posed by Marcin Peczarski

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Let $P(\leq, S)$ be a partial order on the set S. An *extension* of $\leq$ is a partial order $\leq^*$ on S such that $u \leq v$ implies $u \leq^* v$, for all $u, v \in S$. A *linear extension* of P is an extension of $\leq$ that forms a linear order (a.k.a a total order). The number of linear extensions of P is denoted $e(P)$.

Conjecture (The Golden Partition Conjecture). For any finite poset P that is not a chain, there exist two consecutive comparisons such that regardless of their results the inequality

$$e(P) \geq e(P_1) + e(P_2)$$

holds, where P_1 and P_2 are the posets obtained from P after the first comparison and after both comparisons, respectively.

The Golden Partition Conjecture is motivated by conjectures whose root is a famous sorting problem. Imagine that a finite set S with cardinality n has a hidden total order $\prec$ that we would like to uncover by making comparisons between pairs of elements from S. Comparing a pair $u, v \in S$ means uncovering either $u \prec v$ or $v \prec u$. How many comparisons are needed in the worst case? The classical answer is $\Theta(n \log_2(n))$ comparisons are needed and 'merge sort', for example, gives an algorithm to achieve this bound.

How many comparisons are needed if some partial information about $\prec$ is already known? Partial information about $\prec$ can be summarized by a poset $P(\leq, S)$. Let $C(P)$ denote the number of comparisons required to find the hidden linear extension $\prec$, in the worst case, starting from partial information P? Clearly $C(P) \geq \log_2(e(P))$, since each comparison can reduce the number of linear extensions by a factor of at most 2. The Golden Partition Conjecture gets its name because Peczarski [Pec06] has shown that it implies $C(P) \leq \log_\phi(e(P))$, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033988$ is the golden ratio. As Peczarski states, "informally this [bound] means that during the sorting process the number of linear extensions can be decreased in every comparison on average by at least the golden ratio ϕ." Linial [Lin84] has constructed a sequence of posets that show that, if true, this bound would be tight.

The survey article by Brightwell [Bri99] is highly recommended.

Notes:

1. Peczarski [Pec06, Pec08] has proven the conjecture for semi-orders, width two posets, 6-thin posets, posets containing at most 11 elements. The conjecture implies the famous $1/3 - 2/3$ conjecture. The fraction of n-element posets satisfying the conjecture goes to 1 as $n \to \infty$.

2. Brightwell's survey [Bri99] notes that Fredman proved that $C(P) \leq 2n + \log_2(e(P))$.

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posed by Agarwal, Aronov, Koltun, and Sharir
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Motivated by problems in visibility, ray shooting, motion planning and geometric optimization, Agarwal et al. [AAKS05] investigated the space of lines that avoid the interior of a collection of unit-radius balls in $\mathbb{R}^3$. They define the 'combinatorial complexity' of this space to be the number of lines in the space that are tangent to four of the balls. They prove that, for any $\epsilon > 0$, the complexity is at most $O(n^{3+\epsilon})$; they seek a matching lower bound.

Conjecture. There exists a configuration of n (not necessarily disjoint) unit-radius balls in $\mathbb{R}^3$ with $\Omega(n^3)$ lines such that each line is tangent to exactly four of the balls and misses the others.

It is easy to achieve $\Omega(n^2)$ lines and this is the best known lower bound according to O'Rourke's web site

http://maven.smith.edu/~orourke/TOPP/P61.html

In light of the fact that four unit balls with collinear centers have infinitely many common tangent lines, the problem should probably restrict attention to configurations of unit-radius balls in $\mathbb{R}^3$ in which the centers of no four balls are collinear. Or perhaps the problem could be rephrased to ask for a configuration of n (not necessarily disjoint) unit-radius balls in $\mathbb{R}^3$ that maximizes the number of four ball subconfigurations that have a common tangent line missing the other balls.

A related theorem by MacDonald et al. [MPT01] may be of interest:

Theorem. Any four unit balls in $\mathbb{R}^3$ whose centers are not collinear have at most twelve common tangent lines and this bound is tight.

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posed by Lebl and Lichtblau
presented by André Kézdy
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Let us call a polynomial $p(x, y) \in \mathbb{R}[x, y]$ *sharp* if it has all of the following properties:

i) p has no negative coefficients,
ii) $p(x, y) = 1$, for all $x, y \in \mathbb{R}$ such that $x + y = 1$, and
iii) p has exactly $\frac{d+3}{2}$ nonzero terms, where d is the degree of p.

The reason that these polynomials are "sharp" is because D'Angelo et al. [DKR03] have shown that polynomials satisfying (i) and (ii) must have at least $\frac{d+3}{2}$ nonzero monomials; so sharp polynomials achieve the bound. For all odd d, the polynomial f_d below is sharp [D'A88],

$$f_d(x, y) = \left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^d + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^d + y^d.$$

Lebl and Lichtblau [LL10] construct all sharp polynomials of odd degree $d \leq 17$, prove many results about sharp polynomials, and make the following conjecture.

