Summer 2009 REGS in Combinatorics — Final Report — D. B. West

Overview

Our REGS project in 2009 was quite large and quite successful. Varying amounts of progress was made on many problems, the the usual half-dozen or so will become papers submitted for publication. For a general description of the format and rationale of the program, see the reports from 2006 or 2005 or the proposal for 2010.

These reports appear at http://www.math.uiuc.edu/∼west/regs/index.html, the website for the Combinatorics REGS program. Also posted there are the proposed problems from 2009, 2008, and a few of those from 2007. This year most students wrote up the problems they presented for the web page. I edited these writeups.

In 2009, several new features were introduced. The biggest change was the REGS0 inclusion of four entering graduate students. They participated actively in all aspects of the program. Although they were missing occasional bits of terminology or familiarity with some basic results, essentially there were no impediments to their full participation, even though combinatorics was only a side interest for them. Only one of them took Math 580 this fall, although others plan to take combinatorics courses later.

Another change was bringing a funded junior faculty visitor to participate for 10 days. This was Joshua Cooper from the University of South Carolina. Such participation brings a significant outreach component to the program, as visitors learn how to run such programs at their own institutions, in addition to bringing diverse problems and perspectives on how to approach problems, exposing the students to different viewpoints from mine. Cooper's report, included below, indicates his enthusiasm about the experience.

Alumni of the program continued to return for visits, but a new wrinkle here was their bringing or sending students to participate. Seog-Jin Kim returned from Korea and brought two of his students. Michael Pelsmajer sent his research student Hong Liu from Illinois Institute of Technology. Liu was an active and sharp participant and has applied to enter our Ph.D. program.

Last year's difficulty of having several early easier problems attract large groups of interested students was not so strong this year. There was enough variety of accessible problems at the beginning that the collaborating groups remained of manageable size, and students felt comfortable participating in multiple groups.

There were several social events during the summer at my house or area restaurants.

Participants

The solicitation for REGS participants during the spring elicited quite a lot of interest, and the initial meetings of the summer had more than 30 participants. However, most of those who came from Electrical Engineering and some other disciplines melted away, and after a couple of weeks we were down to a large core group of students from mathematics and computer science and visitors. I have listed only the regular participants.

Funding was provided for REGS0 and REGS1 students. REGS2 and international students received less funding than such participants in other REGS groups, because allotments were made to the Combinatorics group to be divided among the eligible participants. Some of the senior students were supported by appointments as TAs, RAs, or graders.

There was difficulty finding an appropriate course number for the REGS students. Letter grades are inappropriate for participation in REGS. Eventually the literature seminar Math 598 was used, but there were other difficulties involving number of credit hours and an Instructor Approval requirement. Also, although students received tuition waivers, the university assessed fees. As a result, some international students had to pay almost as much or more in fees than they received in funding. This difficulty should be remembered for future years, since the academic credit is generally irrelevant for the students.

REGS0: Daniel McDonald, Nate Orlow, Greg Puleo, Ben Reiniger

REGS1: Matthew Yancey

REGS2: Kevin Milans, Bill Kinnersley, Jane Butterfield, John Lenz

International students (funded): Chayapa Darayon, Ping Hu, Suil O, Hehui Wu

Senior students: Ida Kantor, Mohit Kumbhat, Tim LeSaulnier, Lale Ozkahya, Chris Stocker, Evan Vanderzee, Paul Wenger

CS students: Alina Ene, Kyle Fox, Daniel Schreiber, Reza Zamani

Former students: Daniel Cranston, Hemanshu Kaul, Jeong-Hyun Kang, Seog-Jin Kim, Jeong-Ok Choi

Visiting students: Hong Liu, Hee-Jee Cho, Won-Jin Park

Other visiting faculty: Joshua Cooper, Stephen Hartke

Publications Progress

I have not previously reported the publications from past REGS. This may be a partial list, as I don't have the status of all the papers on which I am not a coauthor.

REGS 2004.

Bunde, David P.; Chambers, Erin W.; Cranston, Daniel; Milans, Kevin G.; West, Douglas B.; Pebbling and optimal pebbling in graphs. *J. Graph Theory* 57 (2008), no. 3, 215–238.

Cranston, Daniel W.; Sudborough, I. Hal; West, Douglas B.; Short proofs for cut-and-paste sorting of permutations. *Discrete Math.* 307 (2007), no. 22, 2866–2870.

Liu, Qi; West, Douglas B.; Tree-thickness and caterpillar-thickness under girth constraints. *Electron. J. Combin.* 15 (2008), no. 1, Research Paper 93, 11 pp.

Milans, Kevin G.; Clark, Bryan; The complexity of graph pebbling. *SIAM J. Discrete Math.* 20 (2006), no. 3, 769–798

Vandenbussche, Jennifer; West, Douglas B.; Yu, Gexin; On the pagenumber of k-trees. *SIAM J. Discrete Math.* 23 (2009), no. 3, 1455-1464.

REGS 2005.

Barrus, Michael D.; Kumbhat, Mohit; Hartke, Stephen G.; Graph classes characterized both by forbidden subgraphs and degree sequences. *J. Graph Theory* 57 (2008), no. 2, 131–148.

Bunde, David P.; Milans, Kevin G.; West, Douglas B.; Wu, Hehui; Parity and strong parity edge-coloring of graphs. *Proc. 38th Southeastern International Conference on Combinatorics, Graph Theory and Computing. Congr. Numer.* 187 (2007), 193–213.

Bunde, David P.; Milans, Kevin G.; West, Douglas B.; Wu, Hehui; Optimal strong parity edge-coloring of complete graphs. *Combinatorica* 28 (2008), no. 6, 625–632.

Choi, Jeong-Ok; Hartke, Stephen G.; Kaul, Hemanshu; Distinguishing chromatic number of cartesian products of graphs. Submitted.

Chambers, Erin W.; Kinnersley, Bill; Prince, Noah; Douglas B. West; Extremal problems for Roman domination *SIAM J. Discrete Math.* 23 (2009), no. 3, 1575-1586

Hartke, Stephen G.; Vandenbussche, Jennifer; Wenger, Paul; Further results on bar k-visibility graphs. *SIAM J. Discrete Math.* 21 (2007), no. 2, 523–531.

Liu, Qi; West, Douglas B.; Yu, Gexin; Implications among linkage properties in graphs. *J. Graph Theory* 60 (2009), no. 4, 327–337.

REGS 2006.

Balogh, József; Hartke, Stephen G.; Liu, Qi; Yu, Gexin; First-Fit chromatic number of planar and random graphs. Submitted.

Chambers, Erin W.; Kinnersley, Bill; Prince, Noah; Mobile eternal security in graphs. Submitted.

Choi, Jeong-Ok; West, Douglas B.; Decomposition of regular hypergraphs. Under revision.

Cranston, Daniel W.; Nomadic decompositions of bidirected complete graphs. *Discrete Math.* 308 (2008), no. 17, 3982–3985.

Cranston, Daniel W.; Kim, Seog-Jin; List-coloring the square of a subcubic graph. *J. Graph Theory* 57 (2008), no. 1, 65–87.

Cranston, D. W.; Korula, N.; LeSaulnier, T.; Milans, K. G.; Stocker, C.; Vandenbussche, J.; West, D. B.; Extremal problems for overlap number of graphs. Preprint.

REGS 2007.

Barrus, Michael D.; Hartke, Stephen G.; Jao, Fang-Kai; West, Douglas B.; Thresholds for graphic lists with and without gaps. Preprint.

Butterfield, Jane; Grauman, Tracy; Kinnersley, Bill; Milans, Kevin G.; Stocker, Christopher, West, Douglas B.; Online degree-Ramsey theory. In preparation.

Cranston, D. W.; Kim, S.-J.; Yu, G.; Injective Colorings of Sparse Graphs. Submitted.

Grauman, Tracy; Hartke, Stephen G.; Jobson, Adam; Kinnersley, Bill; West, Douglas B.; Wiglesworth, Lesley; Worah, Pratik; Wu, Hehui; The hub number of a graph. *Inform. Process. Lett.* 108 (2008), no. 4, 226–228.

