can construct a Moser spindle with all algebraic coordinates, which yields the following result.

Theorem. $4 \le \chi(\mathcal{A}) \le 7$.

Conjecture. $\chi(\mathcal{A}) = \chi(\mathbb{C})$

It is relatively easy to see that $\chi(\mathcal{A}) \le \chi(\mathbb{C})$ since any satisfactory coloring of $\mathbb{C}$ is automatically a satisfactory coloring of $\mathcal{A}$. However, demonstrating the reverse inequality could be much more difficult, if such a statement even holds.



$$v_1 = (0,0) \qquad v_2 = (1,0) \qquad v_3 = \left(\tfrac{3-\sqrt{33}}{12},\ \tfrac{3\sqrt{11}+\sqrt{3}}{12}\right)$$

$$v_4 = \left(\tfrac{9+\sqrt{33}}{12},\ \tfrac{3\sqrt{11}+\sqrt{3}}{12}\right) \qquad v_5 = \left(\tfrac{1}{2},\ \tfrac{\sqrt{11}}{2}\right)$$

$$v_6 = \left(\tfrac{9-\sqrt{33}}{12},\ \tfrac{3\sqrt{11}-\sqrt{3}}{12}\right) \qquad v_7 = \left(\tfrac{3+\sqrt{33}}{12},\ \tfrac{3\sqrt{11}-\sqrt{3}}{12}\right)$$

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Definition. A *k-fold coloring* of a graph is a mapping that assigns k distinct colors to each vertex so that adjacent vertices are assigned disjoint sets of colors. The *k-fold chromatic number* of a graph G, denoted $\chi^{(k)}(G)$, is the minimum number of colors in a k-fold coloring of G.

Naturally, $\chi^{(1)}(G)$ equals $\chi(G)$, the chromatic number of G.

Definition. A k-fold coloring is *saturated* if every color occurs $\alpha(G)$ times, where $\alpha(G)$ is the maximum number of pairwise nonadjacent vertices of G.

Question. Is it true that, if every k-fold coloring a graph G is saturated, then

$$\chi^{(k+1)}(G) = \chi^{(k)}(G) + \chi(G)? \tag{1}$$

This problem appears as problem 952 (BCC21.5) in the problems from the 21st British Combinatorial Conference [1].

Notes:

1. It suffices to consider connected graphs.

2. It is easy to see that
$$(k+1)\omega(G) \leq \chi^{(k+1)}(G) \leq \chi^{(k)}(G) + \chi(G),$$
where $\omega(G)$ is the maximum number of pairwise adjacent vertices of G. It follows that (1) is true when $k = 1$ for graphs with $\omega(G) = \chi(G)$ (perfect graphs, for example).

3. The hypothesis that every k-fold coloring a graph G is saturated is necessary because $\chi(C_5) = 3$ but $\chi^{(2)}(C_5) = 5$ (also $\chi^{(3)}(C_5) = 8$).

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A game on a graph was defined by Nowakowski and Winkler [3]: the first player (the "cop") selects a vertex; the second (the "robber") then selects a vertex. The players then take turns selecting a new vertex adjacent to their old position. The cop wins if at any time both players have selected the same vertex. The robber wins if it is possible to evade for arbitrarily many turns. A graph G is called *cop-win* if the optimal play by both the cop and robber leads to a cop victory. Nowakowski and Winkler described a necessary and sufficient condition on the graph G for this property:

Theorem. A graph G is cop-win if and only if there is an ordering of its vertices $v_1, v_2, \ldots, v_n$ such that for every $i < n$, there if a $j > i$ such that

$$N(v_i) \cap \{v_i, v_{i+1}, \ldots, v_n\} \subseteq N(v_j) \cap \{v_i, v_{i+1}, \ldots, v_n\}$$

An extension of this game leads to the possibility of the first player controlling several independently mobile cops; a graph G is called *k-cop-win* if the first player can win a game in which there are k cops, and the *copnumber* (also known as the *search number*) of a graph $cn(G)$ is the least value of k for which the graph is cop-win. The Nowakowski-Winkler criterion clearly describes all graphs of copnumber 1, but other copnumber-related problems remain open. Hahn [2] presents two problems which are logical consequences of the Nowakowski-Winkler results:

- What is a necessary and sufficient condition for a *directed* graph to have copnumber 1?
- What is a necessary and sufficient condition for an undirected graph to have copnumber 2?

There are several established upper bounds for the cop number: Andreae [1] notes that for every graph H, graphs which lack a subgraph H as a minor have copnumber bounded above by a parameter of H. Quilliot [4] determined that if a graph G can be drawn on a surface of genus g, then $cn(G) \leq 2g + 3$. Schroeder [5] improved this bound to $\lfloor \frac{3}{2} g \rfloor + 3$, and believes this bound can be further reduced to $g + 3$.