Conjecture. There exists infinitely many positive integers d such that, up to a swap of variables, f_d is the unique sharp polynomial of degree d.

Uniqueness holds for $d = 1, 3, 5, 9, 17$ and is known to fail [LL10] when

a) $d > 3$ and d is congruent to 3 modulo 4, or
b) $d > 1$ and d is congruent to 1 modulo 6, or
c) d has the form
$$d = \frac{(7 + 4\sqrt{3})^k + (7 - 4\sqrt{3})^k}{2},$$
for some positive integer k.

By the way, it is also known that uniqueness fails for all $d \leq 149$ such that d is not one of the following:

$$1, 3, 5, 9, 17, 21, 33, 41, 45, 53, 69, 77, 81, 93, 105, 113, 117, 125, 129, 141, 149.$$

So the first case for which uniquenss is not known is $d = 21$.

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posed by Reza Naserasr, et al.
presented by Chris Caragianis
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For a finite simple graph G, denote by $Forb_m(G)$ the class of graphs that do not have G as a minor. In [NNŠ09], Reza Naserasr, Yared Nigussie, and Riste Škrekovski use the theory of cuts and bounds in the homomorphism order of graphs to show that every triangle free memeber of $Forb_m(K_5)$ is 3-colorable. The authors exhibit a triangle-free 4-chromatic graph with no K_6 minor, showing that the result cannot be extended, but offer the following conjecture.

Conjecture. Every K_6-minor free graph G of girth at least 5 is 3-colorable

They note also that in personal communication, R. Thomas has made the following conjecture.

Conjecture. Every triangle free graph G in $Forb_m(K_6)$ is 4-colorable.

Naserasr et al. present a K_6-minor free, triangle-free, 4-chromatic graph. Their example is an *apex* graph, a graph that can be made planar by the deletion of a vertex. This is a handy way of ensuring that the graph has no K_6 minor. With this in mind, I propose the following small step toward these conjectures.

Problem. Are the above conjectures true with the additional hypothesis that G is an apex graph?

If the answer is yes, then I propose the next somewhat larger step. Let M be a collection of graphs and define $Ap_m(\mathcal{M})$ to be the class of graphs for which the deletion of some vertex leaves a graph with no minor in $\mathcal{M}$. Then the class of apex graphs is precisely $Ap_m(\{K_5, K_{3,3}\})$.

Problem. Are the above conjectures true with the additional hypothesis that $G \in Ap_m(K_5)$?

The answer to both questions is "yes" in the case of the second conjecture. Adjoining a single vertex will increase the chromatic number by at most one. Grötzsch's theorem tells us that a planar triangle free graph is 3-colorable, while the main result of [NNŠ09] tells us that a K_5-minor free, triangle free graph is 3-colorable.

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Tower of Hanoi Graphs

posed by Danielle Arett & Suzanne Dorée
problem 5
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The Tower of Hanoi is a familiar puzzle in which disks of increasing size are moved from one peg to another, called a *move*, following three rules:

1. Exactly one disk is moved each turn.
2. Only the top disk from a stack can be moved.
3. No disk may be above a disk of smaller size.

The traditional puzzle, as invented by Édourd Lucas in 1883, has 3 pegs and 5 disks, with a four peg version appearing in [Dud59]. One of the first instances in which the number of moves necessary to solve the puzzle was studied shows up in [SBLT39]. More history of the puzzle is available in [Hin89].

In [AD10], Arett and Dorée give an overview of some of the known results for state graphs associated with various generalizations of the Tower of Hanoi puzzle, called *Hanoi graphs*, where H_p^d is the state graph for the puzzle with p pegs and d disks. Hanoi graphs were first introduced by [SGS44] where properties of H_3^d were explored. The H_3^d graphs are similar to special cases of Sierpiński's triangles, which have been discussed by various authors such as [HP06], [JK09], [KM05] and [TG06]. The puzzle (and graph) is only interesting with $d > p$, otherwise, you can put each disk on a separate peg as the first series of moves, then get wherever you want by stacking the disks in the desired order. That is, when $d > p$ you can get from any state to any other state in at most $2d - 1$ moves. If $d = p$, then some careful planning will allow you to get from any state to any other state in at most $2d + 1$ moves. We now assume $d > p$.

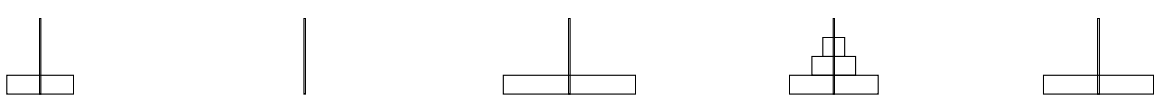
Each vertex in the state graph represents a state of the puzzle. Two vertices are adjacent if a single move will allow you to move from one state to the other. A path in the graph represents a series of moves in the puzzle. A path from a vertex representing the initial state of the puzzle to a solution state of the puzzle represents a solution. A shortest path represents a minimal solution to the Tower of Hanoi puzzle.