Kantor, Ida; Prague dimension of trees. Submitted.

LeSaulnier, Timothy; Prince, Noah; Wenger, Paul; West, Douglas B.; Worah, Pratik; Acquisition number of graphs. Preprint.

Milans, Kevin G.; Stocker, Christopher; West, Douglas B.; Wiglesworth, Lesley; Game acquisition number of graphs. Preprint.

Prince, Noah; Wenger, Paul; Partial Acquisition in Graphs. Preprint.

Wu, Hehui; West, Douglas B.; Packing of S-trees. In preparation.

REGS 2008.

Balogh, József; Lenz, John; Wu, Hehui; On the independence number and clique minors. Submitted.

Busch, Arthur H.; Ferrara, Michael J.; Hartke, Stephen G.; Jacobson, Michael S.; Kaul, Hemanshu; West, Douglas B.; Packing of graphic sequences. Submitted.

Downey, Rod; Greenburg, Noam; Jockusch, Carl; Milans, Kevin G.; Binary subtrees with few path labels. Submitted.

Kinnersley, Bill; Milans, Kevin G.; West, Douglas B.; The degree-Ramsey number of graphs. In preparation.

Kostochka, Alexandr V.; Stocker, Christopher; Domination in cubic graphs. Submitted.

LeSaulnier, Timothy; Wenger, Paul; West, Douglas B.; Acyclic coloring of digraphs and degeneracy coloring of graphs. In preparation.

Milans, Kevin G.; Rautenbach, Dieter; Regen, Friedrich; West, Douglas B.; Cycle spectra of Hamiltonian graphs. Submitted.

REGS 2009.

Cooper, Joshua; Lenz, John; LeSaulnier, Timothy; West, Douglas B.; Uniquely C_4-saturated graphs. Preprint.

Cranston, Daniel W.; Kinnersley, Bill; Milans, Kevin G.; Puleo, Gregory; West, Douglas B.; Chain games on posets. In preparation.

Fox, Kyle; Kinnersley, Bill; McDonald, Daniel; Orlow, Nate; Puleo, Gregory; Spanning paths in Fibonacci-sum graphs. In preparation.

LeSaulnier, Timothy; Stocker, Christopher; Wenger, Paul; West, Douglas B.; Rainbow matchings in edge-colored graphs. Preprint.

Milans, Kevin G.; Schreiber, Daniel; West, Douglas B.; Acyclic sets in k-majority tournament. In preparation.

Yancey, Matthew; Monotone sequence games. In preparation.

Zamani, Reza; West, Douglas B.; Game domination number. In preparation.

2009 Results

This year the ratio of number of results produced to the number of actively studied problems seems to be lower than usual. There are several possible reasons.

It may be that the problems on average were harder than usual, that students were more interested in studying a variety of problems, that there were too many problems, or that the results obtained were not strong enough. In particular, some of the successful resolutions turned out to be counterexamples to conjectures made, which generally do not lead to publishable papers.

It may also be that some of the investigations are still evolving and will mature into papers later. As is evident from the reports, students obtained partial results on many problems not listed among the forthcoming papers above.

In any case, the sessions were quite lively, there definitely were results, and the program succeeded very well in its goal of involving beginning students in research.

To avoid duplication, I will not repeat here the problems and progress described in the participant reports below. Reports were received from most of the students who received funding and from a few others. In fact, most of the problems that received the most attention are described in the reports.

About the first 60% of the problems presented received substantial study during the summer. This is not surprising, since problems presented later were available for less time. Nevertheless, a few of the later problems also received substantial study.

The problems that received more thorough and more successful attention are described below, so it seems a bit off the mark to mention some of the odds and ends first, but here they are.

Boxicity and maximum degree. The *boxicity* of a graph G is the minimum number of interval graphs whose intersection is G. Introduced by Roberts in 1969, the parameter has received moderate attention over the years, but only recently was a bound proved in terms of the maximum vertex degree $\Delta(G)$. The boxicity of G was shown to be at most $2[\Delta(G)]^2$, but it is conjectured that the upper bound should be linear in $\Delta(G)$. Evan Vanderzee thought about this problem, computing boxicity on a few classes of graphs.

Domination game. A *dominating set* is a vertex subset S such that every vertex outside S has a neighbor in S, and $\gamma(G)$ denotes the minimum size of a dominating set in a graph G. In a game version of the problem, Dominator and Staller alternately choose vertices to add to a set that will eventually dominate the graph. Each move must increase the set that has been dominated, and the game ends when no further move is possible. Dominator wants to minimize the size of the chosen dominating set at the end; Staller wants to maximize it. The *game domination number* $\gamma_g(G)$ is the result of optimal play, when Dominator plays first. Always $\gamma(G) \le \gamma_g(G) \le 2\gamma(G) - 1$.

Initial study of the game focused on the difference when Staller moves first, and the students added some to what is known about this. As a result of these discussions, Reza Zamani has adopted this problem as one of his thesis problems. The examples given originally for sharpness of $\gamma_g(G) \le 2\gamma(G) - 1$ had many leaves; he has constructed examples with large minimum degree where this still holds. Also of interest are computations of $\gamma_g(G)$ on special families of graphs. These involve giving strategies for Dominator and Staller to prove upper and lower bounds.

Laborde–Payan–Xuong Conjecture. Laborde, Payan, and Xuong conjectured in 1982 that every digraph has an independent set S of vertices (pairwise nonadjacent vertices) that intersects every longest path. A stronger form is that S can be chosen so that each vertex of S is the initial vertex in some longest path. Both trivially hold when the digraph has a spanning path. The conjectures hold when the longest paths are very long or very short, when the underlying graph is bipartite, etc. A group of REGS students including Jane Butterfield, Tim LeSaulnier, Ben Reiniger, Chris Stocker, Paul Wenger, and Matt Yancey obtained a variety of new partial results on the problem. These may develop into a paper later. Notable is that this was one of the last few problems presented, yet it still received substantial attention.

Participant Reports (alphabetical order)

These reports have been lightly edited to achieve a consistent format, eliminate some duplication, etc. They contain descriptions of many of the problems and statements of progress made. They also summarize participants' (strongly positive) subjective impressions of the program and its benefits.

One surprising item mentioned by several of the students is that making blackboard presentations helped them learn how to use the blackboard instead of making the computerized slide projections they are used to (how quickly technology swallows the new generation!).

Joshua Cooper – Visiting Faculty (U. South Carolina)

From July 13 to July 23 of 2009, I attended and participated in UIUC's Research Experience for Graduate Students (REGS) program, under the direction of Prof. Doug West. My attendance was profitable in several respects. I learned about the mechanics of REGS and met a host of clever graduate students, which gave me the opportunity to work on some interesting mathematics and even make progress on a problem that many found worthwhile. That UIUC has such a large group of strong students invested in discrete mathematical research makes it an ideal setting for this program.

REGS, as I observed it, provides a valuable experience for the students who participate. It teaches them collaborative skills they would not be exposed to in classroom settings, and which they otherwise may only see one aspect of when working with a dissertation adviser. The setting is gently competitive and flexible to a variety of communication styles, pushing students to talk to one another in productive ways about real problems. The requirement that students present a problem (and discuss it before a friendly audience) has the potential to teach them better presentation skills and ease them into public speaking. Naturally, the students learn mathematics at REGS as well, from each other, from the supervisory faculty, and from the research they do while investigating problems. In summary, the pedagogical structure of REGS was clearly advantageous as a mode of mathematical socialization.

Working with groups of graduate students on open problems is something I have never done, and it was therefore a fast education in the challenges and satisfaction gained in the use of supervisory skills. It also gave me greater insight into the types of problems that graduate students might find accessible or exciting. The experience left me wishing I had been exposed to something similar as a student – and wondering how I could bring this sort of program to the University of South Carolina, a math department with a strong cohort of students studying discrete mathematics. Other junior faculty members of my home department could benefit from the conversational atmosphere fostered by REGS. I also learned a good deal of mathematics (mostly graph theory) from my various interactions with students and faculty. (In truth, there were so many mature and sharp graduate students that I hesitate to single out any one.) During the course of my stay, I found that a problem (concerning *uniquely C_4-saturated graphs*) that I brought with me generated considerable attention amongst the students. We made some progress in our conversations, and I am hopeful that the problem can be finished off at some point, perhaps resulting in a publication.

