

Greedy Cycle Structures and Hamiltonian Cycles in Regular Graphs

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Cycles in Graphs, Vanderbilt

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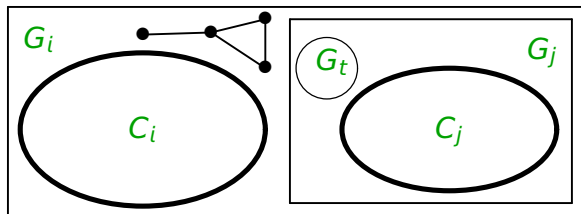
Pf. $\kappa(G) \geq \alpha(G) \Rightarrow$ Lu's Condition: trivial when $\partial S = \bar{S}$.

Otherwise ∂S separates S from $\bar{S} - \partial S$.

Therefore $|\partial S| \geq \kappa(G) \geq \alpha(G) \geq \alpha(G)|\bar{S}|/n$. ■

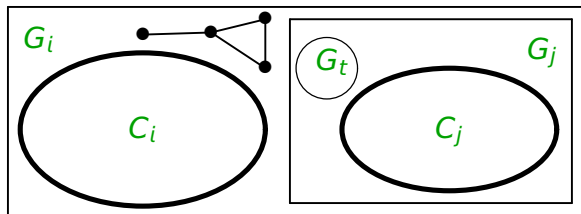
Greedy Cycle Structures

Def. A greedy cycle structure in G is a maximal list $G_0, C_0, G_1, C_1, \dots, G_t$ such that $G_0 = G$, each C_i is a longest cycle in G_i , and G_{i+1} is a component of $G_i - V(C_i)$. Each cycle C_i has a consistent orientation.



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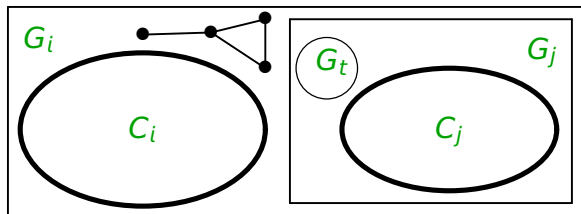
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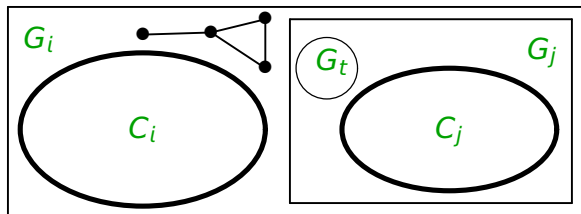
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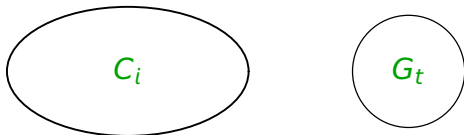
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Hardest case: $G_t = K_1$. Hopping Lemma needed.

Tools

Let $r = |V(G_t)|$ and $A = \bigcup_{i=0}^{t-1} V(C_i)$.

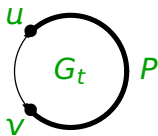
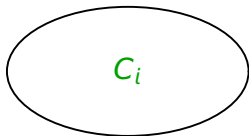


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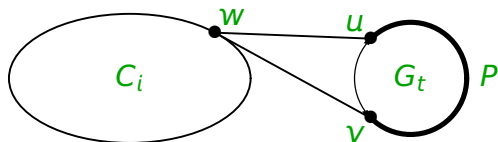
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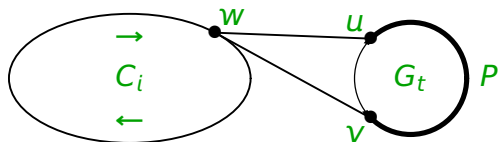
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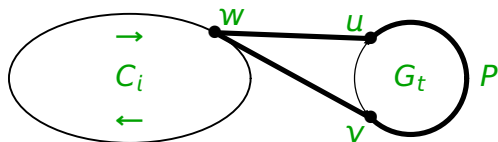
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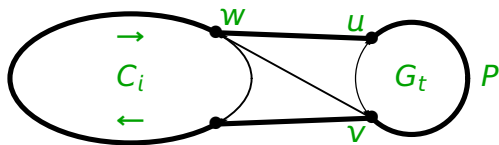
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If fewer than p vertices following w are blank, then a detour along P yields a cycle longer than C_i . ■

More Blank Vertices

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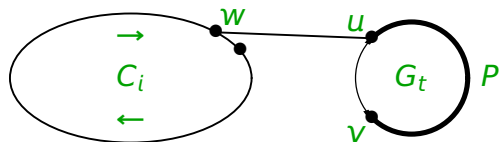
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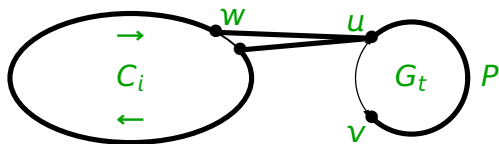
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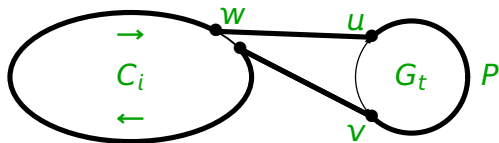
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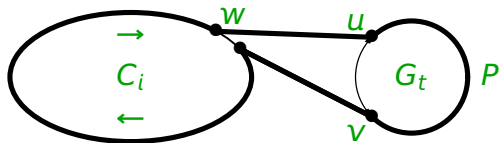
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- If $p \geq 3$, then for $w \in Q$ we can count 2 for each edge from $\{u, v\}$ to w instead of $p + 1$ for the vertex w .

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Again each remaining edge from $\{u, v\}$ to A gives two more vertices, so

$$3k - r \geq |A| \geq 2p - 2 + 2(2k - s). \quad (1)$$