The vertices of the graph can be labeled in such a way to describe the state the vertex represents. Number the pegs $0, 1, \ldots, p-1$ such that peg i is the ith peg from the left. The label for each vertex will be string of length d over the alphabet $0, 1, \ldots, p-1$ such that the jth entry in the string tells you on which peg the jth largest disk sits. Since rule #3 prevents larger disks from being on smaller disks, the order on each peg is prescribed and so each string represents a single state.

Example. In H_5^6, the vertex **143033** represents the largest disk on peg 1, the second largest disk on peg 4, the third largest disk on *the bottom of* peg 3, the fourth largest disk on peg 0, the fifth largest disk on peg peg 3 (on top of the third largest disk), and the sixth largest disk (or smallest disk) on *the top of* peg 3.

If you move the largest disk from peg 1 to peg 2, you have moved from the state vertex **143033** to the state vertex **243033**. Since this is a valid move, the vertices labelled **143033** and **243033** are adjacent in the Hanoi graph.

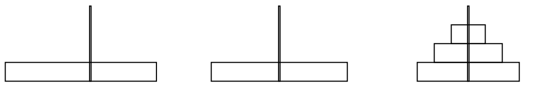
Picture corresponding to Vertex **143033** in H_5^6



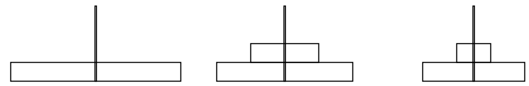
Picture corresponding to Vertex **243033** in H_5^6

Since you can only move one disk at a time, two vertices can only be adjacent if the labels on the vertices differ in exactly one spot. However, just because two labels differ in one spot does not mean the vertices are adjacent.

Example. In H_3^5 the vertex **01222** is not adjacent to the vertex **01212** as this would require moving the middle vertex on peg 2 to peg 1.



Picture corresponding to Vertex **01222** in H_3^5



Picture corresponding to Vertex **01212** in H_3^5

It is easy to see that $H_p^1 \cong K_p$. This puzzle would consist of a single disk which could be moved from any peg to any other peg. The vertices would be labeled $0, 1, \ldots, p-1$.

In [AD10], the following properties of Hanoi graphs are proven or references to proofs are given.

1. They can be constructed recursively. The d-disk graph H_p^d is built from p copies of H_p^{d-1}, based on placement of the largest disk.
2. The graphs are $(p-1)$-connected.
3. The graphs are Hamiltonian when $p \geq 3$.
4. There are p^d vertices in H_p^d.
5. There are $\frac{1}{2}\binom{p}{2}[p^d - (p-2)^d]$ edges.
6. The degree of a vertex is $\binom{p}{2} - \binom{p-k}{2}$ where k is the number of pegs with at least one disk in that state (the second term is zero if $k = p-1$ or $k = p$).
7. The chromatic number is independent of the number of disks, $\chi(H_p^d) = p$.
8. The independence number is $\beta(H_p^d) = p^{(d-1)}$.

It has been shown in 2010 that $Aut(H_p^d) \cong S_p$ (see [Par10]) but not much else about H_p^d is known when $p > 3$. Three open questions when $p > 3$ include:

1. What is the optimal number of moves requires to solve the puzzle? A conjecture (and seemingly incorrect proof) appeared in 1941 in [SF41]. The problem still appears to be open.
2. What is the crossing number for H_p^d? In [AD10], Arett claims H_3^d, H_4^1, and H_4^2 are the only planar Hanoi graphs.
3. What is the diameter of H_p^d?

The diameter question is of particular novelty. The minimum number of moves required to solve the Tower of Hanoi puzzle is bounded by the diameter of the graph, and for $p = 3$, it is equal (see [AD10]). In [Kor08], Korf conjectures for $d \geq 19, p = 4$, the number steps required for moving from vertex $\mathbf{0}$ to $\boldsymbol{p}$ (a solution to the classic puzzle) is less than $diam(H_p^d)$. Specifically, in H_4^{15}, $dist(\mathbf{0}, \mathbf{3}) = 129$ whereas $dist(\mathbf{0}, 102310010331333) = 130$ (where $\mathbf{0}$ means d zeroes). Other results on the diameter appear in [BS06].

There are also further similar problems about which nearly nothing is known. Restricting movements such that a disk can only be moved to an adjacent peg, with or without wrap-around, is one such similar problem. These and other variations are described in [Poo94].

Other papers on Hanoi graphs are listed in the references. I have PDF versions of most of the references listed below.

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posed by Paul Erdős, Dhruv Mubayi, and Jacques Verstraëte
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A Hypergraph is a graph where edges are generalized to vertex sets of any size. A k-graph is a hypergraph in which every edge has size k. Degree and regularity are defined normally. The size of a graph is its number of edges. A star is defined as a hypergraph containing some vertex v, all the possible edges that contain v, and no edges that do not contain v.