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Let G be a graph with k vertices and m edges. Sidorenko offers the following conjecture concerning the Ramsey number $R(3, G)$ [5]:

Conjecture. $R(3, G) \leq k + m$

This conjecture grows out of questions concerning the Ramsey number of a triangle versus a graph of a given size. Sidorenko has shown that $R(3, G) \leq 2m + 1$, positively resolving a conjecture of Harary [2], [6]. A result of Erdos, et al. found in [1] can be used to show the Sidorenko conjecture to be true for graphs with fewer than $17k/15$ edges.

Bert Randerath asked at the recent DIMACS Ramsey theory workshop whether the conjecture might be be true for Turán graphs $T(k, w)$. By applying the following result concerning lower bounds on the independence number of triangle-free graphs it can be shown that the conjecture is true for all $T(k, w)$ with $\ln k \geq \frac{3w-1}{w-1}$.

Theorem (Shearer, 1983 [4]). Let G be a triangle free graph on n vertices with average degree d. Let α be the independence number of G. Let $f(d) = (d \ln d - d + 1)/(d - 1)^2$, $f(0) = 1$, $f(1) = \frac{1}{2}$. Then $\alpha \geq nf(d)$.

This proves the Sidorenko conjecture for all $T(k, w)$ with $k > 149$, some with $k > 20$, and has nothing to say about Turán graphs on fewer than 21 vertices. In fact this shows that any graph with k vertices and at least

$$\frac{k^2(k + 1 - 1/k - \ln k)}{k \ln k - k + 1} \sim \frac{k^2}{\ln k}$$

edges satisfies the Sidorenko conjecture. This relates directly to the bound $R(3, k) = \Theta(k^2/\ln k)$ found in [3].

Question: Can the probabilistic approach of [1] be adjusted to capture more (all) graphs with $O(k)$ edges?

Question: Is the conjecture true for regular graphs? This would show that the conjecture is true for some graphs of any given density.

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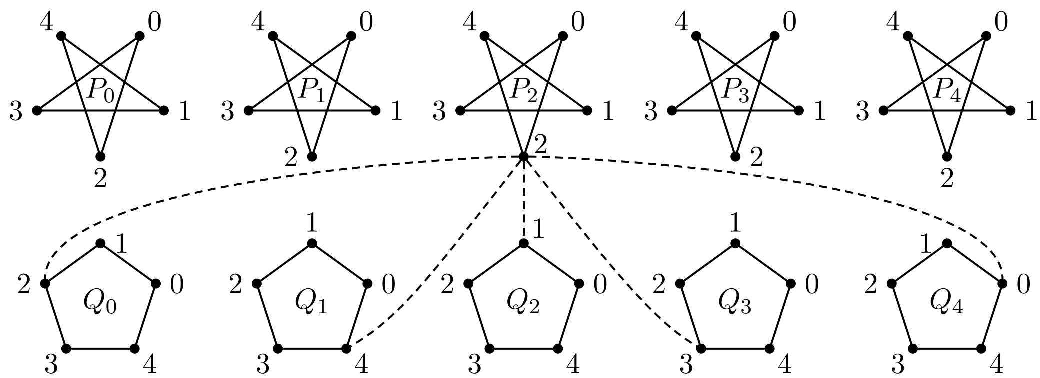
This problem was presented by Alexander Rosa in [2].

The Hoffman-Singleton graph is a $(7, 5)$-cage (that is, it is a 7-regular graph with girth 5 with the fewest possible number of vertices) that can be constructed as follows. Take five 5-cycles $P_0, \ldots, P_4$ and another five 5-cycles $Q_0, \ldots, Q_4$, so that vertex i of P_k is adjacent to vertices $i-2, i+2$ of P_k and vertex i of Q_k is adjacent to vertices $i-1, i+1$ of Q_k (all indices are modulo 5, naturally). Now join vertex i of P_j to vertex $jk+i$ (modulo 5) of Q_k (see below):



Problem. How many copies of the Hoffman-Singleton graph can be packed edge-disjointly into K_{50}?

Meszka and Šiagiová [3] used topological graph theoretic methods to pack five edge-disjoint copies of the Hoffman-Singleton graph into K_{50}. Their packing is maximal, in that no further copies the Hoffman-Singleton graph can be added to their packing. Consequently, the answer is either 5, 6, or 7.

There is a similar question for packing other cages into complete graphs. Many have shown that the maximum number of Petersen graphs ($(3, 5)$-cage) that pack into K_{10} is 2. Most interesting is the eigenvalue argument given by Schwenk [5] that proves that three copies of the Petersen graph can not be packed into K_{10}. It would very interesting to adapt this argument to prove that 7 copies of the Hoffman-Singleton graph can not be packed into K_{50}. Of course, other methods can also be considered. For example, T.S. Michael [4] has recently given a counting argument to prove that three copies of the Petersen graph can not be packed into K_{10}.