My time may have been even more productive if I had been able to meet with the students more. One possible route for ensuring collaboration with visitors outside of the main REGS meetings (Mondays, Wednesdays, and Fridays) is to formalize scheduling group research times for Tuesdays and Thursdays. Then the visitor could drop in on these meetings and help to guide the students' work.

Chayapa Darayon

It was indeed a right decision to participate in the Combinatorics REGS group. Throughout the program, I have improved important research skills from the group discussion and developed interest in particular problems that can be worked on even after the program ended.

Working in groups with people of various backgrounds benefited me greatly. First of all, it broadened my perspective on how I perceived a problem. At the beginning I tended to solve problems as they were stated originally and did not quite know how I should put some relaxation on problems that might help in acquiring the answers or at least partial solutions. But with help and guidance from the students with more experience in the field, I became more flexible on how I proceeded, and more comfortable to work on a variety of each problem. In addition, during the group discussion, many ideas were presented and some of them were ones that I had never thought they could be used in solving a particular problem before. I learned a great deal from this and tried to apply as many methods studied in the previous course as I could.

During the first month, the problems that interested me are bichromatic coloring, lucky labeling and spanning paths in Fibonacci-sum graphs. For the bichromatic coloring problem, our conjecture was that a graph is (2,2)-colorable if and only if the graph itself and its complement do not contain three odd cycles as an induce subgraph. We believed that this is a sufficient condition as we could not find any counterexample yet; the proof is not known still. In the lucky labeling problem, I was working on a bipartite graph by modifying an algorithm from a tree case. In the first half, I did not discover a major result, only some contribution during the group discussion. In the second half of the program, I mainly worked on the Hoffman and Ostenhof conjecture about spanning trees in 3-regular graphs. We have proved that the conjecture is true in the case where a graph is Hamiltonian or is a 2-factor with two components, i.e. it can be decomposed into two disjoint cycles and a matching. We had several ideas on how we might prove this conjecture. Some of them were to generalize and possibly use induction for a graph which is 2-factor with more than two components, to reduce a path of length two to a matching or cycles, or two prove a conjecture on a subcubic graph instead. After the REGS program, I plan to continue working on the bichromatic coloring, lucky labeling and spanning trees in 3-regular graphs problems.

Overall I like how the group discussion was conducted. Having each student presented a problem is a good way to learn how to find a problem to work on and make the topics diversified, although most problems I found interesting during this REGS program were presented by the professors. However, since the program was in a two-month period and not all the participants stayed for the whole time, it might be better if the progress report was divided into two: one after a month and another at the end of the program. By doing so, people who have to leave early get a chance to present their solutions and listen to other ones as well. Besides, this might also prevent the students from forgetting the result they obtain early in the program or the list of colleagues who worked on the problem together, and reduce the time in explaining the definition all over again during the presentation.

Participating in the Combinatorics REGS group was a valuable experience for me. Not only did it enhance my research skill, it also greatly stimulated my curiosity to study more graph theory and assured my decision in choosing combinatorics as my major field of study. If time allowed, I wish to continue participating in the program in the following year as well.

Kyle J. Fox – Computer Science

Participating in the REGS group in combinatorics was certainly a worthwhile experience. One of the most beneficial aspects of REGS was the sheer number of problems presented and people present guaranteed I could find a variety of problems that interested me. Giving a presentation was also worthwhile as it helped me rethink how to phrase my arborally satisfied set problem while practicing giving a "board talk".

I am not sure what recommendations to give to improve the program. It seems there is a good balance between time spent presenting problems and results and time spent working on problems in small groups, although I am a little concerned about what might happen with the current setup if the number of participants grows in future years. The time spent just talking about problems might interfere with time spent working on problems.

I will give a couple results that I was involved in. The first is a positive result involving the Fibonacci-sum graph problem presented by Gregory Puleo. The second is a negative result involving the arborally satisfied point set problem I presented myself.

Fibonacci-sum graphs. The *Fibonacci numbers* are the solutions to the recurrence $F_i = F_{i-1} + F_{i-2}$ with initial values $F_0 = F_1 = 1$. The *Fibonacci sum graph on* $[n]$, denoted G_n, is the graph with vertex set $[n]$ and edge set $\{uv\colon u + v = F_i \text{ for some } i\}$.

Bill Kinnersley, Daniel McDonald, Nate Orlow, Gregory Puleo, and I managed to prove a few facts about Fibonacci-sum graphs. The following theorems may be considered the most interesting of our results.

1) For each natural number k, G_{F_k} has a spanning path.

2) Fix k. If $k \not\equiv 0 \pmod 3$, then G_{F_k} has exactly one spanning path. Otherwise, G_{F_k} has exactly two spanning paths.

3) G_n has a spanning path if and only if $n \in \{F_i, F_i - 1\}$ for some i, or $n = 9$, or $n = 11$.

The first theorem is surprising because the Fibonacci-sum graphs are rather sparse. They have other interesting structural properties that allowed us to find a proof, though. This theorem also has the interesting consequence that for any k it is possible to list the members of $[F_k]$ in such a way that any adjacent pair of values in the list sum to a Fibonacci number. Proving this fact about lists was the original motivation behind defining the Fibonacci-sum graphs. This is a generalization of Problem 2732 in J. Recreational Mathematics, Jan. 2009 which asked for such a list of the members of [34].

Arborally satisfied point sets The definitions originated from Erik D. Demaine, Dion Harmon, John Iacono, Daniel Kane, and Mihai Pătrascu, The geometry of binary search trees, *SODA '09: Proc. 20th ACM/SIAM Symp. on Discrete Algorithms* (2009), 496–505.

Given two points a and b in a point set P in $\mathbb{Z}^2$, let $[a, b]$ denote the axis-aligned rectangle with corners a and b. Say that a pair of points a and b in a point set P are *arborally satisfied* if a and b are in the same row or column, or if $P - \{a, b\}$ has at least one point in $[a, b]$. Say that P is arborally satisfied if all pairs of points in P are arborally satisfied. Given an $m \times n$ grid and a point set X on the grid with exactly one point per row, we seek a smallest superset Y of X such that all pairs of points in Y are arborally satisfied (by Y). Let $f(X)$ denote the minimum number of points that must be added to X to obtain an arborally satisfied set Y.

Computing $f(X)$ exactly corresponds to finding an optimal dynamic binary search tree algorithm. Demaine et al. proposed the following algorithm to find an upper bound. Starting with point set X, sweep the grid using a horizontal line, iteratively increasing the height. At time i, the algorithm places the uniquely defined minimal set of points in row i that makes the point set from the bottom to row i arborally satisfied. Let $g(X)$ be the number of points added by this algorithm.

Demaine et al. gave another algorithm to find a lower bound on $f(X)$. Starting with point set X, sweep the grid using a horizontal line, iteratively increasing the height. At time i, the algorithm places the uniquely defined minimal set of points in row i to arborally satisfy all pairs of points a and b that form the bottom-left and upper-right corners of a rectangle with a in row i. Let $g^+(X)$ be the number of points added by this algorithm. $g^-(X)$ is defined similarly, changing the slope of the line from b to a for the pairs of points to be satisfied.

It is known that $f(X) \geq g^+(X)$ and $f(X) \geq g^-(X)$. Jeff Erickson and I had conjectured that $g(X) \leq g^+(X) + g^-(X)$, hoping this would lead to $g(X) \leq 2f(X)$. Nate Orlow, Evan Vanderzee, and I found a counterexample to this conjecture during REGS, but first we had a lot of practice finding minor facts about the point sets placed by the above algorithms. Hopefully this will be useful in future attempts to find good approximation algorithms for the problem of creating minimal arborally satisfied point sets.