Erdős raised the following question in 1938 [Erd38]: Is there an upper bound on the size of a k-graph if it contains no 2-regular subgraphs? In an abstract sense, 2-regular subgraphs may be thought of as cycles of a hypergraph, and thus hypergraphs containing none of them may be thought of as the forests among hypergraphs. Since we have an upper bound for the size of a forest as a function of its order, it is natural to seek that of a hypergraph, as an analogue. Only k-graphs with $k > 2$ are considered.

Dhruv Mubayi and Jacques Verstraëte show the following in [MV09] (though this is paraphrased):

Theorem. For any even $k > 2$, there exists a finite integer n_k s.t. for all $n \geq n_k$, any k-graph on n vertices with no 2-regular subgraphs has no more than $\binom{n-1}{k-1}$ edges, with equality only in the case of a star.

However, when $k > 2$ is odd, a star is not the extremal configuration, because a matching omitting its center vertex v could be added to the hypergraph, and it will still contain no 2-regular subgraph. Such a matching could contain up to $\lfloor \frac{n-1}{k} \rfloor$ edges. That seems to be the limit, so those authors pose the following conjecture:

Conjecture. For any odd $k > 2$, there exists a finite integer n_k s.t. for all $n \geq n_k$, any k-graph on n vertices with no 2-regular subgraphs has no more than $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ edges, with equality only for the union of a star and a maximal matching omitting that star's center.

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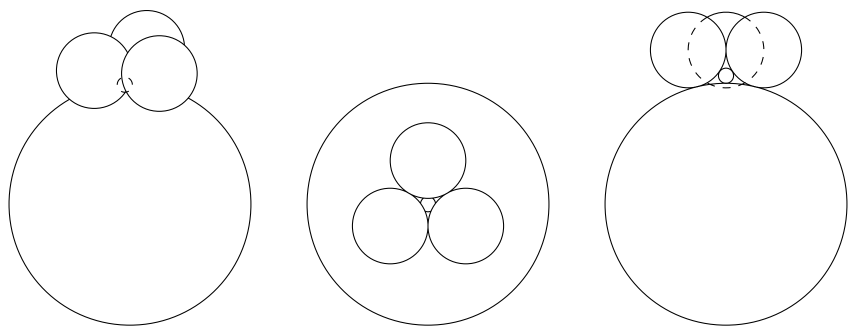
Definition. Consider a collection of (not necessarily congruent) circles in the plane. The *intersection graph* of this collection is the graph formed by creating a vertex for every circle and making vertices adjacent if the corresponding circles intersect.

Definition. A graph is a *disk contact graph* (aka a coin graph) if it can be represented as an intersection graph where the circles may not overlap at more than one point (i.e. they may only touch). We can define a *n-sphere contact graph* over $\mathbb{R}^n$ analogously.

The disk contact graphs yield a wonderful classification:

Theorem (Circle Packing Theorem [Koe36]). A graph is a disk contact graph if and only if it is planar.

Harborth by way of Archdeacon [Arc] asked about the structure of sphere contact graphs over $\mathbb{R}^3$. We can represent K_5 as a 3-sphere contact graph:



But Neil Sloan claims [Arc] that K_6 is not a 3-sphere contact graph.

Problem. Which graphs can be represented in $\mathbb{R}^3$ as sphere contact graphs?

In particular they wonder if you can obtain some type of forbidden substructure (e.g. subdivision) classification.

Problem. Does there exist a "Kuratowski-type" characterization of 3-sphere contact graphs?

The problem also arises in an area known as circle packing. The survey papers [Bez06] and [HK01] contain some results about density and recognition. [HK01] also contains a result of D.G. Kirkpatrick and G. Rote (originally in a personal communication to the authors of that paper) that recognition of 3-sphere contact graphs is an NP-hard problem. Neither paper has any results addressing the questions of Archdeacon and Harborth, though the NP-hard result shows (via the Graph Minor Theorem) that a forbidden-minor classification is probably not possible.

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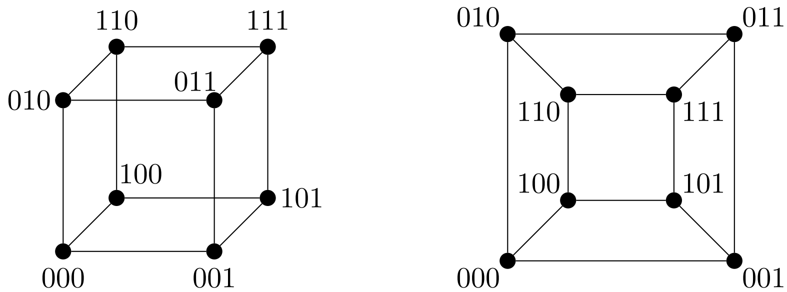
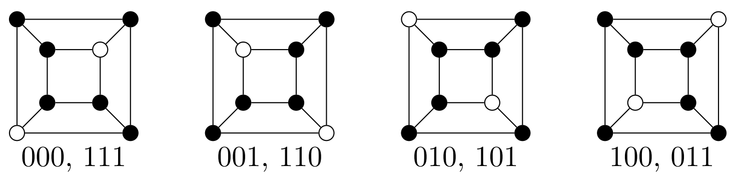
Definition. Q_d is the d-dimensional Hypercube graph: the graph with vertex set consisting of all binary strings of length d and edges between vertices whose strings differ in exactly one position.