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GRÜNBAUM'S GIRTH PROBLEM — PROBLEM 8

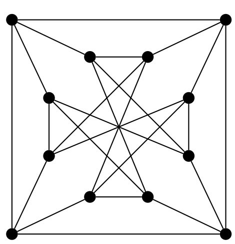
POSED BY BRANKO GRÜNBAUM — *presented by Adam Jobson*

Inspired by a theorem of Erdös that there exist graphs of arbitrarily high girth and chromatic number, Grünbaum [3] conjectured that for all $g \geq 4$ and $k \geq 3$ there exists a k-regular, k-chromatic graph of girth at least g. A few years later Kostochka [1] showed that for any triangle-free graph G, $\chi(G) \leq \frac{2}{3}(\Delta(G)+3)$, which implies Grünbaum's conjecture can only hold for $k \leq 6$. The $k = 2$ and $k = 3$ cases are true due to the existence of even cycles and cages, respectively; the $k = 4$, $k = 5$, and $k = 6$ cases are open. As noted in [4], the $k = 5$ and $k = 6$ cases could be shown impossible by a proof of the triangle-free version of Reed's Conjecture:

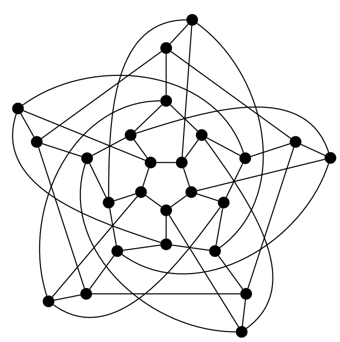
Conjecture (Reed). If G is a triangle-free graph then $\chi(G) \leq \frac{\Delta(G)}{2} + 2$.

Conjecture (Reed, general version). $\chi(G) \leq \left\lceil \frac{\Delta(G)+\omega(G)+1}{2} \right\rceil$ where $\omega(G)$ is the clique number of G.

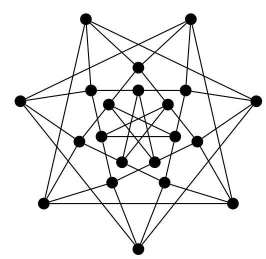
The $k = 4$ case of Grünbaum's problem holds for small g. The Chvátal Graph is 4-regular, 4-chromatic with girth 4, and Grünbaum himself presented a 4-regular, 4-chromatic graph with girth 5 on 25 vertices. The only other example many sources give is the Brinkmann graph, which is provably [2] the smallest 4-regular, 4-chromatic graph with girth 5.



The Chvátal Graph



The Grünbaum Graph



The Brinkmann Graph

Problem. Are there 4-regular, 4-chromatic graphs of higher girth?

Kostochka mentioned a similar problem at this year's Cumberland conference. I do not have any other information about it, but it has the same flavor and so is worth mentioning.

Conjecture. For all $g \geq 3$ there is a graph G with $\Delta(G) \leq 5$ and girth g such that G is not 3-colorable.

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In a classical result in Ramsey theory in 1916, Isaai Schur proved that for any $r \in \mathbb{N}$, there is a least positive integer S_r such that for every r-coloring of $\{1, \ldots, S_r\}$, there is a monochromatic solution to the equation $x_1 + x_2 = x_3$. Now consider a finite system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is a matrix and $\mathbf{b}$ is a column vector, all of whose entries are integers. Richard Rado, a student of Schur, called such a system r-**regular** if for every r-coloring of $\mathbb{N}$, there is a monochromatic solution to $A\mathbf{x} = \mathbf{b}$. If $A\mathbf{x} = \mathbf{b}$ is r-regular for every $r \in \mathbb{N}$, then the system is said to be **regular**. In his thesis (1933), Rado classified which systems of linear equations are regular.

If a system of equations $A\mathbf{x} = \mathbf{b}$ is not regular, then the degree of regularity is defined as the largest r such that $A\mathbf{x} = \mathbf{b}$ is r-regular. This number is denoted by $\mathrm{dor}_{\mathbb{N}}(A\mathbf{x} = \mathbf{b})$.

In [1], Bialostocki et al. studied the equation $x - 2y + z = b$, where $b \in \mathbb{Z}$. They proved that when b is odd, $\mathrm{dor}_{\mathbb{Z}}(x - 2y + z = b) = 1$, and when b is even and $b \not\equiv 0 \pmod 6$, $\mathrm{dor}_{\mathbb{Z}}(x - 2y + z = b) = 2$. Moreover, in the case that $b \equiv 0 \pmod 6$, they showed that $3 \leq \mathrm{dor}_{\mathbb{Z}}(x - 2y + z = b) \leq 7$. Fox and Kleitman showed that this lower bound is tight in [2] by exhibiting a 4-coloring of $\mathbb{N}$ without a monochromatic solution to $x - 2y + z = b$ when $b \equiv 0 \pmod 6$. This gives a specific example of the following result by Fox and Kleitman, which appears as a lemma in [2]:

Lemma. If b is a positive integer, then there exists a $2n$-coloring $c : \mathbb{Z} \to \{0, 1, \ldots, 2n - 1\}$ without any solutions to $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + b$.

Fox and Kleitman conjecture that the lemma is tight.

Conjecture. For any $n \in \mathbb{N}$, there exists $b_n \in \mathbb{N}$ such that the equation $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i + b_n$ is $(2n - 1)$-regular.

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