Ping Hu

This is the first time I attend REGS and when I was an undergraduate I had little experience in research. So at the begining I tried several problems and listened to others to learn how to do research. Fortunately I found although some problems are difficult the REGS itself is not as difficult as I thought before. The REGS experience helps me a lot in the development of my research skills.

After listening to and thinking about several problems, I focus on the conjecture that every 3-regular graph G has a spanning tree T such that $G - E(T)$ consists of isolated vertices, isolated edges, and cycles.

I discussed this conjecture with Chayapa, Hong, Hehui, and Evan.

We have two main results. (1) The conjecture is true for Hamiltonian graphs. (2) The conjecture is true for 3-regular graphs which can be decomposed into matchings and two cycles. But I don't think the same method can be generalized to solve other cases.

We also have some partial results. (1) The conjecture is true if and only if it is true for 3-regular triangle-free graphs. (2) Every 3-regular connected graph has a cycle such that after deleting edges of this cycle, the graph is still connected.

Timothy D. LeSaulnier

During the summer of 2009 I participated in REGS, designated as MATH 598 REW, under the guidance of Professor Douglas West.

The aim of the REGS in which I participated was to research open problems in combinatorics. To this end, many open problems were presented by participants and visitors. Once problems were presented the participants would discuss their ideas in small groups.

I found the exposure to open problems and the chance to collaborate with other participants in this summer's REGS beneficial. Professor West was excellent at encouraging research and matching people who were working on the same or similar problems. I would like to participate in REGS in the future and would suggest doing so to any graduate student looking for exposure to research topics in any area.

Jones' Conjecture. I presented a conjecture attributed to Chuan-Min Lee, known as Jones' Conjecture, which has relations to another well known conjecture of Ryser. Both conjectures can be phrased in terms of packings and transversals of hypergraphs.

A *hypergraph* is a generalization of a graph in which edges can contain any number of vertices. The *packing number* of a hypergraph H, denoted $\nu(H)$, is the maximum number of pairwise disjoint edges in H. A *transversal* of a hypergraph is a vertex subset W containing a vertex of each edge. The size of the smallest transversal of a hypergraph is its *transversal number*, denoted $\tau(H)$. A hypergraph is *r-uniform* if every edge has r vertices, and it is *r-partite* if the vertices consist of r disjoint sets such that no edge has more than one vertex in any set.

Originally appearing in the thesis of his student J. Henderson in 1971, Ryser's Conjecture states that if H is an r-uniform r-partite hypergraph, then $\tau(H) \le (r-1)\nu(H)$. This conjecture is a generalization of the classical König-Egerváry Theorem, which states the same when $r = 2$. Ryser's Conjecture was proved for $r = 3$ in R. Aharoni, Ryser's conjecture for tripartite 3-graphs, *Combinatorica* 21 (2001), no. 1, 1–4.

Jones' Conjecture asks if a similar bound holds in a different class of hypergraphs. The *cycle hypergraph* H of a graph G is the hypergraph whose vertex set is the same as the vertex set of G and whose edges are the vertex sets of cycles in G. Jones' Conjecture states that if G is a planar graph then $\tau(H) \le 2\nu(H)$, where H is the cycle hypergraph of G. This inequality is known to hold for outerplanar graphs, but for arbitrary planar graphs only $\tau(H) \le 5\nu(H)$ is known. The conjecture and these results appear in Kloks, Ton; Lee, C. M.; Liu, Jiping; New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs. *Proceedings of the 28th International Workshop on Graph-Theoretic Concepts in Computer Science (WG2002)*, LNCS 2573, pp. 282-295.

Rainbow matchings. I also discussed other problems with the other participants, including a problem due to Wang and Li presented by Professor West. In an edge-colored graph, a subgraph is *rainbow* if its edges have distinct colors. A *star* is a graph whose edges each contain a fixed vertex. The *color degree* of a vertex v in an edge-colored graph is the size of the largest rainbow star centered at v. Given an edge-colored graph G, let $\hat{\delta}(G)$ denote the smallest color degree of its vertices. A *matching* is a graph whose edges are pairwise disjoint; let $\hat{\alpha}(G)$ be the maximum size of a rainbow matching in an edge-colored graph G. Wang and Li conjectured that, with one exception, if G is an edge-colored graph with $\hat{\delta}(G) = k$, then $\hat{\alpha}(G) \geq \lceil k/2 \rceil$. The exception is a properly 3-edge-colored K_4 (complete graph with four vertices). Wang and Li showed that $\hat{\alpha}(G) \geq \lceil (5k-3)/12 \rceil$ (see Wang, Guanghui; Li, Hao; Heterochromatic matchings in edge-colored graphs. *Electron. J. Combin.* 15 (2008), no. 1, Research Paper 138, 10 pp). This summer we showed that $\hat{\alpha}(G) \geq \lceil k/2 \rceil$ whenever G is triangle-free, G is properly edge-colored, G has more than $5k/4$ vertices, or $k \leq 12$. [Editorial Note: After the end of REGS, LeSaulnier and West improved the bound to $\hat{\alpha}(G) \geq \lfloor k/2 \rfloor$, so the conjecture is nearly proved.]

Uniquely C_4-saturated graphs. Another problem I discussed with other participants was presented by Josh Cooper, a professor visiting from the University of South Carolina. The graph C_4 is a cycle of length 4. A graph G is *C_4-saturated* if G does not contain any cycles of length 4 but the addition of an edge joining any two nonadjacent vertices of G yields a graph that does contain a 4-cycle. A graph is *uniquely C_4-saturated* if it is C_4-saturated and the addition of any edge joining nonadjacent vertices of G yields exactly one 4-cycle. We characterized the uniquely C_4-saturated graphs; there are none with ten or more vertices.

Hong Liu – Visiting Student

I'm a graduate student in Applied Mathematics Department of Illinois Institute of Technology. First I would like to thank Professor West for letting me join REGS. I like this program and learn a lot from it. I used to think that research is big and I haven't learned enough yet. But after this I know that the right thing to do is just get started. Many good problems were presented. After sinking into some of them deeply, I was required to learn more. I think this "just get started" experience is the most valuable thing I learned from REGS. And I like the group study because discussing with other students is not only very helpful when I get stuck but also making the research more fun.

At the beginning, I started with the tree-thickness problem. For a hereditary family F, the F-thickness of a graph G, written $\theta_F(G)$, is the minimum number of subgraphs in a decomposition of G using only subgraphs in F. The conjecture is $\theta_T(G) \leq \lfloor \frac{n}{4} \rfloor + 1$ for any connected triangle-free graph with girth ≥ 4. Qi Liu and Professor West proved it's true when the graph contains no subdivision of $K_{2,3}$. Use basically the same idea, I think it can be proved for a graph with no subdivision of $K_{2,4}$.

Then I got stuck there and was attracted by the covering numbers and hypergraph transversals problem. The covering number $C(n, s, t)$ is the minimum number of s-sets in $[n]$ needed to cover every t-set in $[n]$. The transversal number $\tau(H)$ of a hypergraph H is the minimum size of a set of vertices of H that intersects every edge of H. Chvátal and McDiarmid [CM] proved that if H is an r-uniform hypergraph with n vertices and m edges, then $\tau(H) \leq \frac{\lfloor \frac{r}{2} \rfloor m+n}{\lfloor \frac{3r}{2} \rfloor}$. To relate the Chvátal–McDiarmid Theorem to covering numbers, set $\tau(H) = t+1$ and solve for m; this yields a lower bound on the number of r-sets needed to have every t-set avoided by some such r-set: $C(n, n-r, t) \geq t+1+\frac{r(t+1)-n}{\lfloor \frac{r}{2} \rfloor}$. And when

$\frac{3}{4}r(t+1) \leq n \leq r(t+1)$, the bound is optimal.