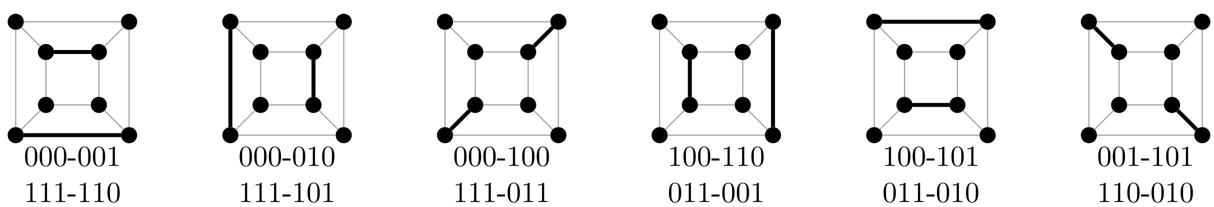
Two drawings of Q_3

Definition. Two vertices in Q_d are called *antipodal* if their strings differ in all d positions.



The antipodal pairs of vertices for Q_3

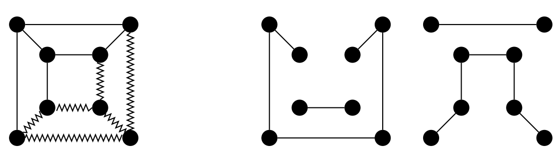
Definition. Two edges uv and xy in Q_d are called *antipodal* if $\{u, x\}$ and $\{v, y\}$ are both pairs of antipodal vertices.



The antipodal pairs of edges for Q_3

Definition. The map $\phi : E(Q_d) \to \{0, 1\}$ is called *edge-antipodal* if for every pair e, e' of antipodal edges, $\phi(e) \neq \phi(e')$.

We can think of this map as a partitioning of the edges of Q_d or as an improper 2-edge-coloring of Q_d.



Two examples of edge-antipodal maps on Q_3

Conjecture. Let $d \geq 2$. If ϕ is an edge-antipodal map on Q_d then there exists a pair v, v' of antipodal vertices joined by a monochromatic path under the edge coloring induced by ϕ.

The conjecture has been verified for $d \leq 5$. Feder and Subi [FS09] prove the conjecture holds for *any* (not necessarily edge-antipodal) map ϕ that does not contain a square $wxyz$ such that $\phi(wx) = \phi(yz) \neq \phi(wy) = \phi(xz)$. Note that the second example above contains such a square: 010, 011, 111, 110.

Notice that in an edge-antipodal map the graphs induced by each color are isomorphic by sending every vertex to its antipodal vertex, so we only need to consider one color class. Feder and Subi note that a counterexample for d can be extended to a counterexample for $d+1$ and that a counterexample for $d = 6$ would need to have at least four components each containing a single edge. They were unable to find such a counterexample. It is unclear from their wording but I believe they intended to say that every map they found with such components still had an antipodal pair joined by a monochromatic path.

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Recall the following definitions:

Definition. A k-fold realizer of a poset is a multiset of its linear extensions, such that every incomparable (ordered) pair is reversed at least k times.

Definition. For each positive integer k, let $t(k)$ be the cardinality of the smallest k-fold realizer. The dimension of a poset is the minimum cardinality of a 1-fold realizer. The fractional dimension (dim*) of the poset is the infimum of the set $\{t(k)/k\}$.

Definition. The standard example S_n is the poset induced by the sets of size 1 and $n-1$ in the subset lattice of $\{1,\dots,n\}$.

In 1973, Trotter conjectured the following.

Conjecture (Trotter). Every finite partially ordered set with three or more points contains a pair whose removal decreases the dimension by at most one.

There are many examples of sufficient conditions which guarantee the existence of such a pair, but progress on the conjecture has slowed to standstill in the recent years.

Conjecture (Trotter). There exists an $\epsilon > 0$, such that every finite partially ordered set with three or more points contains a pair whose removal decreases the fractional dimension by at most $2-\epsilon$.

Fractional dimension relates to dimension in much of the same way as fractional chromatic number of graphs relates to chromatic number. In fact, one can define dimension of posets as a solution of an integer programming problem. The linear relaxation of this problem defines fractional dimension. Clearly $\dim^*(P) \le \dim(P)$. It can be shown that the gap can be arbitrarily large. $\dim(S_n) = \dim^*(S_n) = n$. $\dim^*(P)$ is always a rational number (1, or ≥ 2). It is possible to show that there is always a multirealizer that exhibits the fractional dimension. Also, just like $\dim(P)$, $\dim^*(P)$ is “continuous”, that is, removal of just one point from poset doesn't dramatically change its fractional (or regular) dimension.

In fact, putting together some basic results from the theory, it is easy to see the following.

Theorem. If P is a finite poset, there exists a point x, whose removal decreases the fractional dimension by less than 1, unless P is a standard example.

So it looks like Trotter's conjecture is just slightly stronger.

A related question, which is interesting in its own right is the following.

Conjecture (Biró). For every $t > 1$, but sufficiently close to 1, there is a $c > 0$ such that if a poset has $2n$ points, and its dimension is at least tn, then it contains a standard example of dimension cn.

The best reading is [BS92]. For additional information on dimension, see [Tro92]. For another result on fractional dimension, see [FT94].

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Consider the euclidean plane equipped with a standard x, y coordinate system and say a line segment s in the plane is *regular* if it satisfies the following two properties:

1. One end point of the segment is on the negative half of the x-axis; and
2. The slope of the line segment is positive.

Associated with a regular line segment s is a triple (s_1, s_2, s_3) of real numbers with $s_2 > 0 > s_1$ and $s_3 > 0$ so that the "left" end point of s is at $(s_1, 0)$ and the "right" end point of s is at (s_2, s_3).

Farhad Shahrokhi proposed definitions for two partial orders on a family of regular line segments. These orders will be denoted by $\mathfrak{P}_1$ and $\mathfrak{P}_2$, respectively.

Let $s = (s_1, s_2, s_3)$ and $t = (t_1, t_2, t_3)$ be regular line segments. We we say that $s > t$ in $\mathfrak{P}_1$ if

(i) $s_1 < t_1$;

(ii) $s_2 > t_2$; and

(iii) If x_0 is a real number and the vertical line $x = x_0$ intersects s and t at (x_0, y_1) and (x_0, y_2), respectively, then $y_1 > y_2$.

We say that $s > t$ in $\mathfrak{P}_2$ when conditions (i), (ii') and (iii) are satisfied, where conditions (i) and (iii) are just the same as before but now we have

(ii') $s_2 < t_2$.

Conjecture.

$$\mathfrak{P}_1 \neq \mathfrak{P}_2$$

It is quite intriguing to see that this question is open. Many of the common things were shown about $\mathfrak{P}_1$ and $\mathfrak{P}_2$. They contain all posets of dimension at most 3. They contain large dimensional posets, but they are still sparse in that region. More exactly, for all $k > 3$ integer, they contain infinitely many posets that are of dimension at least k, but only a vanishing fraction all posets of dimension at least k. They contain all interval orders.

The problem is very strongly related to stretchabilty of pseudoline arrangements. In the following definitions, the "plane" means the real projective plane.

Definition. A *pseudoline* is a simple closed curve whose removal does not disconnect the plane. An *arrangement of pseudolines* is a set of pseudolines such that any two intersects at exactly one point, and not all of them intersect in the same point. Two pseudoline arrangements are *isomorphic*, if there is a homeomorphism that maps one to the other. A pseudoline arrangement is *stretchable* if it is isomorphic to a pseudoline arrangement, in which every pseudoline is a straight line.

Biró and Trotter defined a family of pseudoline arrangements, and they proved that $\mathfrak{P}_1 = \mathfrak{P}_2$ if and only if every member of the family is stretchable. If one looks at the proof carefully, it reveals that there is something fundemantal with the Euclidean plane that seems to prevent these two classes to collapse—e.g. in the hyperbolic plane $\mathfrak{P}_1$ and $\mathfrak{P}_2$ would be the same.

The main reading is [BT10]. Basics on pseudoline arrangements may be important to work on the question, see [Goo97]. Stretchability question is NP-hard, see [Sho91] for the proof.

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Definition. A *dominating set* of a graph G is a set of vertices S such that every vertex in G is either in S or adjacent to a member of S. The *domination number* $\gamma(G)$ is the size of the smallest such set.

Definition. The *Cartesian product* $G \times H$ is a graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') if either $u = u'$ and v is adjacent to v' in H, or $v = v'$ and u is adjacent to u' in G.

The products $P_n \times P_m$ are called *grids*, for a visually obvious reason: one can depict them as rectangular grids which have n points along one side, and m points along the other.

The value of $\gamma(P_m \times P_m)$ is not known in general. It has an easy geometric formulation: if we have a rectangle consisting of m squares by n unit squares, how many tiles shaped like five-square crosses are necessary to cover it, if the tiles are allowed to overlap or extend off the edge of the board?

We can prove a lower bound of $\frac{mn}{5}$: with mn squares, and with each tile covering 5 squares, obvviously at least $\frac{mn}{5}$ tiles are necessary to cover every square. In addition, it is easy to see that this bound is never achieved, since every tile must cover precisely five unit squares within the rectangle, and any tile which covers a corner cannot achieve this criterion.