At the beginning we wanted to find the optimal bound when $\frac{2}{3}r(t+1) \leq n \leq \frac{3}{4}r(t+1)$ by improving the upper bound on the transversal number. It turns out really hard to get a better upper bound for $\tau(H)$. We later noticed that in order to improve the upper bound of covering number, we need to find a design such that the edge-loss ratio with respect to n, $(\frac{\Delta m}{\Delta n/r})$ should be as small as possible. For instance, when $n/[r(t+1)]$ is in the interval $[3/4, 1]$, we trade 2 copies of disjoint r-sets for one block design $(3, 2, 1)$, so lose $r/2$ vertices and get 1 more edge, the ratio is 2. This ratio has been proven to be optimal. For the next range we want to trade 3 copies of $(3, 2, 1)$ for 2 copies of $(4, 2, 1)$, the ratio is 6. It turns out there are three designs of the same ratio [table omitted - ed.].

If there are no other designs with ratio less than 6, then we would trade 5 copies of $(3, 2, 1)$ for 2 copies of $(9, 3, 1)$ and $C(n, n-r, t) \geq \frac{3}{2}(t+1) + \frac{3[\frac{3}{4}r(t+1)-n]}{r/2}$ would be optimal when $\frac{3}{5}r(t+1) \leq n \leq \frac{3}{4}r(t+1)$. But it's still hard to prove 6 is the best ratio we can have. Consider a (v, b, s, k, λ)-design B with transversal number τ, where s is the number of appearances of each element. We trade τ copies of $(3, 2, 1)$ for two copies of B to keep the transversal number the same. The number of elements changes from $\frac{3}{2}r\tau$ to $2v\frac{r}{k}$: reduce by $r(3\tau/2 - 2v/k)$ The number of edges changes from 3τ to $2b$: increase by $2b - 3\tau$.

We want $(2b - 3\tau)/(3\tau/2 - 2v/k) < 6$, which is hard because there are too many designs and for most of them we do not know their transversal number. By some calculations, we can show that no Steiner triple system, projective plane, or affine plane does better than $(9, 3, 1)$. So we are now thinking to prove the optimality on a small range using a design of ratio 6. Then use linearity to extend the range to $\frac{3}{5}r(t+1)$. For the bound $C(n, n-r, t) \geq \frac{3}{2}(t+1) + \frac{3[\frac{3}{4}r(t+1)-n]}{r/2}$, it would suffice to prove that $\tau(S) \leq t$ whenever $S \subset \binom{[n]}{r}$ and $|S| = t + k$, where $k = \frac{t+1}{2} + \frac{3[\frac{3}{4}r(t+1)-n]}{r/2}$.

The problem is that using this idea requires $n < r(t+1) - (k-1)\frac{r}{2}$ to get started; that is, $n > \frac{3}{4}(t+1)r - \frac{r}{4}$. Among the three designs $(4, 2, 1), (4, 3, 2)$ and $(9, 3, 1)$, only $(4, 3, 2)$ requires less than $r/4$ room $(r/6)$ for one transformation. That means the bound is optimal on the range $[\frac{3}{4}(t+1)r - \frac{r}{4}, \frac{3}{4}(t+1)r]$. One transformation is not enough and the range is too small to apply linearity. So we are now seeking a big enough range to apply linearity.

If $(9, 3, 1)$ is the best design, then after that the best one I can find so far is $3-(8, 4, 1, 14, 7)$ (a t-design with $t = 3$) on the range $[\frac{1}{2}(t+1)r, \frac{3}{5}(t+1)r]$, which is of ratio 11. This is better than the $(5, 3, 3)$ design with ratio 21.

Daniel McDonald – REGS0

My experience in the combinatorics section of the REGS program was a positive one. Having taken part in two REUs as an undergraduate, the REGS program seemed like the natural step up for a summer math research program for graduate students.

There were several aspects of REGS that I had not experienced in my previous mathematical research. Students had to actually go out and find an open problem and present it, rather than just have one given to them. Furthermore, participants were for the most part responsible for structuring and monitoring progress on their own research projects because there were so many different projects going on. Because there were so many projects, participants could also work in several groups, which proved beneficial when progress stalled on a certain problem or when a problem was completely solved.

Having come from a more enumerative rather than graph theoretical background in combinatorics, I did notice that the vast majority of the problems posed were in graph theory. It would have been nice to see a bit wider variety of problems.

As for what projects I worked on, I bounced around between projects the first few sessions, then worked a bit on the Fibonacci-sum graphs problem (which was solved quickly and given an informal write-up by multiple people), wrote a computer program to see if it could provide meaningful progress in coloring the natural numbers with two colors so that no Pythagorean triple would be monochromatic (little progress was made), and spent most of my time working on the problem I presented, which was to look at necessary and significant conditions for certain permutations (specifically, a particular set of discrete analogs of interval exchange transformations) to be single cycles. I found and proved several necessary conditions. Through a program I wrote, I have made some conjectures about much stronger necessary conditions that I hope to prove in the few days before I turn in my longer report. [Ed. - I seem to have accidentally deleted McDonald's longer report.]

Kevin G. Milans

This summer, I again participated in Professor Douglas West's combinatorics research group. Professor West's research group continues to provide an excellent environment where graduate students of all levels of experience grow academically. Working groups discussed open problems, possible approaches to a problem, and ultimately, solutions.

At the beginning of the program, each student presents an open problem to the group. The student becomes familiar with the relevant journals, has a chance to practice speaking before a knowledgeable but informal audience, and is introduced to the group. Simultaneously, the group is enriched by a collection of open problems to attack throughout the summer. If students are not able to make satisfactory progress on one problem, there are always others available. As the summer progresses, the emphasis shifts from the presentation of open problems to the student groups who work on solutions. As the student groups exchange ideas about their chosen problems, Professor West visits each group to offer advice, new ideas, and evaluation of student proofs.

This summer, I focused on two projects. The first concerns a family of directed graphs, and the second considers two combinatorial games on partially ordered sets.

Acyclic Subsets of Majority Tournaments This section describes joint work with Daniel Schreiber and Douglas B. West. A *tournament* is an orientation of a complete graph. We use the notation $[n]$ for the set $\{1, \ldots, n\}$. Given a set $\{\pi_1, \ldots, \pi_{2k-1}\}$ of permutations of $[n]$, the resulting *majority tournament* is the tournament D on vertex set $[n]$ where $ab \in E(D)$ if at least k of the permutations π_i satisfy $\pi_i(a) < \pi_i(b)$. A tournament arising in this way is a *k-majority tournament.* Alon et al. introduced majority tournaments and showed that there is a constant c such that $\gamma(D) \leq ck \log k$ for every k-majority tournament D, where $\gamma(D)$ is the minimum size of a vertex set dominating every vertex outside that set. They also constructed k-majority tournaments $\{D_k\}$ such that $\gamma(D_k) \ge c'k/\log k$ for some constant k (see N. Alon, G. Brightwell, H. A. Kierstead, A. V. Kostochka, P. Winkler, Dominating sets in k-majority tournaments, *J. Combin. Th. (B)* 96 (2006), 374-387.)

This summer, we studied the behavior of a different parameter for k-majority tournaments. A digraph is *acyclic* if it contains no directed cycle; a set of vertices is *acyclic* if it induces an acyclic subdigraph. If D is an n-vertex acyclic tournament, then the vertices of D can be indexed as $v_1, \ldots, v_n$ so that $v_iv_j \in E(D)$ when $i < j$; hence there is exactly one acyclic tournament on n vertices up to isomorphism. When D is a digraph, let $\alpha_c(D)$ denote the maximum size of an acyclic set in D.

A well-known result in Ramsey theory establishes a constant c such that if D is a tournament on n vertices, then $\alpha_c(D) \ge c \log n$. Also, there is another constant c' such that if D is chosen uniformly at random from all n-vertex tournaments, then with high probability $\alpha_c(D) \le c' \log n$. Hence, in almost every n-vertex tournament the largest acyclic sets have logarithmic size.