For an upper bound, there is an easy construction using $\left\lceil \frac{m}{4} \right\rceil n$ vertices, selecting one out of every four vertices from each row, and offsetting alternate rows. Thus, we know $\frac{mn}{5} < \gamma(P_n \times P_m) \leq \frac{mn}{4}$. The extent to which any given domination number exceeds $\frac{mn}{5}$ is a quantification of the "edge effects"; that is, the extent to which the edges and corners of the grid prevent perfect coverage. Note that in a very large grid, it is easy to achieve a perfect coverage of the grid with pentomino crosses in aribtrary-size regions which do not include the grid's boundary.

Recent results investigating grids have concentrated on variants of domiantion number such as perfect [DD09] and total [Sol10] domination, or on variant problems such as toroidal grid digraphs [NČA09], which are orientations of $C_m \times C_n$.

Investigation suggests, however, that the standard domination number of grids and toroidal grids should be within grasp. El-Zahar and Shaheen [EZS02] have explicitly constructed tight bounds on $\gamma(C_m \times C_n)$ based on the residue classes of m and n modulo 5; for some cases their procedure yields exact values.

Jacobson and Kinch [JK84] determined the exact domination numbers of $P_m \times P_n$ for small values of m; Chang, Clark, and Hare [CCH94] extended their results to determine several exact values and upper bounds for $m \leq 10$; their work sheds significant light on the "fringe effects" seen on a single edge and presents a promising jumping-off point for finding tighter lower and upper bounds for $\gamma(P_n \times P_m)$.

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Imagine that you are a server in a diner; the cook, who is not as cautious as you might like, plates pancakes in messy, arbitrary stacks. You'd like to deliver a plate with pancakes stacked largest-to-smallest, and fortunately, you have a spatula and can fix up your chef's mistakes by lifting some section off the top and flipping it over. At most how many flips will you need to sort a stack of n pancakes?

An explicit construction for a crude upper bound is simple: if you put your spatula under the biggest pancake and flip, it comes to the top, and now you can flip the whole stack to get it on the bottom. Then you flip the second-largest pancake to the top, and flip all but the very bottom of the stack to get it into the second-to-bottom position. Proceeding in this manner it's pretty easy to arrange the whole stack in $2n$ flips (which can be slightly improved to $2n - 3$, observing that the final two pancakes can be oriented with at most a single flip). Likewise, a simple lower bound on the number of necessary flips is n: call a pancake's position "good" if it is adjacent (on either side) to the pancake one larger than itself, or, in the case of the largest pancake, on the bottom. A single flip can make at most one pancake good, namely, the pancake which was on top before the flip, and one can easily construct a stack in which no pancakes are good, so such a stack must requare at least n flips.

We thus know that the number of flips $f(n)$ is linear in the size n of the stack, and fairly tightly constrained, but attempts to tighten up these bounds have moved slowly. The introductory paper [GP79] tightened the above bounds to $\frac{17n}{16} \leq f(n) \leq \frac{5(n+1)}{3}$; the lower bound was further improved in 1997 [HS97] to $\frac{15n}{14} \leq f(n)$, and it was shown quite recently [CFM+09] that $f(n) \leq \frac{18n}{11}$. It is not even known for certain what the "worst" permutations even are for $n > 13$ (exact values have been found for $n \leq 13$ by brute force).

One can approach this as a graph-theoretical question, associating with each permutation a vertex and creating edges among vertices related by prefix-reversal. This so-called *pancake graph* is the Cayley graph for the symmetric group S_n under the generator set consisting of the prefix-reversal permutations; this graph has $n!$ vertices and quickly becomes computationally difficult to work with as a whole; however, pancake labels have been suggested as a viable means for addressing computers on networks, so several questions about pancake graphs, from embeddability of other graphs [Lav02, FH00] to connectivity [LHH05].

A further problem variant: if the pancakes are burned on one side, how many flips will it take to get them all burned-side-down and in order?

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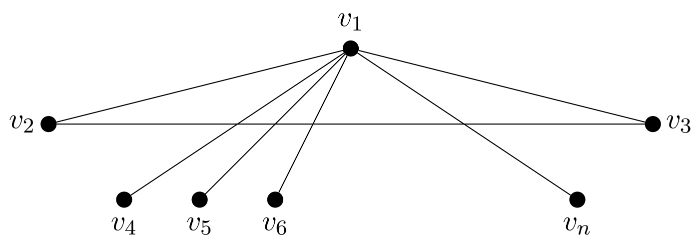
Definition. A thrackle is a drawing of a graph such that every pair of nonadjacent edges cross.[Arc]

To elaborate, a thrackle embedding of a graph is a representation of a graph in which: (1) no edge crosses itself, and (2) for any two edges, either the edges are both incident upon exactly one vertex, or the lines representing the edges cross exactly once.[Weh]

This problem was posed by John Conway in the late 1960's. To answer the question, "What is the maximum number of edges in a thrackle?" Conway proposed this conjecture:

Conjecture. If a graph can be thrackled, then its number of edges is less than or equal to its number of vertices.