The situation is quite different for k-majority tournaments. First consider 2-majority tournaments. The Erdős-Szekeres Theorem states that every permutation of $[n]$ contains a monotone subsequence of length $\sqrt{n}$. Given three permutations π_1, π_2, π_3 generating a 2-majority tournament D, we may assume by relabeling that pi_1 is the identity permutation. By Erdős–Szekeres, there is a sequence $a_1 < \cdots < a_t$ with $t \ge \sqrt{n}$ such that $\pi_2(a_1) < \cdots < \pi_2(a_t)$ or $\pi_2(a_1) > \cdots > \pi_2(a_t)$. In the first case, $\{a_1, \ldots, a_t\}$ is already acyclic in D. If the subsequence is decreasing in π_2, then the order of these elements in π_3 prevails, and again the set is acyclic. Thus $\alpha_c(D) \ge \sqrt{n}$. This bound is best possible up to a multiplicative constant; we also construct an infinite family of 2-majority tournaments in which $\alpha_c(D) \le 2\sqrt{|V(D)|}$.

We also studied k-majority tournaments for larger k. If D is an n-vertex 3-majority tournament, then $\alpha_c(D) \ge n^{1/4}$. Because adding a permutation and its reverse does not change the majority digraph, the construction for 2-majority tournaments yields a family of k-majority tournaments with $\alpha_c(D) \le 2\sqrt{|V(D)|}$ for each $k \ge 2$. We are presently searching for constructions of 3-majority tournaments in which $\alpha_c(D)$ is asymptotically smaller than $\sqrt{|V(D)|}$. For general k, we proved that $\alpha_c(D) \ge n^{1/3^{k-1}}$ when D is an n-vertex k-majority tournament.

Games on Posets. This section describes joint work with Daniel W. Cranston, Bill Kinnersley, Gregory Puleo, and Douglas B. West. We studied two games on partially ordered sets. The first, called the *set Maker-Breaker game*, is played between Maker and Breaker, who alternate moves; Maker goes first. Each move consists of selecting a previously unselected element from P. Maker seeks to maximize the size of a longest chain among the set of elements that Maker has selected, and Breaker tries to limit this size. The second game, called the *sequence Maker-Breaker game*, is similar except that Maker's chain must be consistent with the order in which Maker selected the elements. For example, if $x < y < z$ is a chain in P and Maker selects x followed by z followed by y, then Maker has obtained a chain of length at least 3 in the set Maker-Breaker game, but only gets credit for the chains $x < z$ and $x < y$ in the sequence Maker-Breaker game.

We study the maximum size of a chain that Maker can achieve in two families of posets. Given nonnegative integers $c_1, \ldots, c_d$ with $c_1 \le \cdots \le c_d$, let P_c be the poset whose elements are $\{(x_1,\ldots,x_d)\in\mathbb{Z}^d\colon\ 0\le x_j\le c_j\}$, under the order $x\le y$ if and only if $x_j\le y_j$ for $1\le j\le d$. We show that in the set Maker-Breaker game on P_c, Maker can capture chains of size at least $h(P_c)-\lceil (c_d+1)/2\rceil$, and this is best possible.

Let Q_n^d be the subposet of $P_{n,\ldots,n}$ consisting of all $(x_1,\ldots,x_d)$ such that $\sum x_j\le n$. When $d\ge 14$, a reduction from the famous Angel-Devil game shows that Maker can obtain a chain of size $h(Q_n^d)$ in the sequence Maker-Breaker game on Q_n^d, where $h(Q)$ is the maximum size of a chain in Q. Because the sequence game is more restrictive for Maker, the same strategy achieves a chain of size $h(Q_n^d)$ in the set Maker-Breaker game on Q_n^d for $d\ge 14$. For smaller d, a greedy strategy shows that Maker can obtain a chain of size at least $(1-\frac{1}{d+1})h(Q_n^d)$ in the sequence Maker-Breaker game on Q_n^d.

We believe (but have not yet completed the details) that for fixed d and large n, a more complex potential function strategy shows that Maker can achieve a chain of size at least $(1-\frac{1}{d+1})S-o(S)$ in the sequence Maker-Breaker game on $P_{n,\ldots,n}$, where $S=h(P_{n,\ldots,n})=1+dn$. When $d=2$, these results are tight: there is a strategy for Breaker in the sequence Maker-Breaker game on Q_n^2 ensuring that Maker cannot achieve a chain of size larger than $\frac{2}{3}h(Q_n^2)$.

For $3\le d\le 13$, we do not know the maximum size of a chain that Maker can achieve in both games on Q_n^d, but we believe it may be as large as $h(Q_n^d)$; this is a direction for future work.

Suil O

Combinatorics is the largest group in all REGS groups every year. In this REGS 2009, a lot of students in the department of computer science joined the combinatorics REGS group, and so we enjoyed a lot of problems, which are in the interests of the students in computer science department, and which are a little bit different with the ones in mathematics department.

At this REGS, I was interested in the problem that Prof. West gave about one of the conjectures of Graffiti.pc [Ed. - a computer program that generates conjectures]. To understand the problem, we need several definitions. If S is a subset of $V(G)$, and every vertex has a neighbor in S, then S is a *total dominating set* of G, and the *total domination number* of G is the least size of such a set. If P is a set of vertex disjoint paths, and each vertex is on a path in P, then P is a *path cover* of G, and the *path covering number* of G is the least order of such a set.

Conjecture: If G is a regular graph, then the total domination number of G is at least twice the path covering number of G. Prof. West, Mohit Kumbhat, Reza Zamani, and I studied this conjecture. It is easy to notice that if G is a regular graph with n vertices, then its total domination number is at least n/r. Thus, to show that the conjecture is true, it is enough to prove that the path covering number of G is at most $n/(2r)$.

However, this is not true for $r=5$; the bound must be at least $(6n+4)/55$, which is bigger than $n/10$. In fact, the bound $n/(2r)$ never holds for odd r except when $r=3$. So we study only in regular graphs of even degree and cubic graphs to prove the bound.

In 2007, DeLavina et al. directly showed that if G is cubic, then the total domination number of G is at most twice the path covering number of G. In this REGS, we showed that the path covering number of a cubic graph G is at most $n/7$ with a very short proof, which gives another proof of the conjecture for the cubic case. The bound improves the result of Magnant and Martin in 2009, but in 1996 Reed got the best upper bound for the path covering number of a cubic graph with n vertices, which is $\lceil n/9 \rceil$. Its proof is long, with a lot of cases.

Interestingly, the examples that Prof. West and I constructed in the paper "Balloons, Cut-edges, Matchings, and Total Domination in Regular Graphs of Odd Degree" (just appeared online in JGT) satisfies the bound of Reed.

Now, we are trying to prove the bound $n/8$ for the path covering number of 4-regular graphs with n vertices, to show that the conjecture is true for 4-regular graphs. In fact, we conjecture that if G is a 4-regular graph with n vertices, then the path covering number is at most $n/11$. If the conjecture is true, then we have infinitely many examples achieving equality in the paper "Generalized Balloons and the Chinese Postman Problem in Regular Graphs".

Finally, we want to finish this REGS report with the following statement that we proved recently. If G is a 4-regular graph with n vertices, then the path covering number of G is at most $n/7$, which is an improvement of the result of Magnant and Martin in 2009.

Nate Orlow – REGS0

The past summer of participating in the REGS program was great; it was nice being able to participate and improve both speaking and problem solving skills. I think the introduction through having problems presented was a good idea, as several of the ones presented the first week were not only fairly interesting, but also representative of the types of problems presented through the remaining session. I know it's important to have everyone participate in presenting problems, but I would have liked watching several other people go before presenting my own problem. As someone not previously working on any particular graph theory problem in depth, I went out and looked for a problem. I should probably have waited longer to present a problem.

One way a little more time could be allowed would be a slower influx of problems early on in the summer. When people were presenting new problems, there were several that were interesting, which left me (and possibly others) wondering if people wanted to work on the old problem, or if they were splitting up and thus you would be available to work on new problems. In some ways, it really was one or the other, since if people made progress on a new problem, and you wanted to join later, they would have to give you a summary of the previous day/days they were working on the problem. In addition, you don't know what dead ends or trouble they had with the problem, so to some extent, you weren't able to benefit from the previous time they spent on the problem.