A simple example of a thrackle is shown here. More examples can be found at Stephan Wehner's website, http://www.thrackle.org. [Weh]



The conjecture has been proven true for straight-line thrackles[O'R]. The best known bound for any thrackle graph with V vertices is $1.5(V-1)$, as shown by Cairns and Nikolayevsky in 2000 [CN00]. Research is also being done into the types of graphs that can be thrackled. It was shown that a bipartite graph can be thrackled only if it is planar. It is also known that all cycles other than the 4-cycle can be thrackled [Weh].

According to Wehner [Weh], if a counterexample exists, it must contain two cycles; additionally, any counterexample with a minimal number of vertices must be one of the following: 1- a "dumbbell" ("two cycles connected by a path"), 2- a "theta" ("three paths each connecting the same two points"), 3- a "figure-8" ("two cycles sharing a vertex"). Wehner also references an unpublished proof by Mike Rubinstein which shows that if a counterexample of any one of these three types exists, then there must be counterexamples of all 3 types.

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Conjecture. Suppose k runners, having pairwise distinct constant speeds, start at a common point and run laps around a circular track of unit length. Then each runner will be considered lonely at some point in time.

For the purposes of this problem, 'lonely' must be defined:

Definition. A runner is lonely if it is at a distance of $1/k$ from every other runner on the track, when there are k total runners.

This problem has traditionally been considered in the context of diophantine approximations and as a geometric view obstruction problem. The conjecture has long been known true for $k = 3$ or smaller. The case with four runners was proven in 1972. The case for $k = 5$ was first shown using a computer check, but the $k = 5$ and $k = 6$ cases were simplified and shown to be true from 1998-2001 by various groups. Most recently, Barajas and Serra proved the conjecture to be true for $k = 7$ [BS08]. (Note: Barajas and Serra, along with many other sources, refer to the problem in terms of $k + 1$ instead of k, which can make comparing results tedious and confusing. What their paper refers to as $k = 6$ is the case with seven runners discussed here.) The problem is still considered open for all $k > 7$.

The lonely runner conjecture is often considered in the following manner: assume that all speeds are integers, not divisible by the same prime, and that the designated 'lonely' runner has zero speed. Then an equivalent restatement of the conjecture is as follows:

Conjecture. For any set D of $k - 1$ positive integers with gcd 1, there exists a real number t such that, for all d in D, $||td|| \geq 1/k$ [BS08][BHK01].

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Originally posed by Joseph O'Rourke at the 2005 Canadian Conference on Computational Geometry [EDDJO06], this problem involves rolling a labeled cube with the numbers 1-6 over a labeled "board". Each square of the board has a single label from the numbers 1-6. We restrict the cube's movement so that it can only roll over a square if the label on the square matches the label on the top face of the cube. Cells of the board can either be labeled, blocked, or free depending on the problem. A blocked cell cannot be visited by the cube and a free cell can be visited any number of times with no restrictions on the label of the cube. A board can be solved if there exists a path for the cube to roll over each labeled cell exactly once.

Theorem. It is NP-hard to decide if a board with only labeled and free cells has a solution.

This problem is solved in [KBMBEDD+07] and a stronger conjecture is given.

Conjecture. It is NP-hard to decide if a board with only labeled cells has a solution. Given any $n \times n$ board, the solution is unique for $n > 5$.

The uniqueness part of the conjecture was proven to be false when the board also contains blocked cells. This proof used a tool from graph theory known as a state graph.

Definition. The **state graph** has a vertex for each possible state of the die and an edge for each possible transition between two states. A **state** consists of a board position and the entire orientation of the die.

We define the orientation of the cube to be x^y where x represents the top label of the cube and y represents the label adjacent to the cube facing the top of this page. A rollable orientation is represented as a vertex in the state graph. An important property of the state graph is that a given cell on the board only has two possible states.

With these tools, [KBMBEDD+07] shares some insights into solving the conjecture. A reduction to the known NP-hard problem of finding a Hamiltonian cycle in a grid graph is probable.

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Definition. Let $\alpha \in S_n$ (a permutation of n elements), we say a sequence β is a **supersequence** of α if α is a subsequence of β.

Definition. Let $\Sigma = \{x_1, x_2, ...x_n\}$ be our alphabet of n elements. We say a sequence γ is a **universal supersequence** over Σ if, for all $\alpha \in \Sigma$, γ is a supersequence of α. In earlier formulations of this problem universal supersequences were referred to as *permutation strings*.

For example, consider $\Sigma = \{a, b\}$, $\gamma = (aba)$ is not only a universal supersequence, but it is also the shortest that a universal supersequence could be.

Consider $\Sigma = \{a, b, c\}$, $\gamma = (abcabca)$ is a universal supersequence, and is also as short as one can be for this size alphabet.

Question. How big is the smallest universal supersequence that can be constructed for an alphabet of size n?

Theorem. If $f(n)$ is the length of the smallest universal supersequence on an alphabet of size n, then $f(n) \leq n^2 - 2n + 4$.[KH75]

Theorem. $f(n) \geq n^2 - Cn^{7/4+\Delta}$, where $\Delta > 0$ and C depends on Δ.[KK76]

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