In this way, one feature I felt confused about in general was where credit was due. As one might imagine, working in a group setting, you bounce ideas off of each other or tell each other ideas to help find a solution. When individual progress was made outside the group setting, it is easy to attribute that to your work, but many results seem at least in some part a group effort. Despite this being exactly what is desired by working in a group - having the whole be more than the sum of the parts, with more ideas than working alone - it makes attributing credit (other than yourself) more ambiguous.

One specific result I obtained at the beginning of the session was how to traverse a tree to obtain a lucky labeling. The algorithm is fairly simple. First temporarily assign all nodes the label 2, which makes all nodes have even sum. Then traverse the tree from root to leaf, ignoring the root and leaves; if a node and its parent both have even sum, and in fact are equal, change the left child to 1.

In addition, in the Fibonacci sum group, we are currently working on editing Bill's draft of the proof that the spanning path always exists and is essentially unique (meaning it's unique, or two paths exist which agree up to the final 3 edges). I recently sent some suggestions and received feedback, so we are definitely making progress. Although my particular draft of the proof wasn't as useful, partially due to being less concise, I felt it was a good experience to practice writing up the ideas formally.

In general, I found this a great experience to have, not only in getting pumped in solving several different types of problems, but also working with others in both explaining your ideas and understanding others'. It's interesting being able to work with someone else's approach and adding on to it in order to come up with a better solution than one would get working individually. What also was a help was getting experience explaining things using a chalkboard - most of my presentations have been using some sort of slide format, which meant even when I was quite prepared to explain the topic, I would have to draw the diagrams and write what I was saying as I was giving the talk. I appreciate being part of the REGS and the funding it received, and also thank the faculty and cameo students who helped out for a few sessions.

Gregory Puleo – REGS0

Most of my work this summer was split among three projects: the Fibonacci-sum graphs that I presented, Grytczuk's conjecture on zero-free labelings, and the Chain-sequence Maker/Breaker game, presented to me by Bill Kinnersley and Kevin Milans. [Ed. - see the report of Kyle Fox for the problem and results on Fibonacci-sum graphs.]

I was pleased with the summer's results, especially those relating to the chain-sequence game. More broadly, I enjoyed the opportunity to start settling into the graduate program while developing my mathematical skills.

Zero-free Labelings. A *k-labeling* of a graph G is a function $f\colon V(G) \to \mathbb{Z}_k$. Given a labeling, its associated coloring is the function $F\colon V(G) \to \mathbb{Z}_k$ defined by $F(v) = \sum_{w \in N[v]} f(w)$. The labeling f is *zero-free* if $F(v) \neq 0$ for all $v \in V(G)$. Grytczuk conjectured that, for every k, each graph has a k-zero-free labeling. A classical theorem states that every graph has an odd dominating set – in this context, a 2-zero-free-labeling – but it is not known whether the same statement holds for higher k.

Kinnersley observed that if G has a k-zero-free labeling then, on adding a new vertex adjacent to at most $k-2$ old vertices, the resulting graph also has a zero-free labeling: there are at most $k-1$ labels for the new vertex which will introduce a 0 into the coloring, so at least one available label will give us a k-zero-free labeling. This observation gives us a lower bound of $k-1$ for the minimum degree of a graph of least order with no k-zero-free labeling.

Combining this bound with an existing bound on the domination number in terms of minimum degree, we obtain a bound on the minimum order of a counterexample: if $n(G) < k^2/(1 + \ln k)$, then G has a k-zero-free labeling.

An algebraic approach is also profitable here, since the map from the labeling f to its associated coloring F is a linear transformation – in fact, can be represented by the augmented adjacency matrix of the graph. Using linear algebra, we can prove that if p is prime and the augmented adjacency matrix of G has rank at least $n - (p/2 - 2)$ when considered as a matrix over $\mathbb{Z}_p$, then G has a p-zero-free labeling. [Ed. - This problem is closely related to the Grytczuk's "Lucky Labeling" problem mentioned by several of the students.]

Chain-sequence Maker/Breaker Game. The *chain-sequence Maker/Breaker* game is played on a poset P between two players Maker and Breaker, who take turns claiming the elements of P. Maker's goal is to claim, in increasing order, the elements of a large chain of P, while Breaker wishes to prevent this.

More precisely, we will assume that P is a graded poset with a minimum element. We say that Breaker *wins at level k* if Breaker can prevent Maker from taking a chain with elements on levels $0, 1, \ldots, k$. We will say that Breaker *wins* if he wins at level k for some k, while Maker wins if he can take arbitrarily long contiguous chains.

We can transform the chain-sequence Maker/Breaker game into another game – the *angel/devil game* – by imposing a constraint on Maker's moves. Unlike Maker, the angel has a spatial *position* within the poset. It starts on the minimum element and, on its turn, may move to any unclaimed element that covers its current position. The angel wins if it can keep moving forever. Breaker plays exactly as before and has exactly the same goal, but is now called the *devil*. Maker "plays like an angel" by claiming, on each turn, the element on which the angel currently rests. It is clear that if the angel wins, then Maker wins, as each angel move gives Maker another level in his chain.

More surprisingly, the other direction holds as well: if Maker wins, then the angel wins, and so these games are equivalent. Loosely, we can imagine Maker as possessing an infinite *army* of angels and being allowed to move one stack of these angels each turn. If the devil beats a single angel, then Breaker can beat the army by imitating the devil's strategy against each individual stack.

Beyond establishing this relationship between angel/devil and Maker/Breaker games, we also started studying a class of *robust maps* between angel/devil games. We can use these maps to "simulate" strategies between games, proving theorems of the form "if an angel wins against devil X in this game, then it wins against the stronger devil Y in this other game". Using these techniques together with the existing angel/devil literature, we were able to make substantial progress on some questions about the chain-sequence Maker/Breaker game. However, many questions remain unanswered.

Ben Reiniger – REGS0

As a new student this year, the summer Research Experience for Graduate Students was my first experience with graduate research. I participated in the combinatorics REGS under the supervision of Doug West. While I didn't find any strong results, I did very much appreciate the opportunity to work on several of the problems presented through the summer. I picked up on many concepts that were completely new to me and was also able to work more in depth with concepts I was already familiar with.

While I very much wish I could have proved some strong result during the summer, I feel very happy with the experience. The opportunity to be exposed to the graduate research environment so early is surely priceless. Below I describe the two problems which I felt most personally attached to, but I worked glancingly on several other problems, including combinatorial games, partitioning complete graphs, splitting a graph into two subgraphs with equal degree sequences, finding covers for long paths in digraphs, and others.

Queue Layouts. A *vertex ordering* of a graph G is a linear order on $V(G)$. Given a vertex ordering of a graph, an edge e has left and right endpoints $l(e) < r(e)$. Two edges e and f with $l(e) < r(f)$ are *crossing* if $l(e) < l(f) < r(e) < r(f)$. A *stack layout* of a graph is a vertex ordering along with a coloring of the edges so that in each color class, edges are pairwise noncrossing. The *stack number* of a graph is the minimum number of colors in a stack layout of the graph. This graph parameter has been studied as the *page number* or *book thickness*, as we might think of the vertices set on the spine of a book, with each color class representing a page with edges embedded in a planar fashion. This parameter has been studied since 1973.

To motivate further questions, we can think of the layout in a data processing sense: if we work through the vertices in their order, mark an edge for processing upon reaching its left vertex, and complete its processing upon reaching its right vertex, then the non-crossing restriction for each color class says precisely that the processing structure is a stack. If we prefer instead to have a queue, then we need to forbid *nesting* edges in each color class: those edges e and f with $l(e) < r(f)$ are *nested* if $l(e) < l(f) < r(f) < r(e)$. Call such a layout a *queue layout*, and say the *queue number* of a graph is the minimum number of colors in a queue layout. Queue layouts were introduced in 1992.

Planar graphs have stack numbers at most 4. It is conjectured that they have bounded queue numbers. Bipartite planar graphs have stack numbers at most 2, but again it is unknown whether they have bounded queue numbers. We looked briefly at the problem for planar graphs, but of course planar graphs are more closely associated with the stack layout than with the queue layout. For k-trees (maximal graphs with tree-width k), the stack number is at most $k+1$, and the queue number has been proven to be bounded by $3^k 6^{\left(4^k-3k-1\right)/9} - 1$; this bound is quite unlikely to be sharp. We primarily sought to improve this bound on the queue number of k-trees. The inductive construction of k-trees seemed to be a promising source for a bound which would be polynomial in k, but we were not successful in finding one.

Chvátal's Conjecture. Consider the set of subsets set of $\{1, 2, \ldots, n\}$. A family of sets is an *intersecting family* if the members of the family are pairwise intersecting, and is called a *star* if the members have a common element. If a family is such that the set of maximum-sized intersecting subfamilies contains a star, we say the family has the *star property*. An *ideal* is a family of sets I such that if $B \in I$ and $A \subseteq B$, then $A \in I$ also. Chvátal conjectured in 1974 that every ideal has the star property.

We considered some related questions posed or inspired by Snevily. Let the *translate* of a family F by a set B be the family $F(B) = \{A \triangle B : A \in F\}$, where $\triangle$ denotes symmetric difference. Note that $F(\emptyset) = F$. We observed, and Paul Wenger proved, that the translate of an ideal I by a set B is an ideal if and only if B is contained in every maximal member of I. (In this case, $I(B) = I$.) I observed that the translation of any ideal I by any singleton $\{x\}$ has the star property. To see this, consider the subideal J generated by x (that is, the smallest subideal J such that the members of $I \setminus J$ do not contain x). Then $J(\{x\}) = J$, and $(I \setminus J)(\{x\})$ is a star containing x. It is not hard to show that J has $\frac{|J|}{2}$ members which contain x, so that $I(\{x\}) = (I \setminus J)(\{x\}) \cup J(\{x\})$ has $|I \setminus J| + \frac{|J|}{2}$ members containing x. A previous result says that any intersecting family in an ideal contains at most half of the members of the ideal, so any intersecting family has at most this same number of members.

The original interest in looking at singleton-translates was the question as to whether $\{B : I(B) \text{ has the star property}\}$ is itself an ideal. This was answered in the negative fairly quickly (alas!). Hopefully the result will yet bear some illumination to the original conjecture.

Evan Vanderzee

Although I have already completed six years of graduate school at the University of Illinois, the summer of 2009 was my first time participating in the REGS program in combinatorics. I was encouraged to try the REGS by previous participants who had good experiences in the program. Thus my expectations for the program were fairly high.

My expectations were not disappointed. I learned a lot during the REGS about presentations and collaboration. Most presentations I had given previously used a set of prepared slides and a projector, but in the REGS program I got experience using a chalkboard to help communicate ideas. I was not very comfortable using the chalkboard at first, but I saw improvement throughout the summer and am more comfortable with the chalkboard now. I also had many opportunities to think collaboratively about problems. It can be difficult for me to communicate ideas that are not fully formed, but my attempts to communicate my thoughts helped me clarify my ideas. I also developed a greater appreciation for the ideas of other people. On several occasions one of my collaborators expressed a concept that led me towards a solution.

In terms of mathematical results, my summer was not as fruitful as I had hoped it would be, but I did contribute to answering some of the questions that were posed. I helped construct counterexamples to show that two of the presented conjectures were false. We also got partial positive results on one or two other problems. For the other problems that I considered, I can at least take them with me; I come away from the summer with plenty of interesting things to think about.

As a recommendation for future combinatorics REGS, I suggest that the program not be much larger than it was this year. My schedule constrained me to have very little time outside of the regular REGS meetings to meet with other people and think about problems. Since so many problems were presented during the regular REGS meeting times, there was not very much time to work together on the problems.

Hehui Wu

This is the fifth year that I participate in the REGS project. Similar to the past four years, the program is very valuable in many points.

As the largest REGS group in the department, the Combinatorics research group provides a lot of benefits that are hard to find in other place: three times per week regular meetings make the group extremely active; 41 problems provided by the professors and the students largely extended our knowledge in the research area; cooperation with other students become possible and also important in our big group.

In the seminar, I presented a problem about the edge-graceful labeling. However, after spending some time on it, I found changing to another topic maybe a better choice. Now the advantage of a big group is very clear: there were totally 41 problems available for us to choose. How to pick one or two problems that fit me best became a challenging thing. Finally, I focused on a conjecture about the spanning tree in 3-regular graphs.

The problem was presented by Professor Kostochka. The conjecture states that any 3-regular graph could be decomposed to a spanning tree, disjoint cycles and a matching. At the beginning, there are Chayapa Darayon, Ping Hu, Evan VanderZee and I working on this problem. Most of them are relatively new to Combinatorics. But quickly, they showed their genius and passion during the discussions. Together, we proved that the conjecture is true for some special cases, for example, it's true for the 3-regular graph which contains a 2-factor with at most two cycles. Also, we proved that if the conjecture is true, then it's also true for any 3-subregular graph instead of 3-regular graph. Later, Kevin Milans and Steven Hartke joined us, and proved that any 3-regular graph could be decomposed to a spanning tree, disjoint cycles and some disjoint P_2s or P_3s.

I have benefited a lot from this meaningful REGS program. These benefits include many interesting problems, wonderful collaborative research experience, and some good results. However, it's just the beginning. Now we have enough good research topics for the coming years, and we will keep working on them. Hopefully, it will bear more fruitful results in the future, just like what we did in the past REGS.

Matthew Yancey

This summer at the combinatorics group in REGS over forty problems were presented. I presented one of them and worked on approximately ten of them. Of those problems, I spent extended amounts of time on three problems: Monotonic Sequence Game Version 2, Bichromatic Number, and Maximum Length Paths Intersecting an Independent Set. Progress of academic merit was not made on Bichromatic Number, but was made on the other two. I mostly worked alone on the sequence game, and a full list of results with proof are given below. The paths problem was worked on with Paul, Tim, Jane, Chris, and Ben. As Jane has already written up the most important results, I will leave it to them to present that material.

The group dynamic at REGS was invaluable. For problems like maximum length paths, I usually found myself at a loss towards starting the problem. The upper level grad students set a good pace in terms of exchanging ideas and facts to keep the progression steady. I felt like I had a better understanding of direction of progress when a conversation was taking place, which helped me contribute. There were times where without a leader a group would stay silent for long periods of time.

The wide list of problems was helpful in that I found progress on only a few of the problems I tried. Most problems had already been explored and required a new method to analyze that was not immediately clear. The options provided allowed me to keep the problems that I felt confident on and to discard the problems I felt frustrated about.

Prior to narrowing my vision on which problems to work on, the list of problems was occasionally overwhelming. Early in the program I found it difficult to make results because there would be a new problem to consider within 48 to 72 hours. There was so much material that it was difficult to keep straight which problems I wanted to continue working on in the future.

I like the system as it stands now, although it took time to adjust to. I think I would handle the situation better in a year, now that I have a better mastery over the art of examining many problems while focusing on few to concentrate progress.

Monotonic Sequence Game, Version 2. Define a game as follows: Fix a poset P and natural numbers a and d prior to play. In the game (P,a,b), Players A and B take turns selecting an element of P to append to a growing sequence of distinct elements. The game ends when all of P has been selected or the sequence has a subsequence that is an ascending chain of size a or a descending chain of size d. In the 'normal' form of the game, a player who ends the game by creating such a chain wins. In the 'misere' form of the game, such a player loses. If neither player wins in a finite number of turns, then the game is a draw.

In many cases, we found the winner of the game given perfect strategic play. The problem was originally considered by Harary, Sagan, and David West under the restriction that P is a finite chain. It was generalized by Albert et al. to the form given above, with special attention to the cases where $P = [n]$ and $P = \mathbb{Q}$ (see M. Albert, R. Aldred, M. Atkinson, C. Handley, D. Holton, D. McCaughan, and B. Sagan; Monotonic Sequence Games, Games of No Chance 3 (2009), 309-327; preprint on Sagan's website at http://www.mth.msu.edu/∼sagan/Papers/Old/list.html).

[Ed. – The remainder of Yancey's report was a full paper with proofs, omitted here to save space.]